$L(j,\, k)$-Labelings of Kronecker Products of Complete Graphs

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***Abstract*—For positive integers $j \geq k$, an $L(j, k)$-labeling of a graph G is an integer labeling of its vertices such that adjacent vertices receive labels that differ by at least j and vertices that are distance two apart receive labels that differ by at least k. We determine $\lambda_k^j(G)$ for the case when G is a Kronecker product of finitely many complete graphs, where there are certain conditions on j and k. Areas of application include frequency allocation to radio transmitters.**

***Index Terms*—λ_k^j-labeling, complete graph, frequency allocation, graph theory, interchannel interference, Kronecker product.**

I. Introduction

THE PROBLEM OF $L(j,\, k)$-*labeling of a graph* is a variation of the classical problem of allocating frequencies to radio transmitters; see Hale [8], Roberts [15], and Griggs and Yeh [7]. Vertices of a graph represent the transmitters, and where an edge exists if the transmitters at the two end points are "very close" to each other. It is assumed that signal transmission is isotropic.

Formally, an $L(j,\, k)$-labeling of a graph is an integer assignment L to the vertices of G such that

$$|L(u) - L(v)| \geq \begin{cases} j, & \text{if } dist_G(u,v) = 1 \\ k, & \text{if } dist_G(u,v) = 2 \end{cases}$$

where j and k are fixed integers with $j \geq k \geq 1$. Elements of the image of L are called labels, and the difference between the largest label and the smallest label is called the span of L. Further, the minimum span over all $L(j,k)$-labelings of G is called the λ_k^j-*number* of G, denoted by $\lambda_k^j(G)$. The general problem of determining this graph invariant is known to be NP-hard [6]. The present study consists of determining $\lambda_k^j(G)$ for the case when G is the *Kronecker product* (defined below) of finitely many *complete graphs*. In a similar study, Georges, Mauro and Stein [5] earlier reported results with respect to the Cartesian product of complete graphs. See Adams *et al.* [1], Jha [10], [11], and Jha *et al.* [12] for other results on this topic.

When we speak of a graph, we mean a finite, simple and undirected graph having at least two vertices. The *order* of a (sub)graph refers to the number of vertices in it. For a graph G, let $dia(G)$ denote the *diameter* of G, and let *clique number* refer to the maximum order of a set of vertices that are mutually adjacent to each other in G.

The Kronecker product $G \times H$ of graphs $G = (V, E)$ and $H = (W, F)$ is defined as follows: $V(G \times H) = V \times W$ and $E(G \times H) = \{\{(a,x),(b,y)\}\colon \{a,b\} \in E \text{ and } \{x,y\} \in F\}$. This product is variously known as direct product, cardinal product and tensor product [9]. It is commutative and associative in a natural way. Further, it is distributive with respect to edge-disjoint union of graphs. It is easy to see that $|V(G \times H)| = |V| \cdot |W|$ and $|E(G \times H)| = 2 \cdot |E| \cdot |F|$.

Let K_n denote the *complete graph* on n vertices where $V(K_n) = \{0, \ldots, n-1\}$ and where any two distinct vertices are mutually adjacent. For $r \geq 2$ and $m_i \geq 3$, the graph $K_{m_0} \times \cdots \times K_{m_{r-1}}$ is regular of degree $\Pi_{i=0}^{r-1}(m_i - 1)$, and its clique number is equal to $min\{m_0, \ldots, m_{r-1}\}$ [9].

The following are certain salient characteristics of the graph $K_m \times K_n$ where m and n are greater than or equal to three: 1) This graph is distance-regular if and only if $m = n$ [2]; 2) It admits a vertex partition into largest cliques, each of order $\min\{m, n\}$ [9]; and 3) If $m-1$ or $n-1$ is even, then $K_m \times K_n$ is edge decomposable into Hamiltonian cycles, otherwise it is edge decomposable into Hamiltonian cycles and a perfect matching [3].

Note that two vertices $(u_0, \ldots, u_{r-1})$ and $(v_0, \ldots, v_{r-1})$ are adjacent in $K_{m_0} \times \cdots \times K_{m_{r-1}}$ if and only if $u_i \neq v_i$ for each i.

