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Perfect r-domination in the Kronecker product of two cycles, with an application to diagonal/toroidal mesh

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Abstract

If $r \geqslant 1$, and m and n are each a multiple of $(r+1)^2 + r^2$, then each isomorphic component of $C_m \times C_n$ admits of a vertex partition into $(r+1)^2 + r^2$ perfect r-dominating sets. The result induces a dense packing of $C_m \times C_n$ by means of vertex-disjoint subgraphs, each isomorphic to a diagonal array. Areas of applications include efficient resource placement in a diagonal mesh and error-correcting codes.

Keywords: Combinatorial problems; Perfect r-domination; Error-correcting codes; Diagonal mesh; Toroidal mesh; Kronecker product; Cartesian product

1. Introduction

Consider a computer/communication network that usually has a regular structure. The nodes are distinguishable into resource nodes and user nodes. Each of the former houses replicable items such as power sources, I/O ports and function libraries, while each of the latter is within a distance of r from at least one resource node, where r is a fixed positive integer. Resources are usually limited and expensive, hence the need for minimizing the number of respective nodes.

The foregoing problem of efficient *resource placement* has a natural graph-theoretical formulation, where the objective is to construct a *perfect r-dominating set* (defined below) of the underlying graph. It has been studied with respect to a number of network topologies, including hypercubes, 2D torus and 3D torus [1–3]. The concept has applications in several other areas too, notably, *error-correcting codes*, game theory and frequency assignment [6,7,16,17]. The well-known Hamming code corresponds to a perfect 1-domination in the n-cube, where $n = 2^k - 1$, $k \geqslant 2$ [14,19]. Even when a perfect r-dominating set is not known for a given graph, an analogous information with respect to a related graph may be useful to help construct a near-optimal set.

This paper presents a vertex partition of the *Kronecker product* of two cycles into perfect r-dominating sets, where length of each cycle is a multiple of $(r+1)^2 + r^2$. The result closely parallels the existence of Lee metric code [6]. A particularly useful application consists of an optimal resource placement in a *diagonal mesh*.

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In this paper, when I speak of a graph, I mean a finite, simple and undirected graph. Unless indicated otherwise, graphs are also connected and contain at least two vertices. For $m \geqslant 2$ and $n \geqslant 3$, let P_m (respectively C_n) denote a *path* (respectively a *cycle*) on m (respectively n) vertices, where $V(P_k) = V(C_k) = \{0, \ldots, k-1\}$, and where adjacencies are defined in a natural way.

For a graph $G = (V, E)$, a vertex v is said to r-dominate a vertex w if $0 \leqslant d_G(v, w) \leqslant r$. A vertex subset S is called an *r-dominating set* (respectively a *perfect r-dominating set*) if every vertex of G is r-dominated by some vertex (respectively a unique vertex) in S. The cardinality of a smallest r-dominating set of G is called the *r-domination number* of G, denoted by $\gamma_r(G)$. The general problem of determining $\gamma_r(G)$ is known to be NP-hard even for bipartite graphs [5].

The *Kronecker product* $G \times H$ of graphs $G = (V, E)$ and $H = (W, F)$ is defined as follows: $V(G \times H) = V \times W$ and $E(G \times H) = \{\{(a, x), (b, y)\}$: $\{a, b\} \in E$ and $\{x, y\} \in F\}$. This product is variously known as direct product, cardinal product and tensor product. Another relevant structure is the *Cartesian product* $G \,\square\, H$ defined as follows: $V(G \,\square\, H) = V \times W$ and $E(G \,\square\, H) = \{\{(u, x), (v, y)\}$: either $u = v$ and $\{x, y\} \in F$; or $x = y$ and $\{u, v\} \in E\}$. Each of the two products is commutative and associative in a natural way, and has found applications in a number of areas. In automata theory, for example, closure of regular sets, or closure of context-free sets with a regular set, under intersection (respectively shuffle) is provable by taking the $\times$-product (respectively $\square$-product) of the corresponding machines. It is interesting to note that if G and H are connected graphs, then $G \times H$ is isomorphic to $G \,\square\, H$ if and only if G and H are odd cycles of the same size [18].

Tang and Padubidri [21] study diagonal mesh and toroidal mesh (for connecting communication elements in parallel computers) that are actually representable as $C_{2i+1} \times C_{2j+1}$ and $C_{2i+1} \,\square\, C_{2j+1}$, respectively. Certain common features of the two graphs are: nonplanarity, nonbipartiteness, edge decomposability into Hamiltonian cycles [8], and embeddability on a torus. Except for these similarities, the two have a number of dissimilarities, and hence each merits an individual treatment; see Table 1.

