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Note

Optimal $L(d,1)$-labelings of certain direct products of cycles and Cartesian products of cycles

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Abstract

An $L(d,1)$-labeling of a graph G is an assignment of nonnegative integers to the vertices such that adjacent vertices receive labels that differ by at least d and those at a distance of two receive labels that differ by at least one, where $d \geqslant 1$. Let $\lambda_1^d(G)$ denote the least λ such that G admits an $L(d,1)$-labeling using labels from $\{0, 1, \ldots, \lambda\}$. We prove that (i) if $d \geqslant 1$, $k \geqslant 2$ and $m_0, \ldots, m_{k-1}$ are each a multiple of $2^k + 2d - 1$, then $\lambda_1^d(C_{m_0} \times \cdots \times C_{m_{k-1}}) \leqslant 2^k + 2d - 2$, with equality if $1 \leqslant d \leqslant 2^k$, and (ii) if $d \geqslant 1$, $k \geqslant 1$ and $m_0, \ldots, m_{k-1}$ are each a multiple of $2k + 2d - 1$, then $\lambda_1^d(C_{m_0} \Box \cdots \Box C_{m_{k-1}}) \leqslant 2k + 2d - 2$, with equality if $1 \leqslant d \leqslant 2k$.

Keywords: $L(d,1)$-labeling; λ-number; Direct product of graphs; Cartesian product of graphs; Channel assignment.

1. Introduction and Preliminaries

Consider the problem of assigning frequencies to radio transmitters at various nodes in a territory. Transmitters that are close must receive frequencies that are sufficiently apart, for

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otherwise they may be at the risk of interfering with each other. The spectrum of frequencies is an important resource on which there are increasing demands, both civil and military. This calls for an efficient management of the spectrum. It is assumed that transmitters are of identical type and that signal propagation is isotropic.

The foregoing problem, with the objective of minimizing the span of frequencies, was first placed on a graph-theoretical footing in 1980 by Hale [5]. (Vertices correspond to transmitter locations and their labels to radio frequencies, while adjacencies are determined by geographical "proximity" of the transmitters.) Roberts [13] subsequently proposed a variation to the problem in which distinction is made between transmitters that are "close" and those that are "very close." This enabled Griggs and Yeh [4] to formulate the $L(2,1)$-labeling of graphs. Georges and Mauro [1] later presented a generalization of the concept. The topic has since been an object of extensive research [1–4,7–12,14,15].

Formally, an $L(d,1)$-*labeling* of a graph G is an assignment f of non-negative integers to vertices of G such that

$$|f(u) - f(v)| \geqslant \begin{cases} d; & d(u,v) = 1, \\ 1; & d(u,v) = 2, \end{cases}$$

where $d \geqslant 1$. The difference between the largest label and the smallest label assigned by f is called the *span* of f, and the minimum span over all $L(d,1)$-labelings of G is called the λ_1^d-*number* of G, denoted by $\lambda_1^d(G)$. The general problem of determining $\lambda_1^d(G)$ is NP-hard [3].

When we speak of a graph, we mean a finite, simple undirected graph having at least two vertices. Let P_m and C_n denote a path on m vertices and a cycle on n vertices, respectively, where $V(P_k) = V(C_k) = \{0, \ldots, k-1\}$ and where adjacencies are defined in a natural way. For graphs $G = (V, E)$ and $H = (W, F)$, the *direct product* $G \times H$ and the *Cartesian product* $G \Box H$ of G and H are defined as follows: $V(G \times H) = V(G \Box H) = V \times W$; $E(G \times H) = \{\{(a,x), (b,y)\}: \{a,b\} \in E$ and $\{x,y\} \in F\}$ and $E(G \Box H) = \{\{(a,x), (b,y)\}: \{a,b\} \in E$ and $x = y$, or $\{x,y\} \in F$ and $a = b\}$, cf. [6]. The direct product is also known as Kronecker product, tensor product, cardinal product and categorical product.

The result below consists of a useful lower bound on $\lambda_1^d(G)$, see [1, Theorem 2.9 (ii)].

