1-Perfect Codes Over Dual-Cubes vis-à-vis Hamming Codes Over Hypercubes

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***Abstract*— A 1-perfect code of a graph G is a set $C \subseteq V(G)$ such that the 1-balls centered at the vertices in C constitute a partition of $V(G)$. In this paper, we consider the dual-cube DQ_m that is a connected $(m+1)$-regular spanning subgraph of the hypercube Q_{2m+1}, and show that it admits a 1-perfect code if and only if $m = 2^k - 2$, $k \geq 2$. The result closely parallels the existence of Hamming codes over the hypercube. The algorithm for that purpose employs a scheme by Jha and Slutzki for a vertex partition of Q_{m+1} into Hamming codes using a Latin square, and carefully allocates those codes among various m-cubes in DQ_m. The result leads to tight bounds on domination numbers of the dual-cube and the exchanged hypercube.**

***Index Terms*— 1-perfect codes; Hamming codes; dual-cubes; hypercubes; exchanged hypercubes; Latin square; domination number; resource placement; interconnection networks.**

I. INTRODUCTION

1-*PERFECT codes* constitute a significant field of study by virtue of their applications in multiprocessor systems, communication systems, and a number of other areas in the wide digital world. They have the capability to detect two or fewer errors, and correct a single error. Among various types of 1-perfect codes, the *Hamming codes* [4], based on the topology of the *hypercubes*, are the foremost. Meanwhile the problem of deciding whether or not a graph supports a 1-perfect code is NP-complete [15] even for planar 3-regular graphs.

The *dual-cube* [17], [20] constitutes one of several variants [16], [19], [25] of the hypercube. The idea grew out of the necessity to mitigate the latter's rapid scaling while retaining most of its good characteristics. Indeed, the dual cube exhibits a number of welcome features like vertex transitivity [24], efficient collective communication [17], high connectivity and fault tolerance [24], Hamiltonian decomposability [22], and low diameter and easy routing [24]. The present paper adds another distinctive property to that list, viz., the existence of 1-perfect codes.

A concept closely related to 1-perfect codes is that of *domination* [21]. Indeed, a 1-perfect code, *a fortiori*, corresponds to a *smallest dominating set* in the graph [9]. Applications exist in areas such as *resource placement* in parallel/distributed systems, construction of an efficient backbone for routing, and partition of a network into small clusters [6], [7]. Meanwhile the problem of obtaining a smallest dominating set is known to be NP-hard [21].

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A. Definitions and Preliminaries

Graphs in this paper are simple, undirected and connected. Let $\text{dist}(u, v)$ denote the (shortest) distance between vertices u and v, where the underlying graph will be clear from the context. The *eccentricity* $\text{ecc}(u)$ of a vertex u in a graph G is given by $\max\{\text{dist}(u, v) \mid v \in V(G)\}$. The *radius* $\text{rad}(G)$ and the *diameter* $\text{dia}(G)$ of G are defined to be $\min\{\text{ecc}(u) \mid u \in V(G)\}$ and $\max\{\text{ecc}(u) \mid u \in V(G)\}$, respectively.

A vertex subset S of a graph G constitutes a dominating set if every vertex of G not in S is adjacent to some vertex in S. If, in addition, the distance between any two distinct elements of S is at least three, then S constitutes a 1-perfect code. Thus the closed neighborhoods of the vertices in such a code are mutually exclusive and exhaustive with respect to the vertex set of the graph. The *domination number* $\gamma(G)$ of G is the least cardinality of a dominating set. For undefined terms, see West [21].

For $x, y \in \{0, 1\}^n$, let $H(x, y)$ denote the *Hamming distance* between the two, i.e., the number of bit positions in which they differ from each other. The n-dimensional hypercube Q_n is the graph on the vertex set $\{0, 1\}^n$, where $\{x, y\} \in E(Q_n)$ iff $H(x, y) = 1$. For $a \in \{0, 1\}$, let $\overline{a} := 1 - a$. Meanwhile we view an n-bit string x as $x_{n-1} \ldots x_1 x_0$ as well as $x_n \ldots x_2 x_1$, and use the context to disambiguate the expression.

For binary strings x and y, let $x \cdot y$ denote the concatenation of x and y, and for sets X and Y of binary strings, let $X \bullet Y := \{x \cdot y \mid x \in X \text{ and } y \in Y\}$. Further, let $u_{j:i}$ denote the substring $u_j \ldots u_i$ of a binary string u, where $j \geq i$.

Definition 1: For $n \geq 1$, the dual-cube DQ_n is a spanning subgraph of Q_{2n+1}. Its edge set is given by $E_0 \cup E_1 \cup E_2$, where

a. $E_0 = \{\{u0, v0\} \mid u, v \in \{0, 1\}^{2n}, H(u_{2n:n+1}, v_{2n:n+1}) = 1$ and $u_{n:1} = v_{n:1}\}$
b. $E_1 = \{\{u1, v1\} \mid u, v \in \{0, 1\}^{2n}, u_{2n:n+1} = v_{2n:n+1}$ and $H(u_{n:1}, v_{n:1}) = 1\}$
c. $E_2 = \{\{u0, u1\} \mid u \in \{0, 1\}^{2n}\}$. ■

Note that $|V(DQ_n)| = 2^{2n+1}$; E_0, E_1 and E_2 are pairwise disjoint; and $|E_0| = |E_1| = n2^{2n-1}$, and $|E_2| = 2^{2n}$, so $|E(DQ_n)| = (n+1)2^{2n}$. Meanwhile it is known that DQ_n is connected, $(n+1)$-regular and bipartite [24] (p. 1732).

Intuitively, a vertex $u = u_{2n} \ldots u_{n+1} u_n \ldots u_1\, c$ consists of three "components," viz., the "leading half" $u_{2n} \ldots u_{n+1}$, the

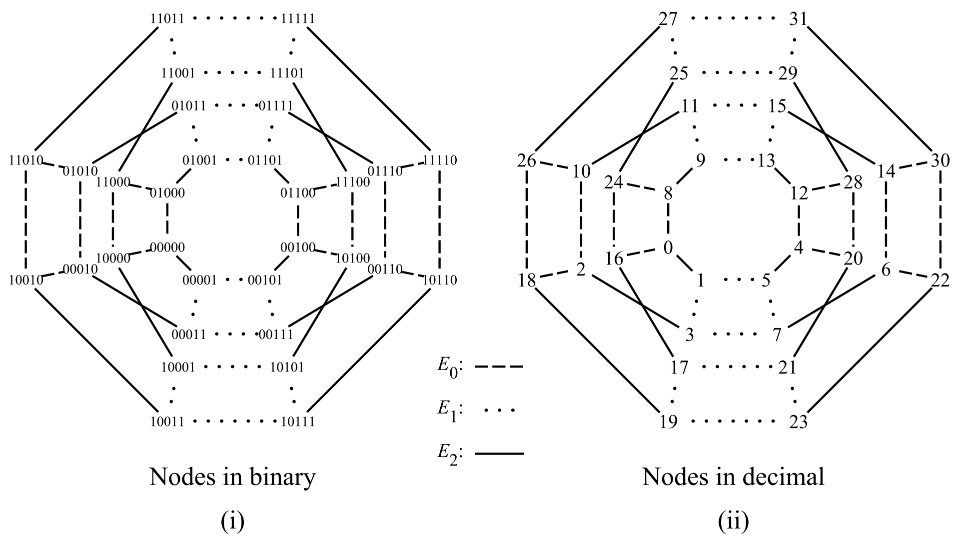

Fig. 1. The graph DQ_2 (cf. Definition 1).

"trailing half" $u_n \ldots u_1$ and the "end bit" c. Let $\langle E_0 \rangle$ and $\langle E_1 \rangle$ denote the subgraphs induced by E_0 and E_1, respectively.