1) Lemma 1.1: If $r \geq 2$ and $m_0, \ldots, m_{r-1} \geq 3$, then $dia(K_{m_0} \times \cdots \times K_{m_{r-1}}) = 2$.

Proof: First consider two vertices $(u_0, \ldots, u_{r-1})$ and $(v_0, \ldots, v_{r-1})$ in $K_{m_0} \times \cdots \times K_{m_{r-1}}$, where $u_0 = v_0$ and $u_1 \neq v_1$. It is clear that these two vertices are nonadjacent, hence $dia(K_{m_0} \times \cdots \times K_{m_{r-1}}) \geq 2$. For the reverse inequality, let $(x_0, \ldots, x_{r-1})$ and $(y_0, \ldots, y_{r-1})$ be two nonadjacent vertices in $K_{m_0} \times \cdots \times K_{m_{r-1}}$. It is easy to see that for each i, there exists a vertex z_i in K_{m_i} such that $x_i \neq z_i \neq y_i$ and $x_i - z_i - y_i$ is a well-defined walk of length two between x_i and y_i. (Note that $m_i \geq 3$.) Therefore, $(x_0, \ldots, x_{r-1})$ and $(y_0, \ldots, y_{r-1})$ are at a distance of two. It follows that $dia(K_{m_0} \times \cdots \times K_{m_{r-1}}) \leq 2$. ∎

Here is a lower bound on $\lambda_k^j(G)$.

2) Lemma 1.2: (Georges and Mauro [4]) Let G be a graph with maximum degree Δ, and suppose that there is a vertex in G having Δ neighbors, each of which is of degree Δ.

1) If $j/k \geq \Delta$, then $\lambda_k^j(G) \geq j + k(2\Delta - 2)$.
2) If $j/k \leq \Delta$, then $\lambda_k^j(G) \geq 2j + k(\Delta - 2)$. ∎

It is also relevant to note that $\lambda_{ck}^{cj}(G) = c\lambda_k^j(G)$ where c is a positive integer [4].

Manuscript received April 22, 2007; revised June 19, 2007, July 16, 2007, and August 13, 2007. This paper was recommended by Associate Editor S. Tsukiyama.

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Digital Object Identifier 10.1109/TCSII.2007.908918

The results of this brief are mostly theoretical in nature, and these particular graphs are not yet known to be specifically important in applications. However, the results may be used to provide a bound on the λ_k^j-number of subgraphs of products of complete graphs.

II. Labeling of $K_{m_0} \times \cdots \times K_{m_{r-1}}$

In this section, we consider the graph $K_{m_0} \times \cdots \times K_{m_{r-1}}$, where r is greater than or equal to two and each m_i is greater than or equal to three.

We first present an inductive scheme, called *procedure indexing*, that assigns a nonnegative integer to each vertex $(i_0, \ldots, i_{r-1})$ of $K_{m_0} \times \cdots \times K_{m_{r-1}}$ in such a way that the indexes assigned to any two adjacent vertices of the graph differ by at least $1 + \Pi_{i=0}^{r-3} m_i$. We will prove that this indexing scheme satisfies this property in Lemma 2.1, and then we will use this property to obtain the λ_k^j-number for these products in Theorem 2.2.

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procedure indexing

Input: $r \geq 2$, and $m_0, \ldots, m_{r-1}$, each greater than or equal to 3

Output: Indexing of the vertices of $K_{m_0} \times \cdots \times K_{m_{r-1}}$

(1) if $(r == 2)$ then assign index $\alpha(i,j)$ to vertex (i,j) of $K_{m_0} \times K_{m_1}$ as follows:

{

(a) for $i = 0$ to $m_0 - 1$

$\alpha(i, 0) = i;$

(b) for $i = 0$ to $m_0 - 1$

$\alpha(i, 1) = (2m_0 - 1) - i;$

(c) for $j = 2$ to $m_1 - 1$

for $i = 0$ to $m_0 - 1$

$\alpha(i, j) = \alpha(i, j-2) + 2m_0;$

return;

}

// in what follows, $r \geq 3$

(2) assume that vertices of $K_{m_0} \times \cdots \times K_{m_{r-2}}$ have been indexed so that each vertex $(i_0, \ldots, i_{r-2})$ receives the index $f(i_0, \ldots, i_{r-2})$;

