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Cycle Kronecker products that are representable as optimal circulants

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ABSTRACT

Broere and Hattingh proved that the Kronecker product of two cycles is a circulant if and only if the cycle lengths are coprime. In this paper, we specify which of these Kronecker products are actually optimal circulants. Further, we present their salient characteristics based on their edge decompositions into Hamiltonian cycles. It turns out that certain products thus distinguished have the added property of being tight-optimal, so their average distances are the least among all circulants of the same order and size. A benefit of the present study is that the existing results on the Kronecker product of two cycles may be used to good effect while putting these circulants into practice. The areas of applications include parallel computers, distributed systems and VLSI.

1. Introduction

Circulant graphs, which we formally define below, constitute a subfamily of Cayley graphs [10]. They possess attractive features such as simplicity, high symmetry, high connectivity and scalability, which lend them to an application as a *network topology* in areas like parallel computers, distributed systems and VLSI [2,3,11].

The question arises as to which *Kronecker products* of circulants are again circulants. Broere and Hattingh [6] attacked this problem in a general setting. Among other things, they proved that the product of two *cycles* is a circulant if and only if the cycle lengths are coprime.

We take the next major step and characterize the Kronecker products of two cycles representable as *optimal circulants*. The products thus distinguished appear in Table 1, which additionally presents certain relevant properties of the graphs. (The implicit claims will be proved later.)

1.1. Definitions and preliminaries

When we speak of a graph, we mean a finite, simple, undirected and connected graph. Let $\mathrm{dist}(u, v)$ denote the shortest distance or path length between vertices u and v, where the underlying graph will be clear from the context. For a given graph G, let $dia(G)$ represent its diameter, i.e., $\max\{\mathrm{dist}(u, v) : u, v \in V(G)\}$. We employ *vertex* and *node* as synonyms, and write "G is isomorphic to H" as $G \cong H$. Let $\alpha(G)$ denote the *independence number* of G, i.e., the largest number of mutually nonadjacent nodes in G.

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Table 1
Products of cycles distinguished as optimal circulants.

Product (a odd)	Odd girth	Distance-wise vertex distribution	Tight-optimal?
$C_a \times C_{2a-1}$	$2a-1$	$1 + \underbrace{4i}_{1\le i\le a-1} + (a-1)$	Yes
$C_a \times C_{2a+1}$	$2a+1$	$1 + \underbrace{4i}_{1\le i\le a-1} + (3a-1)$	Yes
$C_a \times C_{2a+3}$	$2a+3$	$1 + \underbrace{4i}_{1\le i\le a-1} + (3a-1) + (2a)$	No
$a \not\equiv 0 (\mathrm{mod}\ 3)$			

Say that a vertex v is at *level i* relative to a fixed vertex u if $\mathrm{dist}(u, v) = i$. Vertices at a distance of $dia(G)$ from u are called *diametrical* relative to u. A *level diagram* of G relative to u consists of a layout of the graph in which vertices at a distance of i from u appear on a line at height i above u, for $0 \le i \le dia(G)$. If G is known to be vertex transitive (a property held by a circulant), then the form of its level diagram is independent of the choice of the source vertex.

A circulant in the present study connotes a four-regular circulant. To that end, let n, r, s be positive integers, where $n \ge 6$, and $1 \le r < s < n/2$. Then the circulant $\mathcal{C}_n(r, s)$ consists of the vertex set $\{0, \ldots, n-1\}$ and the edge set $\{\{i,\ i \pm r\}, \{i,\ i \pm s\} \mid 0 \le i \le n-1\}$, where $i \pm r$ and $i \pm s$ are each taken modulo n. The parameters r and s are called the *step sizes*. If one of the step sizes is fixed at one, then the circulant is also known as a *chordal ring* or a *double-loop network*.

Proposition 1.1 (*[5]*). *The diameter of a circulant on n vertices is greater than or equal to the least integer c such that* $n \le (c+1)^2 + c^2$. *Hence the diameter is greater than or equal to* $\left\lceil \frac{1}{2}\left(-1 + \sqrt{2n-1}\right)\right\rceil$. ∎

A circulant, say G, is said to be *optimal* (or of *minimal diameter*) if its diameter meets the lower bound from Proposition 1.1 [20]. Meanwhile G may contain a maximum of $4i$ vertices at the ith level relative to a fixed vertex, $1 \le i \le dia(G)$ [5,24], and if that bound is reached at each level from 1 to $dia(G) - 1$, then G is said to be *tight-optimal* [20]. A tight-optimal circulant is necessarily optimal. Clearly, the average distance of a tight-optimal circulant is the least among all circulants of the same order/size. (Lower the average distance, lower the average delay.)

The graphs $\mathcal{C}_{65}(5,\ 6)$ and $\mathcal{C}_{65}(1,\ 14)$ appear in Fig. 1 to illustrate the foregoing. Whereas the two are optimal and of the same order/size, the former is tight-optimal while the latter is not. (As the order goes up, several new pairs appear in which the contrast is more pronounced.)