Remark. Lower diameter, higher independence number and higher odd girth are welcome features of a fault-tolerant communication network. In particular, low diameter ensures low communication delay between two nodes in the worst case, and high odd girth means that the graph is "locally bipartite". By Table 1, therefore, the diagonal mesh outperforms the toroidal mesh in many ways. This is further supported by other findings [21]. Diagonal mesh in some form appeared earlier as the routing network of FAIM-1 computer [4].

Table 1
$C_m \times C_n$ (diagonal mesh) versus $C_m \,\square\, C_n$ (toroidal mesh) m, n both odd; $m \geqslant n \geqslant 3$

	$C_m \times C_n$		$C_m \,\square\, C_n$	References
Diameter	$\max\{(m-1)/2, n\}$	if $m > n$	$(m+n-2)/2$	[8,15]
	$n-1$	if $m = n$		
Independence number	$(m-1)n/2$		$m(n-1)/2$	[9,13]
Odd girth	m		n	[13]

Table 2
Perfect r-dominating sets in products of cycles

$C_{m_0} \times \cdots \times C_{m_{k-1}}$	$C_{m_0} \,\square\, \cdots \,\square\, C_{m_{k-1}}$
$r = 1$, $k \geqslant 2$ and $m_0, \ldots, m_{k-1}$ each a multiple of $2^k + 1$ [11]	$r = 1$, $k \geqslant 2$ and $m_0, \ldots, m_{k-1}$ each a multiple of $2k + 1$ [6]
$r \geqslant 1$, $k = 3$ and m_0, m_1, m_2 each a multiple of $(r+1)^3 + r^3$ [12]	$r \geqslant 1$, $k = 2$ and m_0, m_1 each a multiple of $(r+1)^2 + r^2$ [6]

For any undefined terms or missing references, see the book by Imrich and Klavžar [8].

Table 2 presents certain known results with respect to perfect r-domination in $C_{m_0} \times \cdots \times C_{m_{k-1}}$ and $C_{m_0} \Box \cdots \Box C_{m_{k-1}}$.

Section 2 characterizes the r-ball in $C_m \times C_n$ as well as $C_m \Box C_n$, and presents a lower bound on $\gamma_r(C_m \times C_n)$. Section 3 presents the main result. Concluding remarks appear at end.

2. Preliminaries

It is known that (i) if G and H are not both bipartite, then $G \times H$ is connected, otherwise $G \times H$ consists of two connected components, and (ii) $G \times H$ is bipartite if and only if G or H is bipartite. In particular, $C_{2i+1} \times C_n$ is connected and $C_{2i} \times C_{2j}$ consists of two isomorphic components. The graph $P_7 \times P_7$ appears in Fig. 1. The terms "even component" and "odd component" have been chosen because vertices (p,q) in the former (respectively latter) are exactly those for which $p+q$ is even (respectively odd).

Let $DA(m,n)$ denote the even component of $P_m \times P_n$ where "DA" stands for "diagonal array". It is not difficult to see that $DA(m,n)$ consists of $\lceil mn/2 \rceil$ vertices and $(m-1)(n-1)$ edges [8].

Note. Ramirez and Melhem [20] present a fault-tolerant computational array (called processor switch/voter array) whose underlying graph is essentially $DA(2i+1, 2j+1)$.

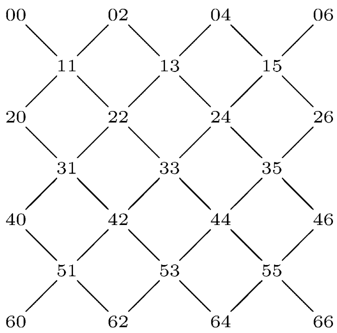
Definition 1. Let G be a graph with radius s. For $0 \leqslant r \leqslant s$, an r-ball centered at a vertex v of G is the set $\{w \in V(G)\colon\ 0 \leqslant d_G(v,w) \leqslant r\}$.

An r-dominating set of G is a spanning of G by r-balls. (In the case of a perfect r-dominating set, the r-balls are mutually exclusive and exhaustive.) In what follows, an "r-ball" will be used also to denote the corresponding induced subgraph. The following is a consequence of a more general result.