(3) for the graph $K_{m_0} \times \cdots \times K_{m_{r-2}} \times K_{m_{r-1}}$, let each vertex $(i_0, \ldots, i_{r-2}, i_{r-1})$ receive the index $g(i_0, \ldots, i_{r-2}, i_{r-1})$ as follows:

(a) $g(i_0, \ldots, i_{r-2}, 0) = f(i_0, \ldots, i_{r-2});$

(b) $g(i_0, \ldots, i_{r-2}, 1) = ((2\Pi_{q=0}^{r-2} m_q) - 1) - f(i_0, \ldots, i_{r-2});$

(c) for $s = 2$ to $m_{r-1} - 1$

$g(i_0, \ldots, i_{r-2}, s) = g(i_0, \ldots, i_{r-2}, s-2) + (2\Pi_{q=0}^{r-2} m_q);$

end of the procedure

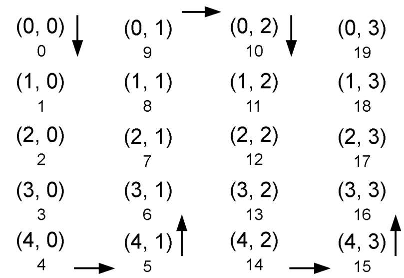

Fig. 1. Working of procedure indexing on $K_5 \times K_4$.

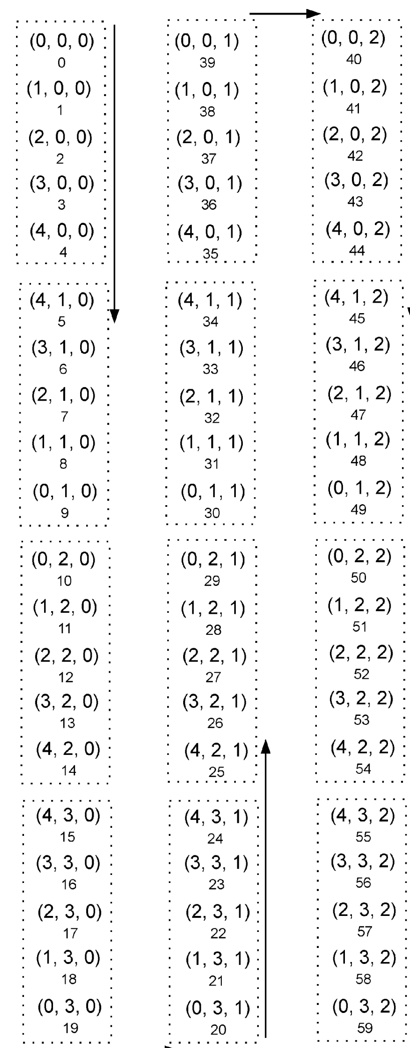

Fig. 2. Working of procedure indexing on $K_5 \times K_4 \times K_3$.

Figs. 1 and 2 illustrate the working of the foregoing procedure on graphs $K_5 \times K_4$ and $K_5 \times K_4 \times K_3$, respectively. Further, Fig. 3 outlines the working on $K_5 \times K_4 \times K_3 \times K_3$. The arrows indicate the sequential order in which indexes are assigned to various vertices.

We now work toward Lemma 2.1, which states that the indexes assigned to adjacent vertices differ by a minimum of a certain value. To that end, here are relevant observations.

- Vertices of $K_{m_0} \times K_{m_1}$ may be viewed to be organized into m_1 "columns," where elements in column c have c as its rightmost co-ordinate, $0 \leq c \leq m_1 - 1$ (see Fig. 1 where