The *Kronecker product* $G \times H$ of graphs $G = (U, D)$ and $H = (W, F)$ is defined as follows: $V(G \times H) = U \times W$, and $E(G \times H) = \{\{(a, x), (b, y)\} \mid \{a, b\} \in D \text{ and } \{x, y\} \in F\}$. It is also known as the tensor product, direct product and cardinal product [9]. Further, the *Cartesian product* $G \Box H$ of graphs G and H is defined as follows: $V(G \Box H) = U \times W$, and $E(G \Box H) = \{\{(a, x), (b, y)\} \mid \{a, b\} \in D \text{ and } x = y, \text{ or } \{x, y\} \in F \text{ and } a = b\}$.

Let C_n denote the cycle having the vertex set $\{0, \ldots, n-1\}$, $n \ge 3$, where adjacencies $\{i, i+1\}$ exist in the natural way. This paper focuses mainly on $C_{2i+1} \times C_{2j+1}$ that is connected and nonbipartite, and occasionally refers to $C_{2i+1} \times C_{2j}$ that is connected and bipartite [9]. ($C_{2i} \times C_{2j}$ is disconnected, hence not relevant in the present study.)

A spanning cycle in a graph (if one exists) is called a *Hamiltonian cycle*. Further, a graph is said to admit a *Hamiltonian decomposition* if its edge set may be partitioned into Hamiltonian cycles. The length of a shortest (induced) odd cycle in a nonbipartite graph G is called its *odd girth*.

Proposition 1.2 (*[18,17]*). *Let m and n be both odd.*

1. $dia(C_m \times C_n) = \begin{cases} m-1 & m = n \\ \max\{m,\ \frac{1}{2}(n-1)\} & m < n. \end{cases}$
2. $\alpha(C_m \times C_n) = \frac{1}{2}m(n-1)$, *where* $m \le n$.
3. $C_m \times C_n$ *admits a vertex partition as well as an edge decomposition into shortest odd cycles, each of which is of length* $\max\{m, n\}$. ∎

Here is the baseline of the present study.

Proposition 1.3 (*[6]*). $C_m \times C_n$ *is a circulant if and only if* $\gcd(m, n) = 1$. ∎

1.2. State of the art

The circulant graphs enjoy a rich literature. Alspach and Parsons [1] studied their isomorphism that was followed by Klin and Pöschel [19] and later by Muzychuk et al. [21]. On the other hand, Boesch and Tindell [4] examined the connectivity of circulants. See Tang et al. [22] for a hierarchy of progressively restricted classes of circulants, and Jha [15] for a family of tight-optimal circulants.

In a seminal piece of work, Wong and Coppersmith [24] earlier presented a geometrical approach for finding shortest paths from a fixed node in a circulant. For related results, see Du et al. [7] and Tzvieli [23], and the surveys [3,11,20].

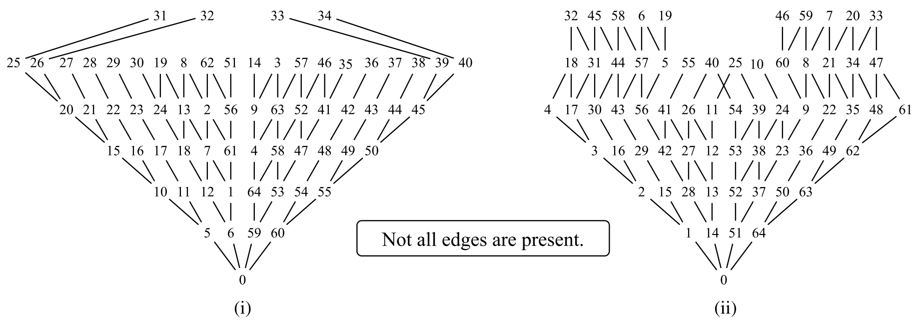


Fig. 1. Level diagrams of (i) $\mathcal{C}_{65}(5,\ 6)$ and (ii) $\mathcal{C}_{65}(1,\ 14)$.

Table 2
Minimality of the diameter of various graphs.

G (a odd)	$\lvert V(G)\rvert$	$dia(G)$ (Proposition 1.2(1))	Lower bound on $dia(G)$ (Proposition 1.1)
$C_a \times C_{2a-1}$	$2a^2 - a$	a	a
$C_a \times C_{2a+1}$	$2a^2 + a$	a	a
$C_a \times C_{2a+3}$ $a \not\equiv 0 (\mathrm{mod}\ 3)$	$2a^2 + 3a$	$a+1$	$a+1$

The Kronecker product is one of the most important products, with numerous applications in areas such as computer networks, perfect codes and algebraic systems [9]. In particular, $C_{2i+1} \times C_{2j+1}$ possesses lower diameter, higher odd girth and higher independence number relative to its closest rival $C_{2i+1} \Box C_{2j+1}$ [14]. For studies on long induced cycles and orthogonal drawings/crossing numbers of $C_m \times C_n$, see [13,16].

What follows: Section 2 characterizes those $C_m \times C_n$ that are representable as optimal circulants, while Section 3 renders their detailed representations resulting in the step sizes associated with them. Further, Section 4 builds the distance-wise level diagrams leading to average distances in the respective graphs and the identification of the tight-optimal cases. Finally, Section 5 presents certain concluding remarks.

2. Characterization

Lemma 2.1. *The following are optimal circulants:*

- $C_a \times C_{2a-1}$, *a odd*
- $C_a \times C_{2a+1}$, *a odd*
- $C_a \times C_{2a+3}$, *a odd and* $a \not\equiv 0 (\mathrm{mod}\ 3)$.