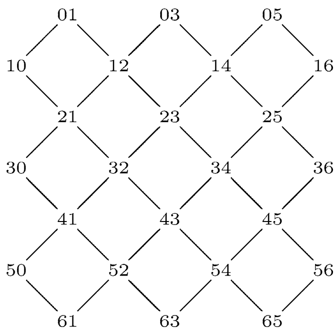
Lemma 2.1 [12]. *If $m,n \geqslant 2r+2$, then an r-ball in $C_m \times C_n$ is isomorphic to $DA(2r+1, 2r+1)$. Accordingly, $\gamma_r(C_m \times C_n) \geqslant mn/((r+1)^2 + r^2)$.*

$DA(7,7)$ appearing in Fig. 1(a) may be viewed as a 3-ball centered at vertex $(3,3)$. A consequence of the main result of this paper is that the lower bound of Lemma 2.1 is achieved if m and n are each a multiple of $(r+1)^2 + r^2$.

Interestingly enough, the subgraph induced by an r-ball in $C_m \Box C_n$ is also isomorphic to $DA(2r+1, 2r+1)$. To see this, let $m,n \geqslant 2r+1$, and $1 \leqslant k \leqslant r$. A vertex at a distance of k from a typical vertex (i,j) in $C_m \Box C_n$ is of the form $(i+p, j+q)$ where $i+p$ is modulo m, $j+q$ is modulo n, and $|p|+|q| = k$. It is easy to check that the number of such vertices is equal to $4k$, and hence the order of an r-ball in this graph is equal to $1 + \sum_{k=1}^{r}(4k) = (r+1)^2 + r^2$. That the induced subgraph is isomorphic to $DA(2r+1, 2r+1)$ may be proved by induction on r. A 3-ball in $C_m \Box C_n$ appears in Fig. 2, and is isomorphic to $DA(7,7)$ that appears in Fig. 1(a).



(a)



(b)

Fig. 1. The graph $P_7 \times P_7$. (a) Even component, (b) odd component.

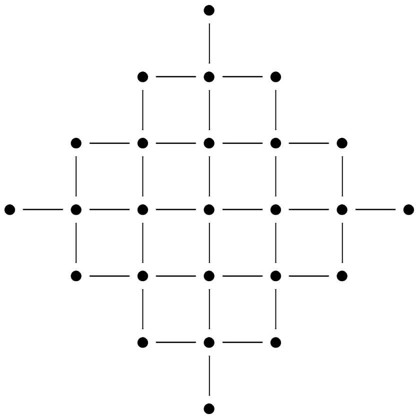


Fig. 2. 3-ball in $C_m \square C_n$.

3. Result

Theorem 3.1. *If $r \geqslant 1$, and m and n are each a multiple of $(r+1)^2 + r^2$, then each connected component of $C_m \times C_n$ admits of a vertex partition into $(r+1)^2 + r^2$ perfect r-dominating sets.*

Proof. Let $s = (r+1)^2 + r^2$, and let vertex (i, j) of $C_m \times C_n$ be assigned the integer label

$$\big[2(r+1)i + 2rj\big] \bmod s.$$

The assignment is clearly well-defined. It suffices to show that a vertex distinct from (i, j) that is within a distance of $2r$ from (i, j) receives a label that is different from that of (i, j).

Let $p \in \{1, \ldots, 2r\}$, and consider a vertex at a distance of p from (i, j). Such a node is of the form $(i + a, j + b)$, where

(i) $a, b \in \{-p + 2k\colon 0 \leqslant k \leqslant p\}$,
(ii) $\max\{|a|, |b|\} = p$, and
(iii) $i + a$ is modulo m and $j + b$ is modulo n.

(Note that a and b are of the same parity.) The label assigned to $(i + a, j + b)$ is

$$\big[2(r+1)i + 2rj + 2(r+1)a + 2rb\big] \bmod s.$$

Accordingly, it needs to be proved that

$$\big[2(r+1)a + 2rb\big] \bmod s > 0, \quad \text{i.e.,}$$
$$\big[(r+1)a + rb\big] \bmod s > 0, \quad \text{since } s \text{ is odd.}$$

The claim follows by a careful case analysis. However, the following argument suggested by a referee is more elegant and intuitive.

Consider the Diophantine equation

$$(r+1)a + rb = \lambda\big((r+1)^2 + r^2\big).$$

If $a_0 = \lambda(r+1)$ and $b_0 = \lambda r$ is a particular solution, then

$$(r+1)a + rb = (r+1)a_0 + rb_0, \quad \text{i.e.,}$$
$$(r+1)(a - a_0) = r(b_0 - b).$$

This means that r divides $(r+1)(a - a_0)$. Since r and $r+1$ are coprime, this implies that r divides $(a - a_0)$, i.e., $(a - a_0) = \mu r$ for some μ. Accordingly, $r(b_0 - b) = (r+1)\mu r$, i.e., $b = b_0 - \mu(r+1)$. Let

$$S = \big\{(a, b)\colon\ a = \lambda(r+1) + \mu r,\\ b = \lambda r - \mu(r+1),\ \lambda,\ \mu \in \mathbb{Z}\big\}.$$