Note: Whereas vertices of Q_n and DQ_n are binary strings, we refer to them as decimal integers as well, where the mapping $\sigma : \{0,1\}^n \to \{0, \ldots, 2^n - 1\}$ is given by the well-known recurrence: $\sigma(0) = 0$, $\sigma(1) = 1$, and $\sigma(x0) = 2\sigma(x)$, $\sigma(x1) = 2\sigma(x) + 1$.

Say that a vertex u (binary) is an *even vertex* if it is 0-ending, and an *odd vertex* otherwise.

Proposition 2 [17]: DQ_n admits a vertex partition into 2^{n+1} n-cubes segregated as follows:

- **Collection 0** (based on $\langle E_0 \rangle$) of 2^n n-cubes, where the i-th n-cube has the vertex set $\{2^{n+1}j + 2i \mid 0 \le j \le 2^n - 1\}$, $0 \le i \le 2^n - 1$. In binary, two vertices u and v in this collection belong to the same n-cube iff $u = xz0$ and $v = yz0$ for a fixed n-bit string z, and $x, y \in \{0,1\}^n$.
- **Collection 1** (based on $\langle E_1 \rangle$) of 2^n n-cubes, where the i-th n-cube has the vertex set $\{2^{n+1}i + (2j+1) \mid 0 \le j \le 2^n - 1\}$, $0 \le i \le 2^n - 1$. In binary, two vertices u and v in this collection belong to the same n-cube iff $u = xy1$ and $v = xz1$ for a fixed n-bit string x, and $y, z \in \{0,1\}^n$. ■

Whereas the n-cubes in Collection 0 are on even vertices, those in Collection 1 are on odd vertices. For $n = 6$, for instance, the vertex sets of some of the (even) 6-cubes in Collection 0 are $\{128 \times 0 + 0, 128 \times 1 + 0, 128 \times 2 + 0, \ldots, 128 \times 63 + 0\}$ and $\{128 \times 0 + 2, 128 \times 1 + 2, 128 \times 2 + 2, \ldots, 128 \times 63 + 2\}$, and the vertex sets of some of the (odd) 6-cubes in Collection 1 are $\{128 \times 0 + 1, 128 \times 0 + 3, 128 \times 0 + 5, \ldots, 128 \times 0 + 127\}$ and $\{128 \times 1 + 1, 128 \times 1 + 3, 128 \times 1 + 5, \ldots, 128 \times 1 + 127\}$.

Proposition 3 [17]:

1) Between each n-cube Q in Collection 0 and each n-cube Q' in Collection 1, there exists a unique edge, one end of which is in Q and the other end is in Q', and that edge itself belongs to E_2 (cf. Definition 1).
2) If u and v are vertices in distinct n-cubes in Collection 0 (resp. Collection 1), then the distance between u and v is at least three. ■

DQ_2 appears in Fig. 1, where a distinction has been made between different edge types. Meanwhile Loh *et al.* [16] recently introduced what is called an *exchanged hypercube* that refines the concept of the dual-cube. In particular, the exchanged hypercube $EH(s,t)$ is a spanning subgraph of Q_{s+t+1}, $s, t \ge 1$. Its edge set is given by $F_0 \cup F_1 \cup F_2$, where

a. $F_0 = \{\{u0, v0\} \mid u, v \in \{0,1\}^{s+t}, H(u_{s+t:t+1}, v_{s+t:t+1}) = 1 \text{ and } u_{t:1} = v_{t:1}\}$
b. $F_1 = \{\{u1, v1\} \mid u, v \in \{0,1\}^{s+t}, u_{s+t:t+1} = v_{s+t:t+1} \text{ and } H(u_{t:1}, v_{t:1}) = 1\}$
c. $F_2 = \{\{u0, u1\} \mid u \in \{0,1\}^{s+t}\}$.

It is easy to see that DQ_n is isomorphic to $EH(n,n)$ [10], hence formulas relating to the distance, radius and diameter of DQ_n are obtainable from those of $EH(s,t)$. The following result deals with the distance function.

Lemma 4 [11]: If $u, v \in V(DQ_n)$, where $u = u_{2n} \ldots u_{n+1} u_n \ldots u_1 c$ and $v = v_{2n} \ldots v_{n+1} v_n \ldots v_1 d$ with $c, d = 0, 1$, then $\mathrm{dist}(u,v)$ is equal to

$$\begin{cases} H(u,v) + 2, & u_{2n:n+1} \neq v_{2n:n+1} \text{ and } c = d = 1, \text{ or} \\ & u_{n:1} \neq v_{n:1} \text{ and } c = d = 0 \\ H(u,v), & \text{otherwise.} \end{cases}$$

■

Note that $\mathrm{dist}(u,v) = H(u,v) + 2$ iff u and v are both odd vertices and they differ in the leading halves, or u and v are both even vertices and they differ in the trailing halves. Meanwhile $\mathrm{rad}(DQ_n) = \mathrm{dia}(DQ_n) = 2n + 2$ [11].

Among various properties that DQ_n inherits from Q_n, one is the existence of a unique *antipodal vertex* relative to every vertex in it, i.e., for every vertex u in DQ_n, there exists a unique v such that $\mathrm{dist}(u,v) = \mathrm{dia}(DQ_n)$.

Lemma 5: (a) If $u \in V(Q_n)$ and $u = u_{n-1} \ldots u_0$, then u's antipodal counterpart is $\overline{u}_{n-1} \ldots \overline{u}_0$. In decimal, u and $2^n - 1 - u$ are mutually antipodal.

(b) If $u \in V(DQ_n)$ and $u = u_{2n} \ldots u_{n+1} u_n \ldots u_1 c$, then u's antipodal counterpart is $\overline{u}_{2n} \ldots \overline{u}_{n+1} \overline{u}_n \ldots \overline{u}_1 c$ [11]. In decimal, if u is an even vertex, then u and $2^{2n+1} - 2 - u$ are mutually antipodal, and if u is an odd vertex, then u and $2^{2n+1} - u$ are mutually antipodal. ■

The 1-perfect code that we build is closed under the property of every element being co-existent with its antipodal counterpart.

B. State of the Art

Hamming codes [4] are based on the topology of the hypercube Q_{2^k-1}, $k \geq 2$. Jha and Slutzki [9] presented a scheme to construct such codes using *Latin squares*. (See Sec. II.)

There exist codes on other topologies, too. For example, the Lee metric codes, devised by Golomb and Welch [3], are r-perfect codes over the Cartesian product of two cycles. Meanwhile Biggs [1] initiated the study of codes based on the topology of general graphs. Kratochvil [14], [15] followed with several useful results.

There have been other interesting developments on this topic during the past two decades. For example, 1-perfect codes have been shown to exist in the Towers-of-Hanoi graphs [2] and Sierpiński graphs [12]. More recently, the existence of r-perfect codes in the Kronecker products (also known as direct products and tensor products) of finitely many cycles was studied by Jha [6], [7], Jerebic *et al.* [5] and Klavžar *et al.* [13], culminating in a complete characterization by Žerovnik [23]. The circulants [18] constitute other useful kinds of graphs that have been a subject of study in this direction.

C. Bounds on $\gamma(Q_n)$ and $\gamma(DQ_n)$

There exists a tight upper bound on $\gamma(Q_n)$ that is within twice the optimal.

Theorem 6 [9]: $\dfrac{2^n}{n+1} \leq \gamma(Q_n) \leq \dfrac{2^n}{2^{\lfloor \log_2(n+1) \rfloor}}$. ■

Corollary 7 below is based on Theorem 6 and the facts that (i) DQ_n is an $(n+1)$-regular graph, and (ii) DQ_n admits a vertex partition into 2^{n+1} copies of Q_n.

