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Note

Smallest independent dominating sets in Kronecker products of cycles

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Abstract

Let $k\geqslant 2$, $n=2^k+1$, and let $m_0,\dots,m_{k-1}$ each be a multiple of n. The graph $C_{m_0}\times\cdots\times C_{m_{k-1}}$ consists of isomorphic connected components, each of which is $(n-1)$-regular and admits of a vertex partition into n smallest independent dominating sets. Accordingly, (independent) domination number of each connected component of this graph is equal to $(1/n)$th of the number of vertices in it.

Keywords: Smallest independent dominating sets; Perfect dominating sets; Codes; Kronecker product; Cycle

1. Introduction

By a graph is meant a finite, simple and undirected graph. The *Kronecker product* $G\times H$ of graphs $G=(V,E)$ and $H=(W,F)$ is defined as follows: $V(G\times H)=V\times W$ and $E(G\times H)=\{\{(u,x),(v,y)\}\colon \{u,v\}\in E \text{ and } \{x,y\}\in F\}$. This product (that is variously known as direct product, cardinal product, categorical product, tensor product and cross product) is one of the most important graph products, with applications in a number of areas. It is commutative and associative in a natural way. Let C_n denote the *cycle* on vertices $0,\dots,n-1$, where adjacencies are defined in the natural way.

Let S be a vertex subset of a given graph $G=(V,E)$. S is said to be a *dominating set* of G if every $x\in V$ is either an element of S or is adjacent to at least one element of S. A dominating set whose elements are mutually nonadjacent in G is called an *independent dominating set* of G, and an independent dominating set of least cardinality is called a *smallest independent dominating set* (s.i.d.s.). Further, if S is

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such that every $x \in V$ is either in S or is adjacent to exactly one element of S, then S is called a *perfect dominating set* of G. By (*independent*) *domination number* of G is meant the cardinality of a smallest (independent) dominating set of G.

The general problem of obtaining a smallest (independent) dominating set is NP-hard even for bipartite graphs [2]. In fact, an s.i.d.s. is not even approximable in polynomial time within a factor of $n^{1-\varepsilon}$ for any $\varepsilon > 0$ unless P = NP [5].

Domination in graphs has a number of applications in areas such as game theory, coding theory, channel assignment and resource placement. Accordingly, it has an extensive literature, cf. Haynes et al. [6]. Perfect dominating sets, in particular, are directly relevant to error-correcting codes. They have been studied in various contexts, Hamming codes being the most important [8,14]. Perfect codes with respect to Cartesian-product graphs have been treated by Kratochvil [11], and Livingston and Stout [12]. Additional references on domination in this product include Klavž ar and Seifter [9] and Gravier and Mollard [4]. Nowakowski and Rall [13] present a systematic approach to graph invariants (including domination number) on graph products.

Domination in Kronecker-product graphs has been studied by several authors [1,3,10]. The present paper presents a vertex partition of Kronecker products of certain cycles into smallest (independent) dominating sets. In fact, each set in the partition is a perfect dominating set.

2. Result

Proposition 1 (Jha [7]). *Let* $m_0,\dots,m_{k-1} \geqslant 3$, *where* $k \geqslant 2$.

1. $C_{m_0} \times \cdots \times C_{m_{k-1}}$ *is a regular graph of degree* 2^k.
2. $C_{m_0} \times \cdots \times C_{m_{k-1}}$ *is bipartite iff at least one* m_i *is even.*
3. *Let* r *be the number of even integers among* $m_0,\dots,m_{k-1}$. *If* r *is at most one, then* $C_{m_0} \times \cdots \times C_{m_{k-1}}$ *is connected, otherwise this graph consists of* 2^{r-1} *connected components that are mutually isomorphic.*
4. *Each component of* $C_{m_0} \times \cdots \times C_{m_{k-1}}$ *is edge-decomposable into Hamiltonian cycles.*

For $r \geqslant 2$, let $m_0,\dots,m_{r-1}$ be all even $\geqslant 4$. The graph $C_{m_0} \times \cdots \times C_{m_{r-1}}$ is such that vertices $(v_0,\dots,v_{r-1})$ and $(w_0,\dots,w_{r-1})$ belong to the same component iff $v_i + v_{i+1}$ and $w_i + w_{i+1}$ are of the same parity, $0 \leqslant i \leqslant r-2$. It is also relevant to note that if j is even, $j/2$ is odd and G is a bipartite graph, then each of the two components of $C_j \times G$ is isomorphic to $C_{j/2} \times G$.

The following is the central result of this paper.