Proof. First note that $\gcd(a, 2a-1) = \gcd(a, 2a+1) = 1$, and if $a \not\equiv 0 (\mathrm{mod}\ 3)$, then $\gcd(a, 2a+3) = 1$. By Proposition 1.3, each graph under consideration is a circulant. For minimality of the diameter, see Table 2. ■

Lemmas 2.2 and 2.3 together establish the converse of Lemma 2.1.

Lemma 2.2. *If m is odd and n is even, then* $C_m \times C_n$ *is not an optimal circulant.*

Proof. Let m be odd and n even, and note that $dia(C_m \times C_n) = \max\{m,\ \frac{1}{2}n\}$ [18]. It suffices to consider only those cases in which $\gcd(m, n) = 1$, for otherwise $C_m \times C_n$ is not even connected. In that light, $m \neq \frac{1}{2}n$.

First suppose that $dia(C_m \times C_n) = m = \frac{1}{2}n + i$, where $i \geq 1$, so $n = 2m - 2i$. Accordingly, $|V(C_m \times C_n)| = mn = 2m^2 - 2mi$ that is smaller than $m^2 + (m-1)^2$, hence the lower bound from Proposition 1.1 is less than or equal to $m - 1$. This means that $C_m \times C_n$ (whose diameter is equal to m) cannot be an optimal circulant. The argument for $dia(C_m \times C_n) = \frac{1}{2}n > m$ is similar. ■

Lemma 2.3. *If m and n are both odd and* $C_m \times C_n$ *is different from the graphs in the statement of Lemma 2.1, then* $C_m \times C_n$ *is not an optimal circulant.*

Proof. Let m and n be both odd, and $\gcd(m, n) = 1$. Without loss of generality, let $m < n$, in which case $\mathrm{dia}(C_m \times C_n) = \max\{m,\ \frac{1}{2}(n-1)\}$.

If $m+2 \leq n \leq 2m-3$, then $\mathrm{dia}(C_m \times C_n) = m$. Also, $m(m+2) \leq mn \leq m(2m-3) < m^2 + (m-1)^2$, hence the lower bound from Proposition 1.1 is at most $m-1$. On the other hand, if $n \geq 2m+5$, then $\mathrm{dia}(C_m \times C_n) = \frac{1}{2}(n-1)$. Also, $mn \leq \frac{1}{2}(n-5)n$, and it is easy to see that $\frac{1}{2}(n-5)n < (\frac{1}{2}(n-3)+1)^2 + (\frac{1}{2}(n-3))^2$, hence the lower bound from Proposition 1.1 is at most $\frac{1}{2}(n-3)$. It follows that $C_m \times C_n$ cannot be an optimal circulant. ■

Unless otherwise indicated, m and n are both odd in each occurrence of $C_m \times C_n$ in the rest of the paper.

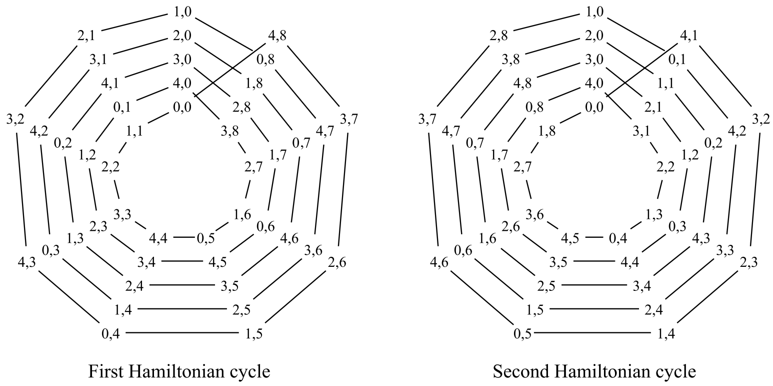


Fig. 2. Edge-disjoint Hamiltonian cycles in $C_5 \times C_9$.

3. Detailed representations of the circulants

The present section determines the precise values of the step sizes associated with the optimal circulants set out in Section 2. To that end, it employs a known result on the Hamiltonian decomposition of $C_m \times C_n$, and carefully indexes the vertices in one of the Hamiltonian cycles to obtain an explicit isomorphism in each case.

Theorem 3.1 ([12]). *If* $\gcd(m, n) = 1$, *then* $C_m \times C_n$ *admits a Hamiltonian decomposition.*

Proof. Consider the following sequences of vertices in $C_m \times C_n$: $x_0, \ldots, x_{mn-1}$ and $y_0, \ldots, y_{mn-1}$, where $x_k = (k \bmod m, k \bmod n)$ and $y_k = (k \bmod m, (-k) \bmod n)$, $0 \le k \le mn - 1$. The two sequences correspond to as many edge-disjoint Hamiltonian cycles in $C_m \times C_n$. ■

Intuitively, the first (resp. the second) Hamiltonian cycle is obtainable as follows: Start at $(0, 0)$, and at each step, increment the first co-ordinate modulo m and simultaneously increment (resp. decrement) the second co-ordinate modulo n. Fig. 2 illustrates the construction in respect of $C_5 \times C_9$.

Note: As far as Hamiltonian decomposition of $C_m \times C_n$ is concerned, there exists a more general result, viz., $C_m \times C_n$ is Hamiltonian decomposable if and only if m and n are not both even [9].