Corollary 7: $\dfrac{2^{2n+1}}{n+2} \leq \gamma(DQ_n) \leq \dfrac{2^{2n+1}}{2^{\lfloor \log_2(n+1) \rfloor}}$. ■

Note that if $n = 2^k - 1$, then $\dfrac{2^{2n+1}}{n+2} \leq \gamma(DQ_n) \leq \dfrac{2^{2n+1}}{n+1}$, where the bounds are pretty close. Based on the forthcoming result in Sec. III-B, it turns out that the lower bound from Corollary 7 corresponds to the exact value if $n = 2^k - 2$, and the upper bound is subject to an improvement.

D. Latin Squares

For a positive integer r, an $r \times r$ Latin square is defined to be a square matrix L over the set $\{0, \ldots, r-1\}$ such that every row and every column of L contains each element of $\{0, \ldots, r-1\}$ exactly once.

The central scheme in this paper critically relies on the employment of a particular kind of Latin square for each power of two. To that end, see the 4×4 Latin square $L_{(4)}$ in Table I, where the "top left" 2×2 submatrix is equal to the "bottom right" submatrix, and the "top right" submatrix is equal to the "bottom left" submatrix. Further, this property

TABLE I
THE LATIN SQUARE $L_{(4)}$

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

TABLE II
THE LATIN SQUARE $L_{(2r)}$ OBTAINABLE FROM $L_{(r)}$

$$L_{(2r)} = \begin{pmatrix} L_{(r)} & r + L_{(r)} \\ r + L_{(r)} & L_{(r)} \end{pmatrix}$$

TABLE III
THE LATIN SQUARE $L_{(8)}$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\ 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\ 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\ 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\ 5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 \\ 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

holds with respect to the entries in each submatrix as well. In general, the Latin square $L_{(2r)}$ is obtainable from $L_{(r)}$ as in Table II, where $r + L_{(r)}$ itself is obtainable from $L_{(r)}$ by systematically adding r to each of its entries. See $L_{(8)}$ in Table III.

What Follows: Sec. II presents the essentials of a scheme [9] to construct Hamming codes. This is in view of the fact that the algorithm to construct a 1-perfect code in DQ_m heavily relies on the employment of the Hamming codes. Sec. III consists of the results. There is a detailed discussion on the algorithm and its correctness. Whereas Sec. IV derives tight bounds on $\gamma(EH(s,t))$ and $\gamma(DQ_n)$, Sec. V presents certain concluding remarks.

II. A SCHEME TO CONSTRUCT HAMMING CODES

The present section recounts the essentials of the scheme by Jha and Slutzki [9] that builds Hamming codes using Latin squares. See Algorithm 1.

Theorem 8 [9]: Algorithm 1 returns a partition of $V(Q_n)$ into Hamming codes, where $n = 2^k - 1$, $k \geq 2$, and the following hold:

(a) Each set in the partition contains as many even vertices as odd vertices.

(b) If x is in a set, then so is its antipodal counterpart. ■

Toward a partition of $V(Q_7)$ into Hamming codes, Table IV presents the initial working of Algorithm 1 on Q_3. Further, Table V presents the details of the construction using the Latin

Algorithm 1 A Scheme to Construct Hamming Codes [9]

Require: $n = 2^k - 1$, $k \geq 2$.
if $k = 2$ **then**
  return $\{\{000, 111\}, \{001, 110\}, \{010, 101\}$ and $\{011, 100\}\}$;
else
  for some $k \geq 2$, let $U_0, \ldots, U_n$ be the sets that constitute a partition of $V(Q_n)$ into Hamming codes, where $|U_i| = \dfrac{2^n}{n+1} = r + 1$ (say);
  let $U_i := \{u_{i,0}, \ldots, u_{i,r}\}$, $0 \leq i \leq n$;
  for $i := 0 \to n$ **do**
    let $C_i := \{u_{i,0} \cdot b_{i,0}, \ldots, u_{i,r} \cdot b_{i,r}\}$ and $D_i := \{u_{i,0} \cdot \overline{b}_{i,0}, \ldots, u_{i,r} \cdot \overline{b}_{i,r}\}$,
    where $b_{i,j} = 0$ (resp. 1) if the no. of 1s in $u_{i,j}$ is even (resp. odd);
  end for
  let $T = (t_{i,j})$ be an $(n+1) \times (n+1)$ Latin square;
  ▹ In this particular scheme, any $(n+1) \times (n+1)$ Latin square will work;
  return $\{V_0, \ldots, V_{2n+1}\}$, where

$$V_i := \begin{cases} (C_0 \bullet U_{t_{i,0}}) \bigcup \ldots \bigcup (C_n \bullet U_{t_{i,n}}), & 0 \leq i \leq n \\ (D_0 \bullet U_{t_{i-n-1,0}}) \bigcup \ldots \bigcup (D_n \bullet U_{t_{i-n-1,n}}), & n+1 \leq i \leq 2n+1; \end{cases}$$

end if

TABLE IV
SETS RELATING TO Q_3 IN THE INITIAL WORKING OF ALGORITHM 1

$U_0 = \{0, 7\}$	$U_1 = \{1, 6\}$	$U_2 = \{2, 5\}$	$U_3 = \{3, 4\}$
$C_0 = \{0, 15\}$	$C_1 = \{3, 12\}$	$C_2 = \{5, 10\}$	$C_3 = \{6, 9\}$
$D_0 = \{1, 14\}$	$D_1 = \{2, 13\}$	$D_2 = \{4, 11\}$	$D_3 = \{7, 8\}$

TABLE V
BUILDING A PARTITION OF $V(Q_7)$ USING ALGORITHM 1

$V_0 =$	$(C_0 \bullet U_0) \cup (C_1 \bullet U_1) \cup (C_2 \bullet U_2) \cup (C_3 \bullet U_3)$
$V_1 =$	$(C_0 \bullet U_1) \cup (C_1 \bullet U_0) \cup (C_2 \bullet U_3) \cup (C_3 \bullet U_2)$
$V_2 =$	$(C_0 \bullet U_2) \cup (C_1 \bullet U_3) \cup (C_2 \bullet U_0) \cup (C_3 \bullet U_1)$
$V_3 =$	$(C_0 \bullet U_3) \cup (C_1 \bullet U_2) \cup (C_2 \bullet U_1) \cup (C_3 \bullet U_0)$
$V_4 =$	$(D_0 \bullet U_0) \cup (D_1 \bullet U_1) \cup (D_2 \bullet U_2) \cup (D_3 \bullet U_3)$
$V_5 =$	$(D_0 \bullet U_1) \cup (D_1 \bullet U_0) \cup (D_2 \bullet U_3) \cup (D_3 \bullet U_2)$
$V_6 =$	$(D_0 \bullet U_2) \cup (D_1 \bullet U_3) \cup (D_2 \bullet U_0) \cup (D_3 \bullet U_1)$
$V_7 =$	$(D_0 \bullet U_3) \cup (D_1 \bullet U_2) \cup (D_2 \bullet U_1) \cup (D_3 \bullet U_0)$

square of Table I, and Table VI depicts the final partition. (Technically, vertices of Q_n are binary strings of length n. For the sake of compactness, they appear in decimal in each of Table IV and Table VI.)

A. Canonical Hamming Codes

Whereas Hamming codes are many, not all are useful in the present study. Say that subsets A and B of $V(Q_n)$ are *cognates* of each other if the following holds: An even vertex v is in A iff the odd vertex $v + 1$ is in B, and an odd vertex w is in A iff $w - 1$ is in B.

Definition 9: For $n = 2^k - 1$, $k \geq 2$, a partition $\{V_0, \ldots, V_n\}$ of $V(Q_n)$ into Hamming codes is said to be canonical if there exists a pairing $\{V_{i_0}, V_{i_1}\}, \ldots, \{V_{i_{n-1}}, V_{i_n}\}$ of the sets such that $V_{i_{2j}}$ and $V_{i_{2j+1}}$ are cognates, $0 \leq j \leq \frac{1}{2}(n-1)$. ■

Lemma 10: For $n = 2^k - 1$, $k \geq 2$, there exists a canonical vertex partition $\{V_0, \ldots, V_n\}$ of Q_n such that V_{2i} and V_{2i+1} are cognates, $0 \leq i \leq \frac{1}{2}(n-1)$.