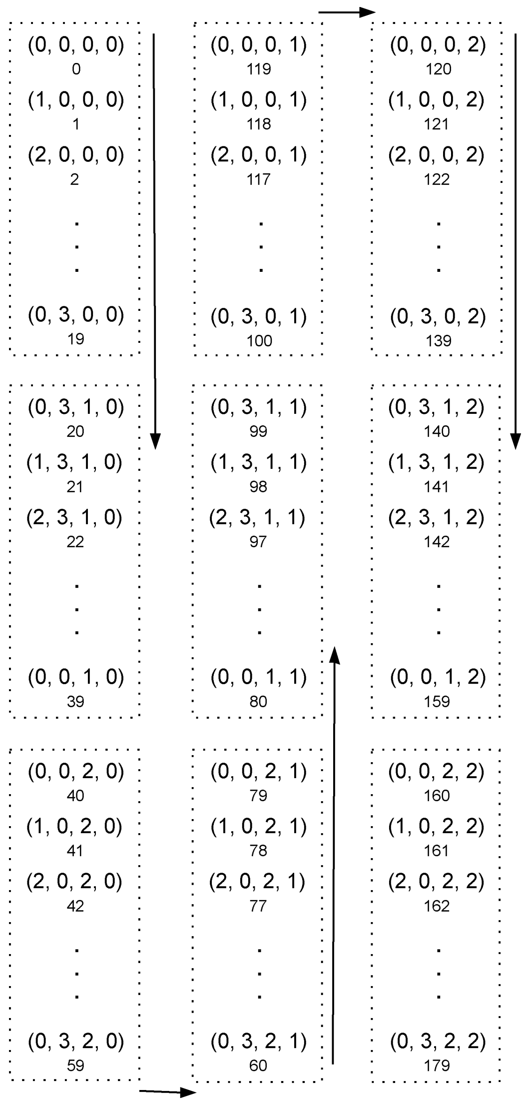

Fig. 3. Working of procedure indexing on $K_5 \times K_4 \times K_3 \times K_3$.

$m_1 = 4$.) To that end, Steps 1(a) and 1(b) of the procedure correspond to column 0 and column 1, respectively. Further, Step 1(c) corresponds to the remaining columns. Note that indexing is such that any two consecutive vertices in the sequence agree in some co-ordinate and are, therefore, nonadjacent. Equivalently, the indexes assigned to any two adjacent vertices in the graph differ by at least two.

- Vertices of $K_{m_0} \times K_{m_1} \times K_{m_2}$ may be viewed to be organized into m_2 columns, where elements in column c have c as its rightmost co-ordinate, $0 \leq c \leq m_2 - 1$. Further, each column may be viewed to be consisting of m_1 "blocks," where a block in this graph corresponds to a column in $K_{m_0} \times K_{m_1}$ (see Fig. 2 where various blocks of $K_5 \times K_4 \times K_3$ appear within dotted rectangles.)
- In general, for $r \geq 3$, vertices of $K_{m_0} \times \cdots \times K_{m_{r-1}}$ may be organized into m_{r-1} columns, where each column consists of m_{r-2} blocks, and where each block consists of $\Pi_{i=0}^{r-3} m_i$ vertices. To that end, Steps 3(a) and 3(b) of the procedure correspond to column 0 and column 1, respectively, and Step 3(c) corresponds to the remaining columns (see Fig. 3.)

For $r \geq 3$, say that a block B_2 of vertices in $K_{m_0} \times \cdots \times K_{m_{r-1}}$ succeeds another block B_1 if the smallest index assigned to a vertex in B_2 is one greater than the highest index assigned to a vertex in B_1. The reader may check to see that the following holds:

- Let B_1 and B_2 be two successive blocks in $K_{m_0} \times \cdots \times K_{m_{r-1}}$. If they belong to the same column, then elements in the two blocks agree in the rightmost co-ordinate. On the other hand, if they belong to different columns, then elements in the two blocks agree in all but the rightmost co-ordinate (see Figs. 2 and 3) It follows that vertices belonging to two successive blocks in $K_{m_0} \times \cdots \times K_{m_{r-1}}$ (where $r \geq 3$) are mutually nonadjacent.

The foregoing discussion leads to the following lemma.

1) Lemma 2.1: If $r \geq 3$, then the indexes assigned to any two adjacent vertices of $K_{m_0} \times \cdots \times K_{m_{r-1}}$ by procedure indexing differ by at least $1 + \Pi_{i=0}^{r-3} m_i$. ∎

Lemma 1.1 and Lemma 2.1 (which relies on procedure indexing) lead us to the following theorem, our central result.

2) Theorem 2.2: If $r \geq 3$, $m_0, \ldots, m_{r-1} \geq 3$ and $1 \leq j/k \leq (\Pi_{i=0}^{r-3} m_i) + 1$, then $\lambda_k^j(K_{m_0} \times \cdots \times K_{m_{r-1}}) = (N-1)k$ where $N = \Pi_{i=0}^{r-1} m_i$.