We are looking for the choice of $(\lambda, \mu) \in \mathbb{Z} \times \mathbb{Z}$ such that the solutions satisfy:

$$a + b \text{ is even} \quad \text{and} \quad |a| + |b| \leqslant 2r.$$

Now, $(a + b)$ is even iff $(2\lambda r + \lambda - \mu)$ is even iff $(\lambda + \mu)$ is even iff $(|\lambda| + |\mu|) = 0$ or $(|\lambda| + |\mu|) \geqslant 2$. If $(a, b) \neq (0, 0)$, then $(|\lambda| + |\mu|) \geqslant 2$.

- For $|\lambda| = 0$ and $|\mu| \geqslant 2$, we have $|b| \geqslant 2(r+1) > 2r$.
- For $|\mu| = 0$ and $|\lambda| \geqslant 2$, we have $|a| \geqslant 2(r+1) > 2r$.
- For $|\lambda| \geqslant 1$, $|\mu| \geqslant 1$ and $\lambda\mu > 0$, we have $|a| \geqslant 2(r+1) > 2r$.
- For $|\lambda| \geqslant 1$, $|\mu| \geqslant 1$ and $\lambda\mu < 0$, we have $|b| \geqslant 2(r+1) > 2r$.

Thus the Diophantine equation has a unique solution for (a, b) that is $(0, 0)$, where $(a + b)$ is even and $(|a| + |b|) \leqslant 2r$. $\square$

To see how the result may be used in practice, consider a parallel computer whose processing units (p.u.'s) and the interconnection network are modeled by $C_m \times C_n$, where each p.u. is associated with a vertex of the graph and a direct link between two p.u.'s is indicated by an edge between the corresponding vertices. Next suppose that there are resource units (r.u.'s) that are to be positioned in such a way that every p.u. is within a distance of r from at least one r.u. A set V_t constructed in the proof of Theorem 3.1 constitutes a collection of vertices where r.u.'s are to be located; this ensures that the number of r.u.'s (and hence the associated cost) is the minimum possible as each vertex is within a distance of r from exactly one "resource vertex". As noted earlier, the diagonal mesh [21] is isomorphic to $C_m \times C_n$, where m and n are both odd.

Theorem 3.1 together with discussions in Section 2 lead to the following result.

Corollary 3.2. *For $r \geqslant 1$, if m and n are each a multiple of $(r+1)^2 + r^2$, then $C_m \times C_n$ as well as $C_m \,\square\, C_n$ admits of a decomposition into $mn/((r+1)^2 + r^2)$ subgraphs, each isomorphic to the diagonal array $DA(2r+1, 2r+1)$.*

The foregoing result may be viewed as a packing of $C_m \times C_n$ (respectively $C_m \,\square\, C_n$) by $mn/((r+1)^2 + r^2)$ vertex-disjoint (and hence edge-disjoint) copies of $DA(2r+1, 2r+1)$, that has $4r^2$ edges. All such copies thus collectively account for $4r^2 \cdot mn/((r+1)^2 + r^2)$ edges of $C_m \times C_n$ (respectively $C_m \,\square\, C_n$), that has $2mn$ edges. Thus the "efficiency" of this packing is equal to

$$\frac{1}{2mn} \cdot \left(4r^2 \cdot \frac{mn}{(r+1)^2 + r^2}\right) = \frac{2r^2}{2r^2 + 2r + 1}$$

that approaches 100% for large r.

4. Concluding remarks

The Kronecker product and the Cartesian product have gained prominence by virtue of their applications in engineering, computer science and related disciplines. Between the two, the latter is relatively simple and intuitive, and hence more widely studied. For example, $d_{G \,\square\, H}((u,x),(v,y))$ is given by the simple formula of $d_G(u,v) + d_H(x,y)$, whereas $d_{G \times H}((u,x),(v,y))$ is given by a rather complicated formula [15].

A number of results with respect to the Kronecker product are amenable to useful applications. In particular, each of $P_m \times P_n$, $C_m \times P_n$ and $C_m \times C_n$ has a rich cycle structure [10]. Accordingly, each is responsive to applications in areas such as VLSI layout, computer and communication networks, and management of multiprocessors.

While $C_m \times C_n$ and $C_m \,\square\, C_n$ are nonisomorphic (with the sole exception of when m and n are equal and odd), they have a similar local structure in that the subgraph induced by an r-ball centered at a particular vertex is isomorphic to a diagonal array that itself has proved to be a useful structure [20].

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