Proof: Use induction on k. For $k = 2$, the partition $\{U_0, U_1, U_2, U_3\}$ of $V(Q_3)$ appearing in Table IV is canonical, where the pairing consists of $\{U_0, U_1\}$ and $\{U_2, U_3\}$. For $k = 3$, similarly, the partition in Table VI is canonical, where the pairing consists of $\{V_0, V_1\}$, $\{V_2, V_3\}$, $\{V_4, V_5\}$ and $\{V_6, V_7\}$.

For some $k \geq 3$, let $\{U_0, \ldots, U_n\}$ be a canonical vertex partition of Q_n, where U_{2i} and U_{2i+1} are cognates, $0 \leq i \leq \frac{1}{2}(n-1)$, and obtain the partition $\{W_0, \ldots, W_{2n+1}\}$ of $V(Q_{2n+1})$ by means of Algorithm 1 using the $(n+1) \times (n+1)$ Latin square $L_{(n+1)}$ that is based on the scheme in Sec. I-D. (For example, $L_{(8)}$ appears in Table III.) That the resulting partition is canonical follows from the following fact: If U_p and U_q are cognates, then so must be $C_i \bullet U_p$ and $C_i \bullet U_q$ (resp. $D_j \bullet U_p$ and $D_j \bullet U_q$). See the last step of Algorithm 1. Illustrations appear in Tables V and VI. ■

Domination of Q_7: Using decimal notation for the vertices, Fig. 2 depicts the domination of Q_7 by means of the set V_0 from Table VI. The even elements of V_0 dominate a total of $8 \times 7 = 56$ even vertices (including themselves) plus eight odd vertices. The odd vertices of V_0 in turn dominate a total of $8 \times 7 = 56$ odd vertices (including themselves) plus eight even vertices. Further, the even vertices not dominated by their "even counterparts" and the odd vertices not dominated by their "odd counterparts" comprise the set V_1 that is the cognate of V_0. This property holds for other pairs of cognates too. Meanwhile the scheme that is forthcoming in Sec. III-B employs an amplified form of the "dovetailing" that exists in Fig. 2.

III. RESULTS

The degree of each vertex of DQ_m being equal to $m + 1$, a 1-perfect code in this graph is feasible iff $m + 2$ evenly divides $|V(DQ_m)| = 2^{2m+1}$. In that light, $m = 2^k - 2$ in the rest of this section, where $k \geq 2$.

Fig. 3 depicts a 1-perfect code (consisting of the vertices that are "circled") in DQ_2. The line types representing the

TABLE VI
A CANONICAL VERTEX PARTITION OF Q_7

i	Elements of V_i															
0	0	7	25	30	42	45	51	52	75	76	82	85	97	102	120	127
1	1	6	24	31	43	44	50	53	74	77	83	84	96	103	121	126
2	2	5	27	28	40	47	49	54	73	78	80	87	99	100	122	125
3	3	4	26	29	41	46	48	55	72	79	81	86	98	101	123	124
4	8	15	17	22	34	37	59	60	67	68	90	93	105	110	112	119
5	9	14	16	23	35	36	58	61	66	69	91	92	104	111	113	118
6	10	13	19	20	32	39	57	62	65	70	88	95	107	108	114	117
7	11	12	18	21	33	38	56	63	64	71	89	94	106	109	115	116

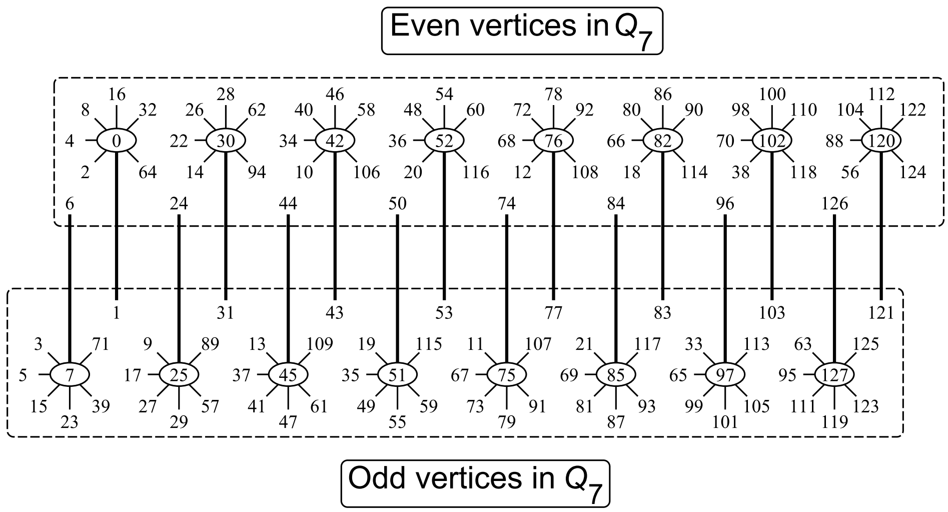

Fig. 2. Domination of Q_7 by means of the Hamming code V_0 (cf. Table VI).

three kinds of edges are same as those in Fig. 1(i). Notice that (i) the graph admits a vertex partition into 2-cubes that contribute equally to the code, and (ii) a vertex and its antipodal counterpart coexist in the code. It turns out that these properties hold in general. In the rest of this section, let $m = 2^k - 2$, $k \geq 3$.

Note: Fig. 1 and Fig. 3 are adaptations of Fig. 2 from Klavžar and Ma [10].

A. A Partition of $V(Q_m)$

We build a partition $\{W_0, \ldots, W_{m+1}\}$ of $V(Q_m)$ obtainable from the canonical partition of $V(Q_{m+1})$ from Sec. II-A, and capture it by means of a "distinguishing" array Δ of 2^m elements. See Algorithm 2, and Lemmas 11 and 12. The idea is useful in the sequel.

Lemma 11: The sets $W_0, \ldots, W_{m+1}$ at the conclusion of Step 7 in Algorithm 2 constitute a partition of $V(Q_m)$, where $m = 2^k - 2$, $k \geq 2$.

Proof: By Theorem 8(b), half of the elements in each V_i at Step 1 of Algorithm 2 are of the form $0x$, and the remaining half are of the form $1x$, where $x \in \{0,1\}^m$. Accordingly, the (twin) sets W_{2i} and W_{2i+1} are well-defined, and $|W_{2i}| = |W_{2i+1}| = \frac{1}{2}|V_i| = \frac{2^m}{m+2} = 2^{m-k}$, $0 \leq i \leq m/2$. In that light, it suffices to show that $W_0, \ldots, W_{m+1}$ are mutually disjoint. Clearly, $W_{2i} \cap W_{2i+1} = \emptyset$, since the Hamming distance between any two distinct elements of V_i is at least three. Further, $V_0, \cdots, V_{m/2}$ being mutually disjoint, so must be $W_0, W_2, W_4, \ldots, W_m$ and, similarly, so must be $W_1, W_3, W_5, \ldots, W_{m+1}$.

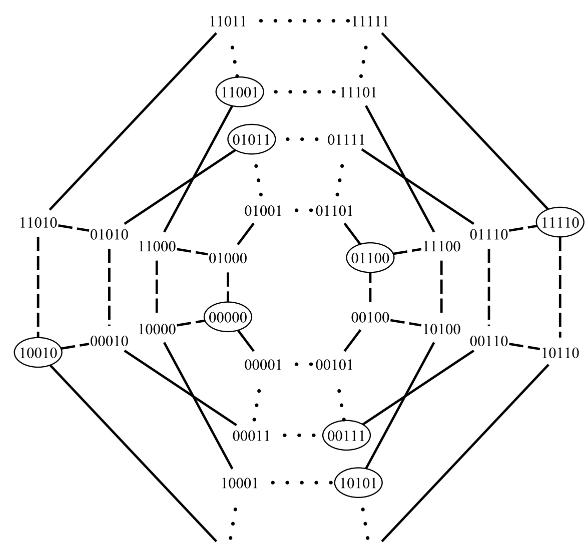

Fig. 3. A 1-perfect code in DQ_2.