3.1. Indexing functions

Consider the sequence $x_0, \ldots, x_{a(2a-1)-1}$ corresponding to the first Hamiltonian cycle of $C_a \times C_{2a-1}$ based on the proof of Theorem 3.1, and let f be the function such that $f(i, j) = k$ if and only if (i, j) is the kth vertex in the foregoing sequence, $0 \le k \le a(2a-1)-1$. The construction of the function is based on the observation that $x_0, \ldots, x_{a(2a-1)-1}$ may be partitioned into a blocks of $2a - 1$ vertices each, where (i, j) is in the rth block if and only if $(j - i) \equiv r (\bmod a)$, $0 \le r \le a - 1$. Note that the argument relies on the fact that $\gcd(a, 2a - 1) = 1$.

Analogous to f, let g and h denote the indexing functions in respect of $C_a \times C_{2a+1}$ and $C_a \times C_{2a+3}$, respectively. The associated partition of $x_0, \ldots, x_{a(2a+1)-1}$ for $C_a \times C_{2a+1}$ is such that (i, j) is in the rth block if and only if $(i - j) \equiv r \ (\bmod a)$. Likewise (i, j) is in the rth block in respect of $C_a \times C_{2a+3}$ if and only if $(i - j) \equiv 3r (\bmod a)$, where $a \not\equiv 0 (\bmod 3)$.

Precise formulations of the indexing functions appear in Table 3, and the illustrations appear in Figs. 3 and 4, where a vertex and its index coexist at each "node". The "dotted" arrows in the figures highlight the sequencing associated with the indices. It is not difficult to check the bijectivity of each of f, g and h.

3.2. Isomorphisms

Theorem 3.2. $C_a \times C_{2a-1} \cong \mathcal{C}_{a(2a-1)}(1, 4a - 1)$, *where a is odd.*

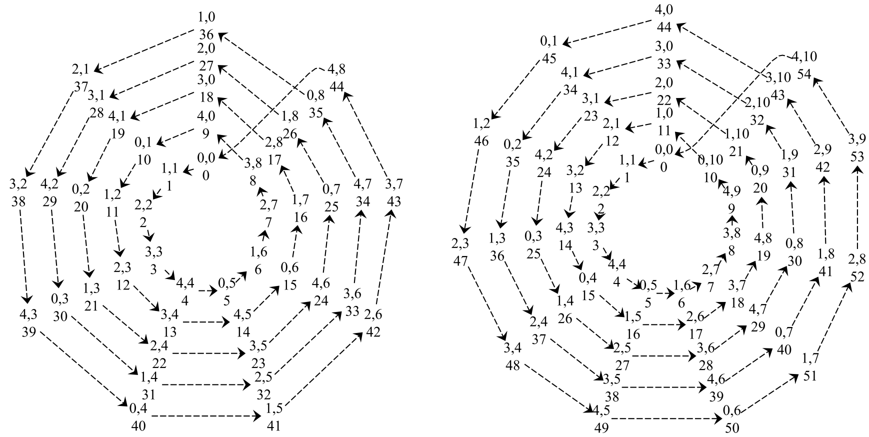
Proof. Consider the first Hamiltonian cycle of $C_a \times C_{2a-1}$ given by the sequence $x_0, \ldots, x_{a(2a-1)-1}$ from the proof of Theorem 3.1. It suffices to show that x_k and x_{k+4a-1} are adjacent, where $0 \le k \le a(2a - 1) - 1$, and $k + 4a - 1$ is computed modulo $a(2a - 1)$. To that end, let $x_k = (i, j)$, so $x_{k+4a-1} = (i + 4a - 1, j + 4a - 1)$, where $i + 4a - 1$ is computed modulo a and $j + 4a - 1$ is computed modulo $2a - 1$. It is easy to see that x_{k+4a-1} is given by $((i + a - 1) \bmod a, (j + 1) \bmod 2a - 1)$, which is clearly adjacent to $x_k = (i, j)$ in $C_a \times C_{2a-1}$. ■

Theorem 3.3. $C_a \times C_{2a+1} \cong \mathcal{C}_{a(2a+1)}(1, 4a + 1)$, *where a is odd.*

Proof. Similar to the proof of Theorem 3.2. ■

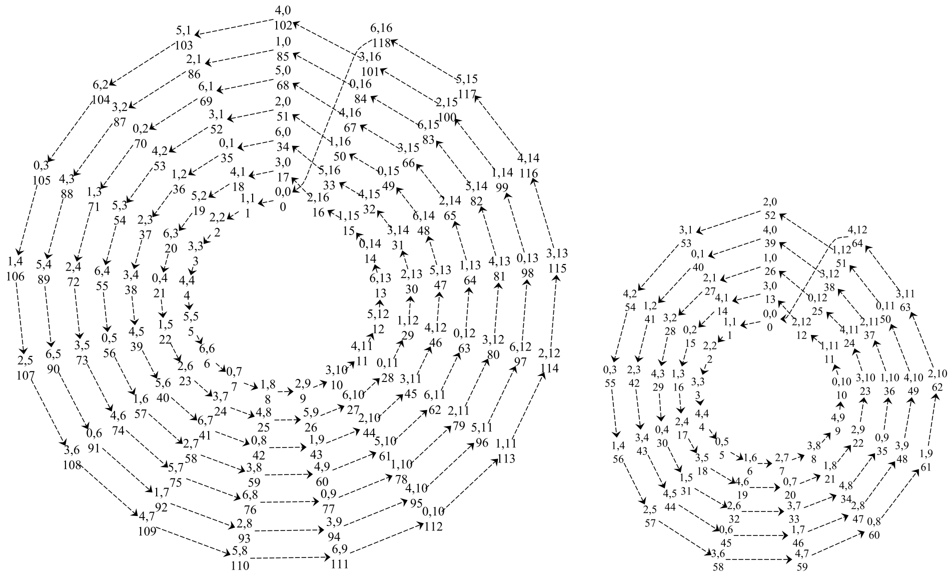
Table 3
Indexing functions.