Lemma 1. If G is a graph with maximum degree Δ and G includes a vertex with Δ neighbors, each of which is of degree Δ, then $\lambda_1^d(G) \geqslant \Delta + 2d - 2$, where $1 \leqslant d \leqslant \Delta$. $\Box$

The central message of this paper is that the preceding lower bound corresponds to the exact value with respect to $C_{m_0} \times \cdots \times C_{m_{k-1}}$ and $C_{m_0} \Box \cdots \Box C_{m_{k-1}}$ where there are certain conditions on d and on $m_0, \cdots, m_{k-1}$. Analogous result is known with respect to λ_1^2-numbering of the strong products of cycles [8]. For results with respect to Cartesian products, see [2,7,10,11,14,15]. The following fact will be useful in the sequel.

Claim 2. If a, b and n are integers with $n \geqslant 1$, then $|(a \bmod n) - (b \bmod n)| = (|a-b| \bmod n)$ or $n - (|a-b| \bmod n)$. $\Box$

Section 2 deals with the λ_1^d-numbering of direct products of cycles while Section 3 presents the analogous result with respect to Cartesian products of cycles. Methods of attack are similar. Concluding remarks appear in Section 4.

2. $L(d,1)$-labeling of $C_{m_0} \times \cdots \times C_{m_{k-1}}$

Theorem 3. If $d \geqslant 1$, $k \geqslant 2$, and $m_0, \ldots, m_{k-1}$ are each a multiple of $2^k + 2d - 1$, then $\lambda_1^d(C_{m_0} \times \cdots \times C_{m_{k-1}}) \leqslant 2^k + 2d - 2$, with equality if $1 \leqslant d \leqslant 2^k$.

Proof. Let $n = 2^k + 2d - 1$, and let a vertex $v = (v_0, \cdots, v_{k-1})$ be assigned the integer

$$f(v) = \left[\frac{1}{2}(n-1)\sum_{i=0}^{k-1} 2^i v_i\right] \bmod n.$$

The assignment is clearly well-defined. Let w be a vertex adjacent to v, so w is of the form $(v_0 + a_0, \ldots, v_{k-1} + a_{k-1})$, where $a_i \in \{+1, -1\}$ and $v_i + a_i$ is modulo m_i, $0 \leqslant i \leqslant k-1$. It is clear that

$$f(w) = \left[\left(\frac{1}{2}(n-1)\sum_{i=0}^{k-1} 2^i v_i\right) + \left(\frac{1}{2}(n-1)\sum_{i=0}^{k-1} 2^i a_i\right)\right] \bmod n.$$

To show that $|f(v) - f(w)| \geqslant d$, it is enough to show that

$$d \leqslant \left(\left|\frac{1}{2}(n-1)\sum_{i=0}^{k-1} 2^i a_i\right| \bmod n\right) \leqslant n - d$$

since by Claim 2,

$$|f(v) - f(w)| = \left|\frac{1}{2}(n-1)\sum_{i=0}^{k-1} 2^i a_i\right| \bmod n \text{ or } n - \left(\left|\frac{1}{2}(n-1)\sum_{i=0}^{k-1} 2^i a_i\right| \bmod n\right).$$

Note that $\left|\sum_{i=0}^{k-1} 2^i a_i\right|$ is odd and

$$\left|\sum_{i=0}^{k-1} 2^i a_i\right| \leqslant \sum_{i=0}^{k-1} 2^i |a_i| = \sum_{i=0}^{k-1} 2^i = 2^k - 1.$$

Hence $\left|\sum_{i=0}^{k-1} 2^i a_i\right| = 2p + 1$ where $0 \leqslant p \leqslant 2^{k-1} - 1$, and consequently,

$$\left|\frac{1}{2}(n-1)\sum_{i=0}^{k-1} 2^i a_i\right| = \frac{1}{2}(n-1)\left|\sum_{i=0}^{k-1} 2^i a_i\right| = \frac{1}{2}(n-1)(2p+1)$$
$$= \left(\frac{1}{2}(n-1) - p\right) + np.$$