Algorithm 2 Building Array $\Delta[0..2^m-1]$, $m = 2^k - 2$, $k \geq 2$

consider the sets $V_0, \ldots, V_{m/2}$ in a canonical partition $\{V_0, \ldots, V_{m+1}\}$ of $V(Q_{m+1})$ based on Algorithm 1;
$\triangleright$ The elements of each V_i are viewed as $(m+1)$-bit binary strings as well as integers between 0 and $2^{m+1} - 1$;
for $i := 0 \to m/2$ **do**
  derive the sets W_{2i} and W_{2i+1} from V_i as follows:
    $W_{2i} = \{w \mid 0w \in V_i\}$;
    $W_{2i+1} = \{w \mid 1w \in V_i\}$;
end for
$\triangleright$ $|W_i| = 2^{m-k}$, $0 \leq i \leq m+1$
for $i := 0 \to m+1$ **do**
  for each j (decimal) in W_i **do**
    $\Delta[j] := i$;
  end for
end for
$\triangleright$ $0 \leq \Delta[i] \leq m+1$, where $0 \leq i \leq 2^m - 1$
return Δ;

TABLE VII
ILLUSTRATING THE PROOF OF LEMMA 11 ($m = 6$ AND $k = 3$)

i	Elements of W_i							
0	0	7	25	30	42	45	51	52
1	11	12	18	21	33	38	56	63
2	1	6	24	31	43	44	50	53
3	10	13	19	20	32	39	57	62
4	2	5	27	28	40	47	49	54
5	9	14	16	23	35	36	58	61
6	3	4	26	29	41	46	48	55
7	8	15	17	22	34	37	59	60

What remains to show is that $W_{2i} \cap W_{2j+1} = \emptyset$, where $0 \leq i \neq j \leq m/2$. To that end, let $u \cdot v \in W_{2i}$ and $x \cdot y \in W_{2j+1}$, where $u, v, x, y \in \{0,1\}^{m/2}$. At this point, note that the following hold: (i) $0u \cdot v \in V_i$, (ii) $1x \cdot y \in V_j$, and (iii) $0u$ and $1x$ are of the same parity, cf. Algorithm 1. It is clear that u and x are of different parities, hence $u \cdot v \notin W_{2j+1}$ and $x \cdot y \notin W_{2i}$. ■

Note that each of $0, \ldots, m+1$ appears equally often in Array Δ. In particular, $|\{p \mid \Delta[p] = i\}| = 2^{m-k}$, where $0 \leq i \leq m+1$. Using decimal notation, Table VII presents the sets $W_0, \ldots, W_7$ based on the canonical partition of $V(Q_7)$ appearing in Table VI. The resulting array itself appears in Table VIII.

Lemma 12: The following hold relative to array Δ returned by Algorithm 2:

(a) If $\Delta[p] = \Delta[q]$ and $p \neq q$, then $H(p,q) \geq 3$
(b) If $\Delta[p] = 2i$ and $\Delta[q] = 2i+1$, then $H(p,q) \geq 2$, and
(c) For every p such that $\Delta[p] = 2i$, there exists a q such that $\Delta[q] = 2i+1$ and $p + q = 2^m - 1$, and vice versa,

where $0 \leq p, q \leq 2^m - 1$, and $0 \leq i \leq m/2$.

Proof: It is clear that the Hamming distance between two distinct elements in W_{2i} (resp. W_{2i+1}) is at least three. Next, if $w, x \in V_i$, where $0 \leq w \leq 2^m - 1$ and $2^m \leq x \leq 2^{m+1} - 1$, then the fact that $H(w,x) \geq 3$ implies that $H(w, x - 2^m) \geq 2$.

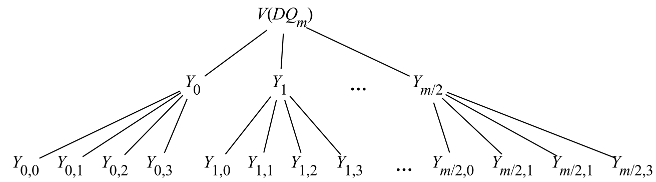

Fig. 4. Successive vertex partition of DQ_m as per Defintion 13.

Finally, by Theorem 8(b), the elements p and $2^{m+1} - 1 - p$ coexist in each V_i, where $0 \leq p \leq 2^m - 1$, hence $p \in W_{2i}$ iff $2^m - 1 - p \in W_{2i+1}$.

The foregoing properties of the elements of W_{2i} and W_{2i+1} are seamlessly inherited by the elements in the array Δ. ■

B. 1-Perfect Code

In the rest of this section, let $\{V_0, \ldots, V_{m+1}\}$ be a canonical vertex partition of Q_{m+1} into Hamming codes, and let $\{W_0, \ldots, W_{m+1}\}$ be the corresponding vertex partition of Q_m obtainable by means of Algorithm 2, where $m = 2^k - 2$. Note that $|V_i| = 2^{m-k+1}$ and $|W_i| = 2^{m-k}$.

Definition 13: Let $Y_0, \ldots, Y_{m/2}$ be the vertex subsets of DQ_m, where Y_i is the (disjoint) union of the sets $Y_{i,0}$, $Y_{i,1}$, $Y_{i,2}$ and $Y_{i,3}$ that appear in Eq. (1), and let G_i be the subgraph induced by Y_i, $0 \leq i \leq m/2$. ■

$$\left.\begin{aligned} Y_{i,0} &= \bigcup_{\substack{r \in V_{2i} \\ r \text{ even}}} \{2^{m+1}j + r \mid 0 \leq j \leq 2^m - 1\} \\ Y_{i,1} &= \bigcup_{s \in W_{2i}} \{2^{m+1}s + 2j + 1 \mid 0 \leq j \leq 2^m - 1\} \\ Y_{i,2} &= \bigcup_{\substack{r \in V_{2i+1} \\ r \text{ even}}} \{2^{m+1}j + r \mid 0 \leq j \leq 2^m - 1\} \\ Y_{i,3} &= \bigcup_{s \in W_{2i+1}} \{2^{m+1}s + 2j + 1 \mid 0 \leq j \leq 2^m - 1\} \end{aligned}\right\} \quad (1)$$

Fig. 4 depicts the successive vertex partition of DQ_m as per Defintion 13. The following are salient features of the partition.

a. Whereas each of $Y_{i,0}$ and $Y_{i,2}$ consists of even vertices, each of $Y_{i,1}$ and $Y_{i,3}$ consists of odd vertices. Further, V_{2i} and V_{2i+1} being disjoint, the sets $Y_{i,0}$ and $Y_{i,2}$ must themselves be disjoint. Similarly, $Y_{i,1}$ and $Y_{i,3}$ are disjoint. It follows that $Y_{i,0}$, $Y_{i,1}$, $Y_{i,2}$ and $Y_{i,3}$ are mutually disjoint for each i.
b. For each i, $|Y_{i,0}| = |Y_{i,2}| = \frac{1}{2}|V_i|2^m = 2^{2m-k}$, and $|Y_{i,1}| = |Y_{i,3}| = |W_i|2^m = 2^{2m-k}$. Accordingly, $Y_{i,0}$, $Y_{i,1}$, $Y_{i,2}$ and $Y_{i,3}$ are equinumerous. Indeed, each may be viewed as consisting of 2^{m-k} m-cubes. Accordingly, G_i may be laid out as a $2^{m-k} \times 4$ array of m-cubes. (See Fig. 7 that is forthcoming.)
c. The facts that $V_0, \ldots, V_{m+1}$ are mutually disjoint, and $W_0, \ldots, W_{m+1}$ are mutually disjoint ensure that $Y_0, \ldots, Y_{m/2}$ are mutually disjoint.