Graph (a odd)	Function
$C_a \times C_{2a-1}$	$f(i,j) = (2a-1)r + j$, where $r = (j-i) \bmod a$ Illustration: Fig. 3(i)
$C_a \times C_{2a+1}$	$g(i,j) = (2a+1)r + j$, where $r = (i-j) \bmod a$ Illustration: Fig. 3(ii)
$C_a \times C_{2a+3}$ $a \not\equiv 0 \pmod 3$	$h(i,j) = (2a+3)r + j$, where $r = \begin{cases} \frac{1}{3}c & c \equiv 0 \pmod 3 \\ \frac{1}{3}(c+2a) & c \equiv a \pmod 3 \\ \frac{1}{3}(c+a) & c \not\equiv a \pmod 3 \end{cases} \Bigg\vert\, c = (i-j) \bmod a$ Illustration: Fig. 4(i) and (ii)



(i) (ii)

Fig. 3. Functions f and g in respect of $C_5 \times C_9$ and $C_5 \times C_{11}$.



(i) (ii)

Fig. 4. Function h in respect of $C_7 \times C_{17}$ and $C_5 \times C_{13}$.

Theorem 3.4. 1. *If* $a \equiv 1(\text{mod } 6)$, *then* $C_a \times C_{2a+3} \cong \mathcal{C}_{a(2a+3)}(1, \frac{1}{3}(2a+1)(a+3))$.
2. *If* $a \equiv 5(\text{mod } 6)$, *then* $C_a \times C_{2a+3} \cong \mathcal{C}_{a(2a+3)}(1, \frac{1}{3}(2a-3)(a+1))$.

Proof. Let $a \equiv 1(\text{mod } 6)$, and $c = \frac{1}{3}(2a+1)(a+3)$. It suffices to show that (i, j) and $(i+c, j+c)$ are adjacent in $C_a \times C_{2a+3}$, where $i + c$ is modulo a and $j + c$ is modulo $2a + 3$. Check to see that

- $c = \frac{1}{3}(2a+7)a + 1$, so $c \equiv 1(\text{mod } a)$, and
- $c = \frac{1}{3}(a-1)(2a+3) + (2a+2)$, so $c \equiv 2a + 2(\text{mod } 2a+3)$.

It is clear that i and $i + c$ are adjacent in C_a, and j and $j + c$ are adjacent in C_{2a+3}. Accordingly, (i, j) and $(i + c, j + c)$ are adjacent in $C_a \times C_{2a+3}$.

Next, let $a \equiv 5(\text{mod } 6)$, and $d = \frac{1}{3}(2a-3)(a+1)$. Check to see that

- $d = \frac{1}{3}(2a-4)a + (a-1)$, so $d \equiv a - 1(\text{mod } a)$, and
- $d = \frac{1}{3}(a-2)(2a+3) + 1$, so $d \equiv 1(\text{mod } 2a+3)$.

It follows that (i, j) and $(i + d, j + d)$ are adjacent in $C_a \times C_{2a+3}$. ■

4. Distance-wise vertex distributions

This section builds the distance-wise level diagrams of the circulants, leading to vertex distributions, which in turn yield the average distances in the respective graphs. To that end, the following technical result is useful.

Proposition 4.1 ([18]). *Let* m *and* n *be both odd,* $0 \le i \le m-1$; $0 \le j \le n-1$ *and* $(i, j) \ne (0, 0)$, *and consider the graph* $C_m \times C_n$.

1. *If* i *and* j *are of the same parity, then* $\text{dist}((0, 0), (i, j)) = \min\{\max\{i, j\}, \max\{m-i, n-j\}\}$.
2. *If* i *and* j *are not of the same parity, then* $\text{dist}((0, 0), (i, j)) = \min\{\max\{i, n-j\}, \max\{m-i, j\}\}$. ■

In what follows, all distance-related discussions will be relative to $(0, 0)$ in $C_m \times C_n$. (Vertex transitivity of a circulant affords this freedom of choice.)

Table 4 details top-level vertices in various graphs under discussion. The claims in respect of $C_a \times C_{2a-1}$ and $C_a \times C_{2a+1}$ are proved in Lemmas 4.2 and 4.3, respectively. The arguments being similar, the claims in respect of $C_a \times C_{2a+3}$ are left to the reader. Illustrations appear in Fig. 5(i)–(iv).

Lemma 4.2. *There are a total of* $a - 1$ *diametrical vertices in* $C_a \times C_{2a-1}$.

Proof. Let (i, j) be a diametrical node, i.e., $\text{dist}((0, 0), (i, j)) = a$. For i and j of the same parity, $\min\{\max\{i, j\}, \max\{a-i, 2a-1-j\}\} = a$.