The desired result follows since $\frac{1}{2}(n-1)-p$ is not a multiple of n. To verify this claim, first observe that

$$\tfrac{1}{2}(n-1)-p \geqslant \tfrac{1}{2}(n-1)-(2^{k-1}-1) \geqslant d.$$

On the other hand,

$$\tfrac{1}{2}(n-1)-p \leqslant \tfrac{1}{2}(n-1) \leqslant n-d.$$

Finally, let x be a vertex at a distance of two from $(v_0, \ldots, v_{k-1})$. It is clear that x is of the form $(v_0+b_0, \ldots, v_{k-1}+b_{k-1})$, where $b_i \in \{+2,\ 0,\ -2\}$, $b_0, \ldots, b_{k-1}$ are not all zero, and v_i+b_i is modulo m_i. Note that

$$f(x)=\left[\left(\frac{1}{2}(n-1)\sum_{i=0}^{k-1} 2^i v_i\right)+\left(\frac{1}{2}(n-1)\sum_{i=0}^{k-1} 2^i b_i\right)\right] \bmod n.$$

We claim that $\frac{1}{2}(n-1)\sum_{i=0}^{k-1} 2^i b_i$ is not a multiple of n. Since $\frac{1}{2}(n-1)$ and n are coprime, we need only show that $\sum_{i=0}^{k-1} 2^i b_i$ is not a multiple of n:

- Letting r be the largest integer such that $b_r \neq 0$, it is easy to see that $\sum_{i=0}^{k-1} 2^i b_i$ is of the same sign as b_r, so $\sum_{i=0}^{k-1} 2^i b_i \neq 0$.
- $|\sum_{i=0}^{k-1} 2^i b_i| \leqslant \sum_{i=0}^{k-1} 2^i |b_i| \leqslant \sum_{i=0}^{k-1} 2^{i+1} = 2^{k+1}-2 < 2n$.

Since $\sum_{i=0}^{k-1} 2^i b_i$ is necessarily even and n is odd, it follows that $\sum_{i=0}^{k-1} 2^i b_i$ is not a multiple of n. Accordingly, two vertices that are at a distance of two from each other receive different labels.

Claims are valid even if v_i is of the form m_i-2 or m_i-1, since m_i itself is a multiple of n, and the arithmetic is modulo n. Accordingly, $\lambda_1^d(C_{m_0} \times \cdots \times C_{m_{k-1}}) \leqslant 2^k+2d-2$. Further, $C_{m_0} \times \cdots \times C_{m_{k-1}}$ being a regular graph of degree 2^k, an application of Lemma 1 to the preceding statement shows that $\lambda_1^d(C_{m_0} \times \cdots \times C_{m_{k-1}}) = 2^k+2d-2$, if $1 \leqslant d \leqslant 2^k$. $\square$

The foregoing scheme is illustrated in Fig. 1 where an $L(3,1)$-labeling of $P_9 \times P_{18}$ appears toward that of $C_9 \times C_{18}$.

3. $L(d,1)$-labeling of $C_{m_0} \square \cdots \square C_{m_{k-1}}$

Theorem 4. If $d \geqslant 1$, $k \geqslant 1$ and $m_0, \ldots, m_{k-1}$ are each a multiple of $2k+2d-1$, then $\lambda_1^d(C_{m_0} \square \cdots \square C_{m_{k-1}}) \leqslant 2k+2d-2$, with equality if $1 \leqslant d \leqslant 2k$.

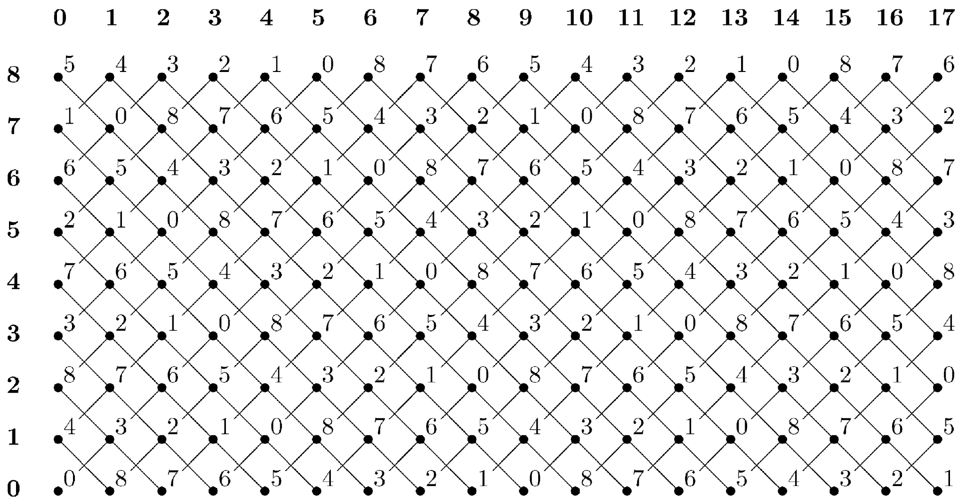

Fig. 1. $L(3, 1)$-labeling of $P_9 \times P_{18}$ toward that of $C_9 \times C_{18}$