1) For each i, $|Y_i| = |Y_{i,0}| + |Y_{i,1}| + |Y_{i,2}| + |Y_{i,3}| = 2^{2m-k+2}$. Further, $2^{2m-k+2}(1 + \frac{m}{2}) = 2^{2m+1} = |V(DQ_m)|$. This and the fact that $Y_0, \ldots, Y_{m/2}$ are mutually disjoint together imply that $Y_0, \ldots, Y_{m/2}$ constitute a vertex partition of DQ_m.

TABLE VIII
ARRAY $\Delta[0..63]$ ($m = 6$ AND $k = 3$)

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\Delta[i]$	0	2	4	6	6	4	2	0	7	5	3	1	1	3	5	7
i	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
$\Delta[i]$	5	7	1	3	3	1	7	5	2	0	6	4	4	6	0	2
i	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47
$\Delta[i]$	3	1	7	5	5	7	1	3	4	6	0	2	2	0	6	4
i	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63
$\Delta[i]$	6	4	2	0	0	2	4	6	1	3	5	7	7	5	3	1

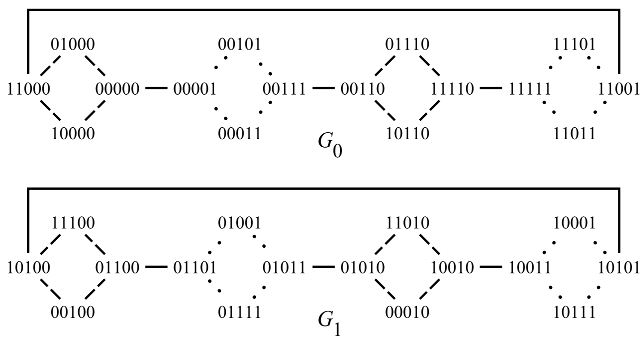

Fig. 5. The induced subgraphs G_0 and G_1 in respect of DQ_2.

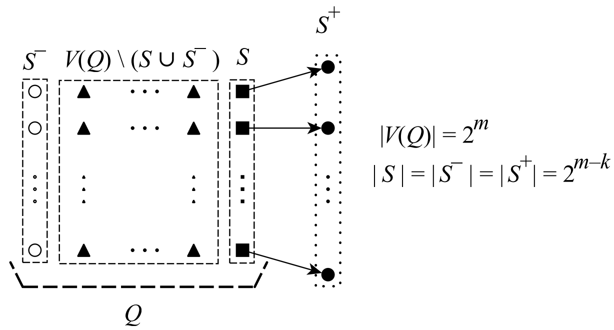

Fig. 6. Sets associated with an m-cube, cf. Definition 16.

It is clear that $|V(G_i)| = 2^{2m-k+2}$. By symmetry, $G_0, \ldots, G_{m/2}$ are mutually isomorphic. We later prove that $|E(G_i)| = m2^{2m-k+1} + 2^{2m-2k+2}$. See Fig. 5 that depicts G_0 and G_1 in respect of DQ_2. (The line types representing the three kinds of edges are same as those in Fig. 1(i).) It turns out that each G_i itself admits a 1-perfect code. See Algorithm 3.

Lemma 14: The following hold relative to an iteration of the outer "for" loop in Algorithm 3:

- $(\cup_{j=0}^{2^{m-k}-1} A_j) \subseteq Y_{i,0}$
- $(\cup_{j=0}^{2^{m-k}-1} B_j) \subseteq Y_{i,1}$
- $(\cup_{j=0}^{2^{m-k}-1} C_j) \subseteq Y_{i,2}$ and
- $(\cup_{j=0}^{2^{m-k}-1} D_j) \subseteq Y_{i,3}$

where $Y_{i,0}$, $Y_{i,1}$, $Y_{i,2}$ and $Y_{i,3}$ are as in Eq. (1).

Proof: By Line 13 of Algorithm 3, $\cup_{j=0}^{2^{m-k}-1} A_j = \cup_{j=0}^{2^{m-k}-1}\{2^{m+1}p_a + u_j \mid 0 \le a \le 2^{m-k} - 1\}$ where (i) $p_0, \ldots, p_{2^{m-k}-1}$ are indices of the array Δ such that $\Delta[p_0] = \ldots = \Delta[p_{2^{m-k}-1}] = 2i$, and (ii) $u_0, \ldots, u_{2^{m-k}-1}$ are the even elements in V_{2i}. Since $|\Delta| = 2^m$, it is clear that $0 \le p_0, \ldots, p_{2^{m-k}-1} \le 2^m - 1$. By Eq. (1),

$$Y_{i,0} = \bigcup_{\substack{r \in V_{2i} \\ r \text{ even}}} \{2^{m+1} j + r \mid 0 \le j \le 2^m - 1\}.$$

It is easy to see that every element of $(\cup_{j=0}^{2^{m-k}-1} A_j)$ belongs to $Y_{i,0}$, i.e., $(\cup_{j=0}^{2^{m-k}-1} A_j) \subseteq Y_{i,0}$. Note also that $|\cup_{j=0}^{2^{m-k}-1} A_j| = 2^{2m-2k}$ and $|Y_{i,0}| = 2^{2m-k}$. The other cases are similar. ■

Lemma 15: The set returned by Algorithm 3 is such that the code elements in each m-cube dominate a total of $(m+1)2^{m-k}$ vertices (including themselves) within the cube plus 2^{m-k} vertices outside the cube.

Proof: In binary, $p_0, \ldots, p_{2^{m-k}-1}$ and $q_0, \ldots, q_{2^{m-k}-1}$ that appear on Lines 5 and 6, respectively, are each an m-bit string. Accordingly, each of $2^{m+1}p_a$ and $2^{m+1}q_a$ is an $(2m+1)$-bit string whose rightmost $(m+1)$ bits are all zeros, where $0 \le a \le 2^{m-k} - 1$. Further, $u_0, \ldots, u_{2^{m-k}-1}$, $v_0, \ldots, v_{2^{m-k}-1}$, $w_0, \ldots, w_{2^{m-k}-1}$ and $x_0, \ldots, x_{2^{m-k}-1}$ that appear on Lines 7, 8, 9 and 10, respectively, are each an $(m+1)$-bit string, so $H(2^{m+1}p_a + u_j, 2^{m+1}p_b + u_j) = H(p_a, p_b)$. By Lemma 12(a), $H(p_a, p_b) \ge 3$ if $\Delta[p_a] = \Delta[p_b]$, where $p_a \ne p_b$.

Consider the set A_j on Line 13. It is clear that (i) the closed neighborhoods of its elements are mutually disjoint, and (ii) one neighbor of each element is an odd vertex that lies outside the (even) cube, and the remaining m neighbors lie within the cube itself. The argument is similar in respect of the set C_j on Line 15.