- If $\max\{i, j\} = a \le \max\{a-i, 2a-1-j\}$, then the equality suggests that $j = a$ (since $i \le a-1$), which, applied to the inequality, leads to $i = 0$ that is even while $j = a$ is odd, contradicting the parity condition. On the other hand, if $\max\{a-i, 2a-1-j\} = a \le \max\{i, j\}$, then the inequality suggests that $j \ge a$, which, applied to the equality, leads to $i = 0$. That along with the parity condition implies that $(0, a+1), (0, a+3), \ldots, (0, 2a-2)$ are diametrical nodes.

For i, j not of the same parity, $\min\{\max\{i, 2a-1-j\}, \max\{a-i, j\}\} = a$.

- If $\max\{i, 2a-1-j\} = a \le \max\{a-i, j\}$, then the equality suggests that $j = a-1$ (even), which, applied to the inequality, leads to $i = 0$ (also even), contradicting the (dis)parity requirement. On the other hand, if $\max\{a-i, j\} = a \le \max\{i, 2a-1-j\}$, then the inequality implies that $0 \le j \le a-1$, which, applied to the equality, leads to $i = 0$. That along with the (dis)parity condition implies that $(0, 1), (0, 3), \ldots, (0, a-2)$ are diametrical nodes. ■

Lemma 4.3. *There are a total of* $3a - 1$ *diametrical vertices in* $C_a \times C_{2a+1}$.

Proof. Let (i, j) be such that $\text{dist}((0, 0), (i, j)) = a$. For i and j of the same parity, $\min\{\max\{i, j\}, \max\{a-i, 2a+1-j\}\} = a$.

- If $\max\{i, j\} = a \le \max\{a-i, 2a+1-j\}$, then the equality suggests that $j = a$ (since $i \le a-1$), which, applied to the inequality, leads to $i \ge 0$, so $(1, a), (3, a), \ldots, (a-2, a)$ are diametrical nodes. (i and j are of the same parity.)
- If $\max\{a-i, 2a+1-j\} = a \le \max\{i, j\}$, then the equality suggests that $i \ge 0$ and $j = a+1$, or $i = 0$ and $j \ge a+1$, so $(0, a+1), (2, a+1), \ldots, (a-1, a+1)$, and $(0, a+1), (0, a+3), \ldots, (0, 2a)$ are diametrical nodes. (Each case is consistent with the inequality.)

For i, j not of the same parity, $\min\{\max\{i, 2a+1-j\}, \max\{a-i, j\}\} = a$.

- If $\max\{i, 2a+1-j\} = a \le \max\{a-i, j\}$, then the equality suggests that $j = a+1$ (since $i \le a-1$), which, applied to the inequality, leads to $i \ge 0$, so $(1, a+1), (3, a+1), \ldots, (a-2, a+1)$ are diametrical nodes.
- If $\max\{a-i, j\} = a \le \max\{i, 2a+1-j\}$, then the equality suggests that either $i = 0$ and $j \le a$, or $i > 0$ and $j = a$. so $(0, 1), (0, 3), \ldots, (0, a)$ and $(2, a), (4, a), \ldots, (a-1, a)$ are diametrical nodes. (Each case is consistent with the inequality.) ■

Table 4
Top-level vertices in $C_a \times C_{2a+d}$, $d = -1, 1, 3$ (a odd).

$C_a \times C_{2a-1}$	Vertices at the ath (top) level: $(0, 1), (0, 3), \ldots, (0, a-2)$ $(0, a+1), (0, a+3), \ldots, (0, 2a-2)$	Total: $a - 1$
$C_a \times C_{2a+1}$	Vertices at the ath (top) level: $(0, 1), (0, 3), \ldots, (0, a)$ $(1, a), (2, a) \cdots, (a-1, a)$ $(0, a+1), (1, a+1), \ldots, (a-1, a+1)$ $(0, a+3), (0, a+5), \ldots, (0, 2a)$	Total: $3a - 1$
$C_a \times C_{2a+3}$	Vertices at the $(a+1)$th (top) level: $(0, a+1), (1, a+1), \ldots, (a-1, a+1)$ $(0, a+2), (1, a+2) \cdots, (a-1, a+2)$	Total: $2a$
$a \not\equiv 0 (\text{mod } 3)$	Vertices at the ath (second from the top) level: $(0, 1), (0, 3), \ldots, (0, a-2)$ $(0, a), (1, a) \cdots, (a-1, a)$ $(0, a+3), (1, a+3), \ldots, (a-1, a+3)$ $(0, a+5), (0, a+7), \ldots, (0, 2a+2)$	Total: $3a - 1$

Table 5
Vertex distribution in $C_a \times C_{2a+d}$, level 0 upward.

Graph (a odd)	Vertex distribution (approx.)	Av. distance	Illustration
$C_a \times C_{2a-1}$	$1 + \underbrace{4i}_{1 \le i \le a-1} + (a-1)$	$\frac{4a-1}{6}$	Fig. 5(i)
$C_a \times C_{2a+1}$	$1 + \underbrace{4i}_{1 \le i \le a-1} + (3a-1)$	$\frac{4a+1}{6}$	Fig. 5(ii)
$C_a \times C_{2a+3}$ $a \not\equiv 0 (\text{mod } 3)$	$1 + \underbrace{4i}_{1 \le i \le a-1} + (3a-1) + (2a)$	$\frac{4a+3}{6}$	Fig. 5(iii)–(iv)

Lemma 4.4. *There are* $2a$ *diametrical vertices and* $3a - 1$ *pre-diametrical vertices in* $C_a \times C_{2a+3}$.