Proof. Let $n = 2k + 2d - 1$. For $k = 1$, there is a single cycle C_{nt}, $t \geqslant 1$, for which the claim is easily seen to be true. In what follows, let $k \geqslant 2$ and let a vertex $v = (v_0, \ldots, v_{k-1})$ be assigned the integer

$$f(v) = \left[\sum_{i=0}^{k-1} (d + 2i)v_i\right] \bmod n.$$

The assignment is clearly well-defined. Let $w = (w_0, \ldots, w_{k-1})$ be a vertex adjacent to v, so v and w differ in exactly one coordinate, say i, such that v_i and w_i are adjacent in C_{m_i}, whence $|v_i - w_i| \bmod n = 1$.

To show that $|f(v) - f(w)| \geqslant d$, it is enough to show that

$$d \leqslant (d + 2i) \bmod n \leqslant n - d,$$

since by Claim 2,

$$|f(v) - f(w)| = (d + 2i) \bmod n \text{ or } n - ((d + 2i) \bmod n).$$

The desired result follows since $d \leqslant d + 2i \leqslant d + 2(k - 1) \leqslant n - d$.

Next, let $x = (x_0, \ldots, x_{k-1})$ be a vertex at a distance of two from v, so either (i) v and x differ in exactly one coordinate, say i, such that $|v_i - x_i|\, mod\, n = 2$, or (ii) v and x differ in exactly two coordinates, say i and j, such that $|v_i - x_i| \bmod n = 1$ and $|v_j - x_j| \bmod n = 1$, where $i \neq j$.

Let $|v_i - x_i| \bmod n = 2$. To show that $|f(v) - f(x)| \geqslant 1$, it is enough to show that

$$0 < 2(d + 2i) \bmod n < n,$$

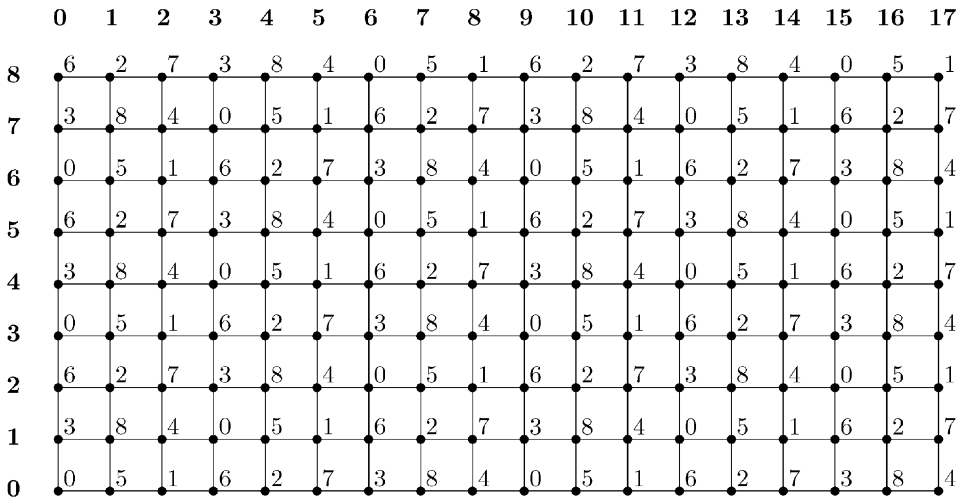

Fig. 2. $L(3,1)$-labeling of $P_9\Box P_{18}$ toward that of $C_9\Box C_{18}$

since by Claim 2,

$$|f(v)-f(x)| = 2(d+2i) \bmod n \text{ or } n-(2(d+2i) \bmod n).$$

The desired result follows since $0<2(d+2i)\leqslant 2(d+2(k-1))<n$.