For the elements in the set B_j on Line 14 in respect of an odd cube, note that $H(2^{m+1}p_j + v_b, 2^{m+1}p_j + v_c) = H(v_b, v_c)$ that is greater than or equal to three, where $v_b \ne v_c$. For D_j on Line 16, similarly, $H(x_b, x_c) \ge 3$ where $x_b \ne x_c$. The rest of the reasoning is as in the preceding paragraph. ■

Definition 16: For a set S of code elements in a particular m-cube Q in DQ_m, let S^- denote the set of vertices in Q not dominated by S, and let S^+ denote the set of vertices outside Q dominated by S. ■

Fig. 6 depicts the sets S, S^- and S^+ associated with an m-cube in DQ_m. Its correctness follows from Lemma 15. An arrowhead from, say, u to v is relevant to the extent that u dominates v. Note that each element of S^+ is dominated by a unique element of S. Meanwhile elements of S^+ belong to

Algorithm 3 Building a 1-Perfect Code in DQ_m, $m = 2^k - 2$, $k \geq 3$

Require: Sets $V_0, \ldots V_{m+1}$, cf. Table VI, and Array $\Delta[0..2^m - 1]$, cf. Table VIII
Ensure: Sets V_{2i} and V_{2i+1} are cognates, $0 \leq i \leq m/2$
1: $\triangleright$ Each V_s consists of 2^{m-k} even elements and as many odd elements.
2: $Z := \emptyset$ $\triangleright$ Initialization
3: **for** $i := 0 \to m/2$ **do**
4: $\triangleright$ Build the set of code elements corresponding to the subgraph G_i.
5: let $p_0, \ldots, p_{2^{m-k}-1}$ be such that $\Delta[p_0] = \ldots = \Delta[p_{2^{m-k}-1}] = 2i$;
6: let $q_0, \ldots, q_{2^{m-k}-1}$ be such that $\Delta[q_0] = \ldots = \Delta[q_{2^{m-k}-1}] = 2i + 1$;
7: let $u_0, \ldots, u_{2^{m-k}-1}$ be the even elements in V_{2i};
8: let $v_0, \ldots, v_{2^{m-k}-1}$ be the odd elements in V_{2i};
9: let $w_0, \ldots, w_{2^{m-k}-1}$ be the even elements in V_{2i+1};
10: let $x_0, \ldots, x_{2^{m-k}-1}$ be the odd elements in V_{2i+1};
11: $\triangleright$ $u_j = x_j - 1$ and $w_j = v_j - 1$, $0 \leq j \leq 2^{m-k} - 1$.
12: **for** $j := 0 \to 2^{m-k} - 1$ **do**
13: $A_j := \{2^{m+1} p_a + u_j \mid 0 \leq a \leq 2^{m-k} - 1\}$; $\triangleright$ within an even cube
14: $B_j := \{2^{m+1} p_j + v_a \mid 0 \leq a \leq 2^{m-k} - 1\}$; $\triangleright$ within an odd cube
15: $C_j := \{2^{m+1} q_a + w_j \mid 0 \leq a \leq 2^{m-k} - 1\}$; $\triangleright$ within an even cube
16: $D_j := \{2^{m+1} q_j + x_a \mid 0 \leq a \leq 2^{m-k} - 1\}$; $\triangleright$ within an odd cube
17: $Z := Z \cup A_j \cup B_j \cup C_j \cup D_j$;
18: **end for**
19: $\triangleright$ Note that each G_i houses $2^{2m+2-2k}$ code elements.
20: **end for**
21: **return** Z;

TABLE IX
SETS IN THE j-th ITERATION OF THE INNER "FOR" LOOP IN ALGORITHM 3

S	S^-	S^+
A_j	$A_j^- = \{2^{m+1}q_a + u_j \mid 0 \leq a \leq 2^{m-k} - 1\}$	$A_j^+ = \{2^{m+1}p_a + u_j + 1 \mid 0 \leq a \leq 2^{m-k} - 1\}$
B_j	$B_j^- = \{2^{m+1}p_j + x_a \mid 0 \leq a \leq 2^{m-k} - 1\}$	$B_j^+ = \{2^{m+1}p_j + v_a - 1 \mid 0 \leq a \leq 2^{m-k} - 1\}$
C_j	$C_j^- = \{2^{m+1}p_a + w_j \mid 0 \leq a \leq 2^{m-k} - 1\}$	$C_j^+ = \{2^{m+1}q_a + w_j + 1 \mid 0 \leq a \leq 2^{m-k} - 1\}$
D_j	$D_j^- = \{2^{m+1}q_j + v_a \mid 0 \leq a \leq 2^{m-k} - 1\}$	$D_j^+ = \{2^{m+1}q_j + x_a - 1 \mid 0 \leq a \leq 2^{m-k} - 1\}$

A_j, B_j, C_j and D_j are as on Lines 13, 14, 15 and 16, respectively, in Algorithm 3.

mutually distinct m-cubes.

Lemma 17: The equations relative to Algorithm 3 in Table IX are correct.

Proof: First consider A_j that is a subset of the vertex set of an even cube, say Q, where $V(Q) = \{2^{m+1}r + u_j \mid 0 \leq r \leq 2^m - 1\}$. The elements of A_j dominate $(m+1)2^{m-k}$ vertices including themselves (all of which are within Q itself) plus 2^{m-k} vertices, each of which belongs to a unique odd cube. Thus there are 2^{m-k} vertices within Q that are not dominated by A_j. The collection of such vertices is equal to $\{2^{m+1}q_c + u_j \mid 0 \leq c \leq 2^{m-k} - 1\}$. To see this, note that $H(2^{m+1}p_a + u_j, 2^{m+1}q_c + u_j) = H(p_a, q_c)$ that is greater than or equal to two, cf. Lemma 12(b). Next, it is easy to see that the set of vertices outside Q dominated by A_j is equal to $\{2^{m+1}p_a + u_j + 1 \mid 0 \leq a \leq 2^{m-k} - 1\}$.

The argument is similar for each of the remaining statements. ■

Lemma 18: The set returned by Algorithm 3 allocates a 1-perfect code to each (induced) subgraph G_i of DQ_m, $0 \leq i \leq m/2$.

Proof: It suffices to show that the following four identities hold:

- $\cup A_j^+ = \cup B_j^-$
- $\cup B_j^+ = \cup C_j^-$
- $\cup C_j^+ = \cup D_j^-$, and
- $\cup D_j^+ = \cup A_j^-$,

where the subscript j runs from 0 to $2^{m-k} - 1$ in each union, and the sets themselves are as in Table IX. To that end, observe that

$$\begin{aligned}
&\cup_{j=0}^{2^{m-k}-1} A_j^+ \\
&\quad = \cup_{j=0}^{2^{m-k}-1} \{2^{m+1}p_a + u_j + 1 \mid 0 \leq a \leq 2^{m-k} - 1\} \\
&\quad = \cup_{j=0}^{2^{m-k}-1} \cup_{a=0}^{2^{m-k}-1} \{2^{m+1}p_a + x_j\} \\
&\quad = \cup_{j=0}^{2^{m-k}-1} \cup_{a=0}^{2^{m-k}-1} \{2^{m+1}p_j + x_a\} \\
&\quad = \cup_{j=0}^{2^{m-k}-1} B_j^- .
\end{aligned}$$

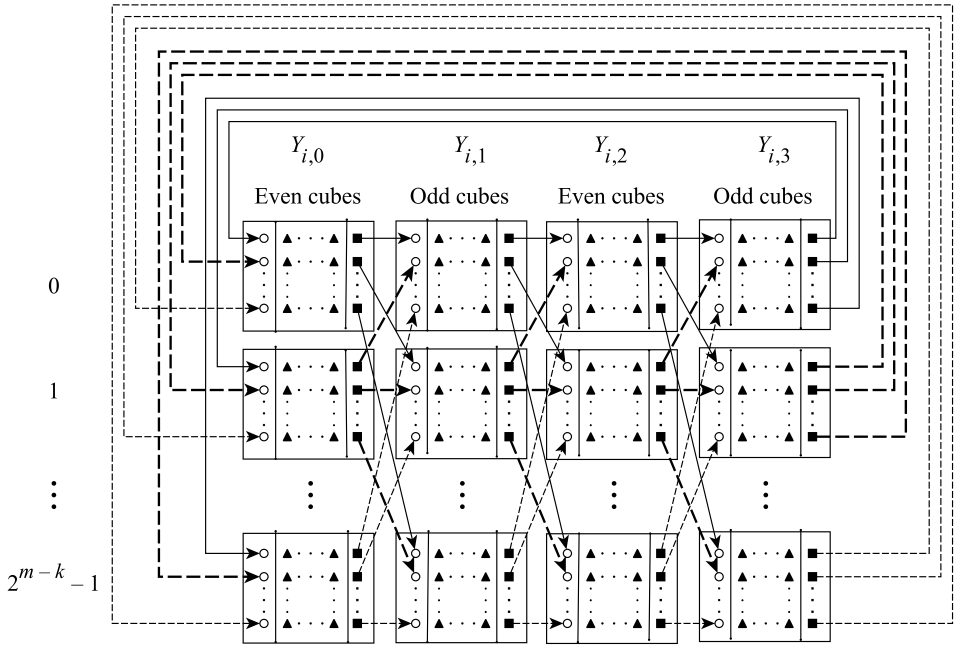

Fig. 7. Illustration of the proof of Lemma 18.