Theorem 2. *If* $k \geqslant 2$, $n = 2^k + 1$, *and* $m_0,\dots,m_{k-1}$ *are each a multiple of* n, *then each connected component of the graph* $C_{m_0} \times \cdots \times C_{m_{k-1}}$ *admits of a vertex partition into smallest independent dominating sets.*

Proof. $C_{m_0} \times \cdots \times C_{m_{k-1}}$ is a regular graph of degree $2^k = n-1$, hence an (independent) dominating set of each component of this graph must include at least $(1/n)$th of the vertices. Therefore, it suffices to label the vertices with integers $0,\dots,n-1$ such that any two distinct elements that are labeled alike are at a distance of at least three.

Let a vertex $v=(v_0,\dots,v_{k-1})$ be assigned the integer

$$\left[\sum_{i=0}^{k-1} 2^{i+1} v_i\right] \bmod n.$$

The assignment is clearly well defined.

A vertex adjacent to $(v_0,\dots,v_{k-1})$ is of the form $(v_0+a_0,\dots,v_{k-1}+a_{k-1})$, where $a_i\in\{+1,-1\}$ and v_i+a_i is modulo m_i. It is clear that $(v_0+a_0,\dots,v_{k-1}+a_{k-1})$ receives the label

$$\left[\left(\sum_{i=0}^{k-1} 2^{i+1} v_i\right)+\left(\sum_{i=0}^{k-1} 2^{i+1} a_i\right)\right] \bmod n.$$

Note that

- $(\sum_{i=0}^{k-1} 2^{i+1} a_i)$ is an even integer,
- $(\sum_{i=0}^{k-1} 2^{i+1} a_i)$ is of the same sign as a_{k-1}, so $(\sum_{i=0}^{k-1} 2^{i+1} a_i)\neq 0$, and
- $|(\sum_{i=0}^{k-1} 2^{i+1} a_i)|\leqslant(\sum_{i=0}^{k-1} 2^{i+1}|a_i|)=2^{k+1}-2<2n.$

Since n itself is odd, $(\sum_{i=0}^{k-1} 2^{i+1} a_i)$ is not a multiple of n. It follows that adjacent vertices receive different labels.

A vertex at a distance of two from $(v_0,\dots,v_{k-1})$ is of the form $(v_0+b_0,\dots,v_{k-1}+b_{k-1})$, where $b_0,\dots,b_{k-1}$ are not all zero, $b_i\in\{+2,\ 0,\ -2\}$ and v_i+b_i is modulo m_i. Vertex $(v_0+b_0,\dots,v_{k-1}+b_{k-1})$ receives the label

$$\left[\left(\sum_{i=0}^{k-1} 2^{i+1} v_i\right)+\left(\sum_{i=0}^{k-1} 2^{i+1} b_i\right)\right] \bmod n.$$

Note that

- $(\sum_{i=0}^{k-1} 2^{i+1} b_i)$ is of the form $4p$ for some p,
- letting r be the largest integer such that $b_r\neq 0$, $(\sum_{i=0}^{k-1} 2^{i+1} b_i)$ is of the same sign as b_r, so $(\sum_{i=0}^{k-1} 2^{i+1} b_i)\neq 0$, and
- $|(\sum_{i=0}^{k-1} 2^{i+1} b_i)|\leqslant(\sum_{i=0}^{k-1} 2^{i+1}|b_i|)\leqslant(\sum_{i=0}^{k-1} 2^{i+2})=2^{k+2}-4=4n-8<4n.$

Since n (and hence $3n$) is odd and $2n$ is of the form $4t+2$, $(\sum_{i=0}^{k-1} 2^{i+1} b_i)$ is not a multiple of n. It follows that vertices that are at a distance of two receive different labels. (Conclusions are valid even if v_i is of the form m_i-2 or m_i-1, since each m_i itself is a multiple of n, and the arithmetic is modulo n.)

For any (isomorphic) component of $C_{m_0}\times\cdots\times C_{m_{k-1}}$, let V_r denote the set of vertices that receive label r, where $0\leqslant r\leqslant n-1$. The sets $V_0,\dots,V_{n-1}$ constitute a vertex partition of that component into smallest independent dominating sets. □

Each V_r in the proof of Theorem 2 is also a smallest dominating set of that component. Also, each such set corresponds to a vertex decomposition into $K_{1,n-1}$'s.

Corollary 3. *Let* $k \geqslant 2$, $n = 2^k + 1$, *and let* $m_0, \ldots, m_{k-1}$ *each be a multiple of* n. *If the number* r *of integers among* $m_0, \ldots, m_{k-1}$ *is at most one, then the graph* $C_{m_0} \times \cdots \times C_{m_{k-1}}$ *is connected, otherwise it consists of* 2^{r-1} *connected components. The (independent) domination number of each connected component of* $C_{m_0} \times \cdots \times C_{m_{k-1}}$ *is equal to* $(1/n)$*th of the number of vertices in it.*

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