Now suppose that v and x differ in the ith and jth coordinates, whence $|v_i-x_i| \bmod n=1$, $|v_j-x_j| \bmod n=1$ and $0\leqslant i<j\leqslant k-1$. To show that $|f(v)-f(x)|\geqslant 1$, it is enough to show that $|d(A+B)+2(Ai+Bj)|$ is not a multiple of n, with $A,\ B$ in $\{1,\ -1\}$ since by Claim 2,

$$\begin{aligned}|f(v)-f(x)| = {} & |d(A+B)+2(Ai+Bj)| \bmod n \text{ or}\\ & n-(|d(A+B)+2(Ai+Bj)| \bmod n).\end{aligned}$$

Clearly $A+B$ is even, so $|d(A+B)+2(Ai+Bj)|$ is even and hence different from n. If $A=B$, then

$$0<|d(A+B)+2(Ai+Bj)|=|2d+2(i+j)|<2d+4(k-1)<2n.$$

On the other hand, if $A=-B$, then

$$0<|d(A+B)+2(Ai+Bj)|=|2(i-j)|\leqslant 2(k-1)<n.$$

In each case, $|d(A+B)+2(Ai+Bj)|$ cannot be a multiple of n.

It follows that $\lambda_1^d(C_{m_0}\Box\cdots\Box C_{m_{k-1}})\leqslant 2k+2d-2$. Further, $C_{m_0}\Box\cdots\Box C_{m_{k-1}}$ being a regular graph of degree $2k$, an application of Lemma 1 to the preceding statement shows that $\lambda_1^d(C_{m_0}\Box\cdots\Box C_{m_{k-1}})=2k+2d-2$, if $1\leqslant d\leqslant 2k$. $\Box$

The foregoing scheme is illustrated in Fig. 2 where an $L(3,1)$-labeling of $P_9\Box P_{18}$ appears toward that of $C_9\Box C_{18}$.

4. Concluding remarks

It is known that if $k \geqslant 2$ and $m_0, \ldots, m_{k-1}$ are each a multiple of 2^k+1, then the graph $C_{m_0} \times \cdots \times C_{m_{k-1}}$ admits a vertex partition into smallest independent dominating sets [9]. That result easily follows from the proof of Theorem 3 for $d=1$. Similarly, it is known that (i) if $k \geqslant 1$ and $m_0, \ldots, m_{k-1}$ are each a multiple of $2k+1$, then the graph $C_{m_0} \Box \cdots \Box C_{m_{k-1}}$ admits a vertex partition into smallest independent dominating sets, and (ii) if $k \geqslant 1$ and $m_0, \ldots, m_{k-1}$ are each a multiple of $2k+3$, then $\lambda_1^2(C_{m_0} \Box \cdots \Box C_{m_{k-1}}) = 2k+2$ [7]. These results follow from Theorem 4 for $d=1$ and $d=2$, respectively.

$L(d,1)$-labeling and the associated λ_1^d-numbering of a graph have been studied in a more general setting of $L(j,k)$-labeling and λ_k^j-numbering, where $j \geqslant k \geqslant 1$. In particular, Georges and Mauro [1] proved that $\lambda_{ck}^{cj}(G) = c\lambda_k^j(G)$. An application of this statement to Theorems 3 and 4 leads to the following result.

Corollary 5. Let $c,\ d \geqslant 1$.

(1) If $k \geqslant 2$ and $m_0, \ldots, m_{k-1}$ are each a multiple of 2^k+2d-1, then $\lambda_c^{cd}(C_{m_0} \times \cdots \times C_{m_{k-1}}) \leqslant c(2^k+2d-2)$, with equality if $1 \leqslant d \leqslant 2^k$.