The remaining three identities follow by analogous arguments. ■

Fig. 7 illustrates the proof of Lemma 18. Each m-cube in the figure is as per the template in Fig. 6.

Lemma 19: The number of edges in each induced subgraph G_i of DQ_m is equal to $m2^{2m-k+1} + 2^{2m-2k+2}$, $0 \leq i \leq m/2$.

Proof: Note that G_i consists of 2^{m-k+2} m-cubes. As depicted in Fig. 6 and Fig. 7, the vertex set of each m-cube Q admits a partition into S^-, $V(Q) \setminus (S^- \cup S)$ and S, where $|S^-| = |S| = 2^{m-k}$. Whereas each vertex in each of S^- and S has all of its $m+1$ neighbors contained in G_i itself, each vertex in $V(Q) \setminus (S^- \cup S)$ has its m neighbors within the m-cube and the remaining one neighbor outside G_i. Accordingly, the total number of edges in G_i is equal to $2^{m-k+2}(\frac{1}{2}(m+1)(|S^-|+|S|) + \frac{1}{2}m|V(Q) \setminus (S^- \cup S)|)$ that is $m2^{2m-k+1} + 2^{2m-2k+2}$. ■

By Lemma 19, $|E(G_i)| = (\frac{m}{2} + \frac{1}{2^k})|V(G_i)|$, i.e., G_i is almost m-regular, i.e., almost all vertices of G_i are of degree m each. This and Lemma 18 yield the following: DQ_m that is an $(m+1)$-regular graph admits a vertex partition into $\frac{1}{2}(m+2)$ subgraphs that are mutually isomorphic, where each subgraph is almost m-regular, and each admits a 1-perfect code.

Lemma 20: If a code element z is in the subgraph G_i, then so is its antipodal counterpart.

Proof: Consider an iteration of the outer "for" loop of the algorithm.

First suppose that z is an even vertex that is added to the code on Line 13, so $z = 2^{m+1}p_a + u_{j_1}$, where $0 \leq a \leq 2^{m-k} - 1$, $\Delta[p_a] = 2i$, u_{j_1} is even, and $u_{j_1} \in V_{2i}$. By Lemma 12(c), there exists a q_b such that $\Delta[q_b] = 2i+1$ and $q_b = 2^m - 1 - p_a$. Next, V_{2i} and V_{2i+1} being cognates, $u_{j_1}+1 \in V_{2i+1}$. By Theorem 8(b), $2^{m+1}-1-(u_{j_1}+1) \in V_{2i+1}$. Accordingly, $2^{m+1}q_b + (2^{m+1} - u_{j_1} - 2)$ gets added to the code on Line 15 during some iteration of the inner "for" loop. Check to see that $2^{m+1}q_b + (2^{m+1} - u_{j_1} - 2)$ is equal to $(2^{2m+1}-2)-z$ that is antipodal to z, cf. Lemma 5(b). Similarly, if z is an even vertex that is added to the code on Line 15, then its antipodal counterpart gets added to the code on Line 13, not necessarily during the same iteration of the inner "for" loop.

An analogous argument leads to the following: If z is an odd vertex that gets added to the code on Line 14, then its antipodal counterpart gets added to the code on Line 16, and vice versa. ■

By the proof of Lemma 19, if there are two code elements, one of which is in G_i and the other in G_j with $i \neq j$, then the (shortest) distance between the two code elements in DQ_m is at least three. In the process, the set Z returned by Algorithm 3 is a 1-perfect code of DQ_m. The following result is immediate.

Theorem 21: If $m = 2^k - 2$, $k \geq 2$, then DQ_m admits a 1-perfect code. ■

Corollary 22: If $m = 2^k - 2$, $k \geq 2$, then DQ_m admits a vertex partition into 1-perfect codes.

Proof: Use vertex transitivity of DQ_m [24] and Theorem 21 to build the necessary vertex-disjoint 1-perfect codes in the graph. ■

Corollary 23: If $m = 2^k - 2$, $k \geq 2$, then $\gamma(DQ_m) = \dfrac{2^{2m+1}}{m+2} = 2^{2m-k+1}$. ■

IV. Bounds on $\gamma(EH(s,t))$ and $\gamma(DQ_n)$

As stated in Sec. I-A, the exchanged hypercube $EH(s,t)$ refines the concept of the dual-cube. In particular, $EH(n,n)$ is isomorphic to DQ_n [10]. Meanwhile $EH(s,t)$ is isomorphic to $EH(t,s)$ [16]. This section presents tight bounds on $\gamma(EH(s,t))$ and $\gamma(DQ_n)$ that are based on the results of Sec. III-B. See [8], [10] for recent results on $\gamma(EH(s,t))$.

Lemma 24: If $2 \leq s \leq t$, and S is a dominating set of $EH(s,t)$, then each of $EH(s+1,t)$ and $EH(s,t+1)$ admits a dominating set of cardinality $2 \cdot |S|$.

Proof: Each of $EH(s+1,t)$ and $EH(s,t+1)$ admits a vertex partition into two copies of $EH(s,t)$ [16]. ∎

Theorem 25: If $2 \leq s \leq t$, then $\gamma(EH(s,t)) \leq \frac{2^{s+t+1}}{2^{\lfloor log_2(s+2)\rfloor}}$.

Proof: Let m be the largest integer such that $m = 2^k - 2$ and $m \leq s$, so $m = \lfloor log_2(s+2)\rfloor$. By Corollary 23, $EH(m,m)$ admits a dominating set, say S, of cardinality $\frac{2^{2m+1}}{m+2}$. By repeated applications of Lemma 24, $EH(s,t)$ admits a dominating set of cardinality $2^{s-m} \cdot 2^{t-m} \cdot |S|$ that is $\frac{2^{s+t+1}}{2^{\lfloor log_2(s+2)\rfloor}}$. ∎

It is clear that the upper bound from Theorem 25 corresponds to the exact value if $s = t = 2^k - 2$. This is also the case if $s = 2$ and $t \geq 3$ [10].

Corollary 26: If $n \geq 2$, then $\frac{2^{2n+1}}{n+2} \leq \gamma(DQ_n) \leq \frac{2^{2n+1}}{2^{\lfloor log_2(n+2)\rfloor}}$. ∎

Note that the upper bound from Corollary 26 is slightly better than that from Corollary 7. Further, there is a striking similarity between Theorem 6 and Corollary 26, and the upper bound in each case is within twice the optimal.

V. Concluding Remarks

The dual-cube DQ_m is an $(m+1)$-regular connected spanning subgraph of the hypercube Q_{2m+1} [17], [20]. It is endowed with a number of welcome features that are of immediate relevance to a network topology in various areas of computing and communications [17], [20], [22], [24]. The present paper further enhances its importance by showing that DQ_m admits a 1-perfect code iff $m = 2^k - 2$. The algorithm for that purpose carefully employs a scheme by Jha and Slutzki [9] for constructing Hamming codes using a Latin square. The result leads to tight bounds on domination numbers of the dual-cube and the exchanged hypercube.

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