Proof. Similar to the proof of Lemmas 4.2/4.3.

Lemma 4.5. *For* $1 \le i \le a - 1$, *there are* $4i$ *vertices at level* i *in each of* $C_a \times C_{2a-1}$, $C_a \times C_{2a+1}$ *and* $C_a \times C_{2a+3}$.

Proof. First consider $C_a \times C_{2a-1}$. By Table 4, it has $a - 1$ diametrical vertices, so the cumulative number of vertices at the lower levels is equal to $(2a-1)a - (a-1) = 2a^2 - 2a + 1$.

It is known that the maximum number of vertices at a distance of k from a fixed vertex in a four-regular circulant is equal to $4k$ [5,24]. Accordingly, the number of vertices between level 0 and level $a - 1$ in $C_a \times C_{2a-1}$ is at most $1 + \left(4 \sum_{k=1}^{a-1} k\right)$, which coincides with $2a^2 - 2a + 1$. The claim follows. Arguments are similar for $C_a \times C_{2a+1}$ and $C_a \times C_{2a+3}$. ■

Table 4 and Lemma 4.5 lead to vertex distributions and average distances that appear in Table 5. The following theorem summarizes the central message of this paper.

Theorem 4.6. *Let* a *be odd.*

1. *Each of* $C_a \times C_{2a-1}$ *and* $C_a \times C_{2a+1}$ *is a tight-optimal circulant.*
2. *For* $a \not\equiv 0 (\text{mod } 3)$, $C_a \times C_{2a+3}$ *is an optimal circulant, but it is not tight-optimal.* ■

5. Concluding remarks

The present paper distinguishes the Kronecker product of two cycles representable as an optimal circulant. The independence number and the odd girth in each such graph are approximately equal to $\frac{1}{2}n$ and $\sqrt{2n}$, respectively, where n denotes the order of the graph (cf. Proposition 1.2). Note that high independence number and high odd girth are welcome features of a network.

Non-isomorphism vis-a-vis other circulants

Beivide et al. [2] presented a class of tight-optimal circulants, called midimew networks. Among them, the following are similar to certain graphs in this paper: $\mathcal{C}_{a(2a-1)}(a-1,\ a)$ and $\mathcal{C}_{a(2a+1)}(a,\ a+1)$. The following questions arise:

1. Is $\mathcal{C}_{a(2a-1)}(1,\ 4a-1)$ isomorphic to $\mathcal{C}_{a(2a-1)}(a-1,\ a)$?
2. Is $\mathcal{C}_{a(2a+1)}(1,\ 4a+1)$ isomorphic to $\mathcal{C}_{a(2a+1)}(a,\ a+1)$?