(2) If $k \geqslant 1$ and $m_0, \ldots, m_{k-1}$ are each a multiple of $2k+2d-1$, then $\lambda_c^{cd}(C_{m_0} \Box \cdots \Box C_{m_{k-1}}) \leqslant c(2k+2d-2)$, with equality if $1 \leqslant d \leqslant 2k$. $\Box$

Another measure of labeling a graph G with a condition at distance two is called the *circular-$L(d,1)$-labeling* that is an assignment g of integers $0, \ldots, r-1$ to the vertices of G such that

$$|g(u) - g(v)|_r \geqslant \begin{cases} d; & d(u,v) = 1, \\ 1; & d(u,v) = 2, \end{cases}$$

where $|x|_r := \min\{|x|,\ r-|x|\}$ [12]. The least r for which G has a circular-$L(d,1)$-labeling is denoted by $\sigma_1^d(G)$. It is easy to see that $\sigma_1^d(G) \geqslant \lambda_1^d(G)+1$. The following result is a simple consequence of the constructions in the proofs of Theorems 3 and 4.

Corollary 6.

(1) For $k \geqslant 2$, if $1 \leqslant d \leqslant 2^k$ and $m_0, \ldots, m_{k-1}$ are each a multiple of 2^k+2d-1, then $\sigma_1^d(C_{m_0} \times \cdots \times C_{m_{k-1}}) = 2^k+2d-1$.

(2) For $k \geqslant 1$, if $1 \leqslant d \leqslant 2k$ and $m_0, \ldots, m_{k-1}$ are each a multiple of $2k+2d-1$, then $\sigma_1^d(C_{m_0} \Box \cdots \Box C_{m_{k-1}}) = 2k+2d-1$. $\Box$

In this paper, we demonstrate that direct products of cycles and Cartesian products of cycles admit optimal $L(d,1)$-labelings if certain conditions are imposed on d and on the lengths of the cycles. Is optimality still achievable if these conditions are relaxed? To that end, we employed a backtracking algorithm to compute $\lambda_1^d(C_m \times C_n)$ and $\lambda_1^d(C_m \Box C_n)$ for $1 \leqslant d \leqslant 4$ and $4 \leqslant m,\ n \leqslant 10$. The results appear in Table 1.

Note that by Lemma 1 each of $\lambda_1^2(C_m \times C_n)$ and $\lambda_1^2(C_m \Box C_n)$ is greater than or equal to 6; each of $\lambda_1^3(C_m \times C_n)$ and $\lambda_1^3(C_m \Box C_n)$ is greater than or equal to 8; and each of $\lambda_1^4(C_m \times C_n)$ and $\lambda_1^4(C_m \Box C_n)$ is greater than or equal to 10.

Table 1
$L(d,1)$-numbers of $G = C_m \times C_n$ and $H = C_m \square C_n$

m, n	$\lambda_1^2(G)$	$\lambda_1^3(G)$	$\lambda_1^4(G)$	$\lambda_1^2(H)$	$\lambda_1^3(H)$	$\lambda_1^4(H)$
4, 4	7	9	10	7	9	10
4, 5	8	9	11	7	9	11
4, 6	7	9	11	7	9	11
4, 7	7	9	11	7	9	11
4, 8	7	9	10	7	9	10
4, 9	7	9	11	7	9	11
4, 10	8	9	11	7	9	11
4, 11	7	9	11	7	9	11
5, 5	8	10	12	8	10	12
5, 6	7	10	12	8	9	11
5, 7	8	10	12	7	10	11
5, 8	8	9	11	7	9	11
5, 9	8	10	12	8	9	11
5, 10	8	10	11	8	9	11
5, 11	8	10	12	7	9	11
6, 6	8	10	12	7	9	11
6, 7	7	10	12	8	9	11
6, 8	7	9	11	7	9	11
6, 9	7	10	12	7	8	10
6, 10	7	10	12	7	9	11
6, 11	7	10	12	8	9	11
7, 7	6	9	11	6	9	11
7, 8	7	9	11	7	9	11
7, 9	8	9	11	8	9	11
7, 10	7	9	11 or 12	7	9	11
7, 11	7	9	11	7	9	11
8, 8	7	9	10	7	9	10
9, 9	7	8	10	7	8	10
10, 10	8	10	11	7	9	11

It is clear from Table 1 that for $2 \leqslant d \leqslant 4$ and $4 \leqslant m,\ n \leqslant 10$, if the conditions of Theorems 3 and 4 are not satisfied, then there are very few cases where $\lambda_1^d(C_m \times C_n)$ and $\lambda_1^d(C_m \square C_n)$ are equal to the lower bound.

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