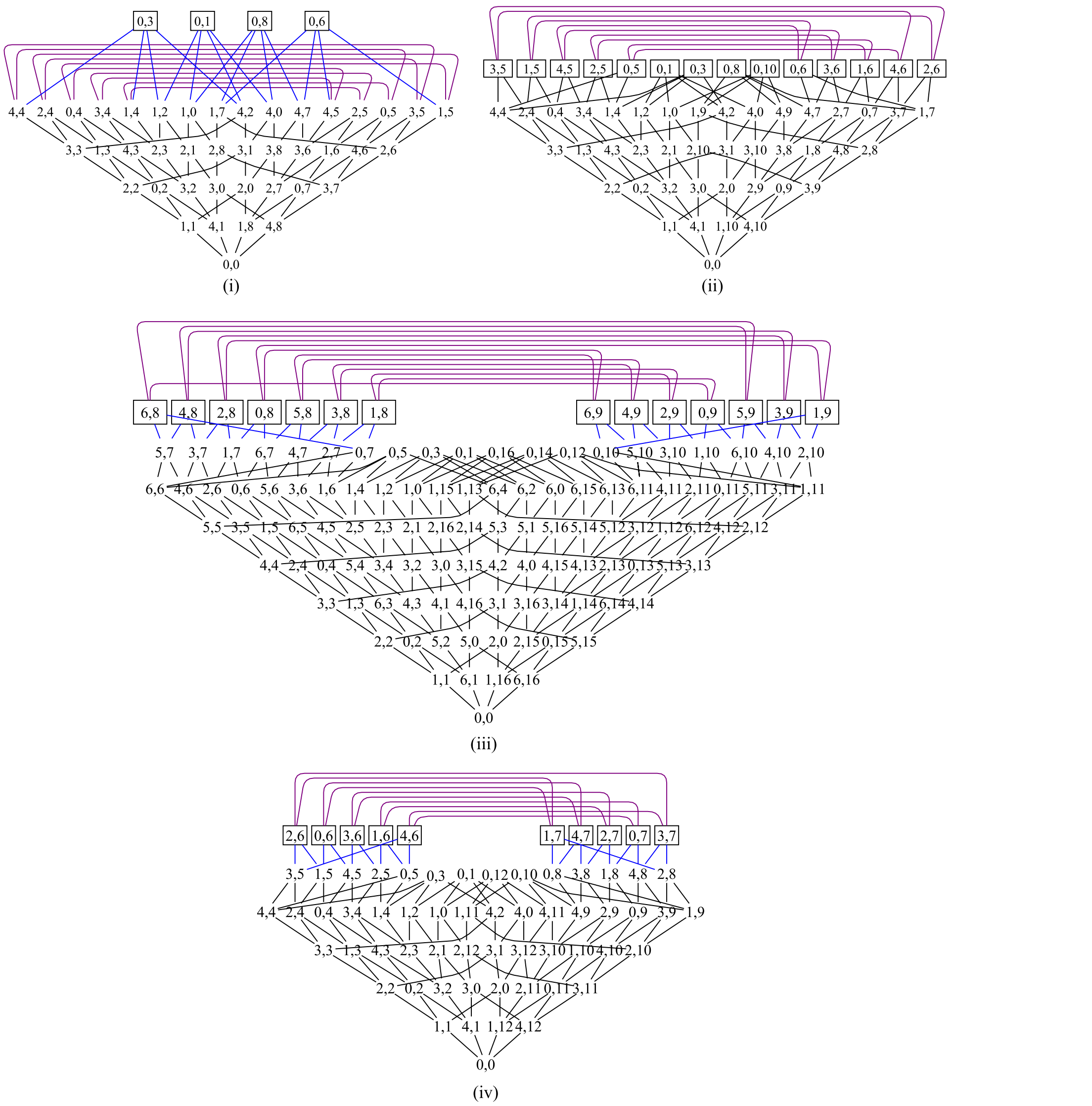


Fig. 5. Level diagrams of $C_5 \times C_9$, $C_5 \times C_{11}$, $C_7 \times C_{17}$ and $C_5 \times C_{13}$.

It turns out that the answer to each of the foregoing is in the negative. The proof is based on Lemma 5.1. (Verification is easy.)

Lemma 5.1. 1. *If* $\gcd(r, n) = 1$, *then* $\mathcal{C}_n(r, s) \cong \mathcal{C}_n(1, r^{-1}s \mod n)$.
2. *If* $s_1 \neq s_2$, $s_1 \neq n - s_2$, *and* $s_1 s_2 \not\equiv \pm 1 \pmod n$, *then* $\mathcal{C}_n(1,\ s_1) \not\cong \mathcal{C}_n(1,\ s_2)$ [8]. ■

In separate studies, Du et al. [7] and Tzvieli [23] presented several infinite families of optimal circulants. Unfortunately, there is no unanimity on the usage of the term "optimal". For example, Du et al. [7, p. 179] use *tight* for what we call *optimal*, while they use *optimal* in a slightly different setting. On the other hand, Tzvieli's definition [23] of an *optimal circulant* coincides with ours.

It is important to note that the concept of a *tight-optimal circulant* (as it appears in the present paper) does not feature in Du et al. [7] or in Tzvieli [23]. Nevertheless some of the circulants in their studies bear similarities to $\mathcal{C}_{a(2a-1)}(1,\ 4a-1)$ and $\mathcal{C}_{a(2a+1)}(1,\ 4a+1)$ in the present study. The following comparisons are based on the referee's remarks.

1. Theorem 4 in [7, p. 178] yields the circulants $\mathcal{C}_{a(2a-1)}(1,\ 2a-2)$ and $\mathcal{C}_{a(2a+1)}(1,\ 2a)$ under the settings $(k, h) = (3, 1)$ and $(k, h) = (5, 3)$, respectively. An application of Lemma 5.1 shows that (i) $\mathcal{C}_{a(2a-1)}(1,\ 2a-2)$ is not isomorphic to $\mathcal{C}_{a(2a-1)}(1,\ 4a-1)$, and (ii) $\mathcal{C}_{a(2a+1)}(1,\ 2a)$ is not isomorphic to $\mathcal{C}_{a(2a+1)}(1,\ 4a+1)$.
2. Theorem 4.2(ii) in [23, p. 402] yields the circulants $\mathcal{C}_{a(2a-1)}(1,\ 2a)$ and $\mathcal{C}_{a(2a+1)}(1,\ 2a)$ under the setting $i = 0$. By Lemma 5.1 again, the two are different from $\mathcal{C}_{a(2a-1)}(1,\ 4a-1)$ and $\mathcal{C}_{a(2a+1)}(1,\ 4a+1)$, respectively.

Circulants based on $C_m \Box C_n$

It is interesting to note that an exact analogue of Proposition 1.3 holds true in respect of the Cartesian product as well, viz., $C_m \Box C_n$ is a circulant if and only if $\gcd(m, n) = 1$ [6]. Now, $C_m \Box C_n$ being itself a four-regular graph, the question arises whether or not there exist optimal circulants representable as $C_m \Box C_n$. To that end, it is known that $\mathrm{dia}(C_m \Box C_n) = \lfloor \frac{1}{2}m \rfloor + \lfloor \frac{1}{2}n \rfloor$ [9]. Further, by Proposition 1.1, the lower bound on the diameter of a four-regular circulant on mn vertices is equal to $\left\lceil \left(-\frac{1}{2} + \frac{1}{2}\sqrt{2mn-1}\right) \right\rceil$ [5].

It is easy to verify that $\lfloor \frac{1}{2}m \rfloor + \lfloor \frac{1}{2}n \rfloor$ is strictly greater than the foregoing lower bound for all $m,\ n$ with $\gcd(m, n) = 1$. Accordingly, there do not exist optimal circulants representable as $C_m \Box C_n$.

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