Proof: By Lemma 1.1, $K_{m_0} \times \cdots \times K_{m_{r-1}}$ is a graph of diameter two, hence labels assigned to any two distinct vertices of this graph in an $L(j,k)$-labeling must differ by at least k. It follows that the span of such a labeling must be at least $(N-1)k$ where N denotes the number of vertices in $K_{m_0} \times \cdots \times K_{m_{r-1}}$. In other words, $\lambda_k^j(K_{m_0} \times \cdots \times K_{m_{r-1}}) \geq (N-1)k$ where $j \geq k \geq 1$. In what follows, we prove the reverse inequality for the case when $j \leq ((\Pi_{i=0}^{r-3} m_i) + 1)k$

Perform procedure indexing on $K_{m_0} \times \cdots \times K_{m_{r-1}}$, and if a vertex of this graph receives the index l, then assign the label $l \cdot k$ to it, where $0 \leq l \leq N-1$. It is clear that the labels assigned to any two distinct vertices differ by at least k. Further, by Lemma 2.1, the labels assigned to any two adjacent vertices differ by at least $((\Pi_{i=0}^{r-3} m_i) + 1)k$. ∎

Note: We may assume without loss of generality that $m_0 \geq m_1 \geq \cdots \geq m_{r-1}$ in the statement of Theorem 2.2. In the process, there may be a slight improvement in the span of the labels.

Observe that the number of factor graphs in the statement of Theorem 2.2 is at least three. The following result takes care of the remaining case when the number of factor graphs is two. The proof of Theorem 2.3 is similar to the proof of Theorem 2.2.

3) Theorem 2.3: If m_0, $m_1 \geq 3$ and $1 \leq j/k \leq 2$, then $\lambda_k^j(K_{m_0} \times K_{m_1}) = (m_0 m_1 - 1)k$. ∎

III. Labeling of $K_n \times K_2$

In this section, we consider $K_n \times K_2$, where $n \geq 3$. This graph is different from the product considered earlier. In particular, graphs in the previous section are nonbipartite and of diameter two whereas $K_n \times K_2$ is bipartite and of diameter three. Here are certain other distinguishing characteristics of $K_n \times K_2$.

- It is a distance-regular graph [2].
- It is an antipodal graph. In particular, for each vertex (i, j), there exists a unique vertex, viz., $(i, 1-j)$ whose distance from (i, j) is equal to the diameter of the graph.

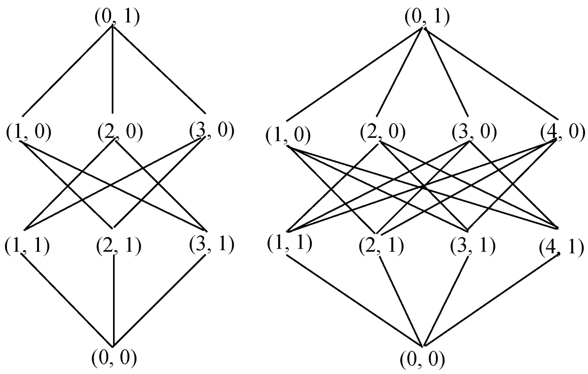

Fig. 4. Graphs $K_4 \times K_2$ and $K_5 \times K_2$.

- It is isomorphic to $K_{n,n}$ minus a perfect matching. Indeed, if n is odd, then $K_n \times K_2$ is edge decomposable into $(n-1)/2$ Hamiltonian cycles, and if n is even, then $K_n \times K_2$ is edge decomposable into $(n-2)/2$ Hamiltonian cycles and a perfect matching.
- For a vertex (i, j) of this graph, let $N[i, j]$ denote the closed neighborhood of (i, j), where $0 \le i \le n-1$ and $0 \le j \le 1$. Then $N[i, 0] \bigcup N[i, 1] = V(K_n \times K_2)$ and $N[i, 0] \bigcap N[i, 1] = \emptyset$.

Note that $K_3 \times K_2$ is isomorphic to C_6 while $K_4 \times K_2$ is isomorphic to Q_3, the hypercube of dimension three. Graphs $K_4 \times K_2$ and $K_5 \times K_2$ appear in Fig. 4.

1) Theorem 3.1: If $n \ge 3$ and $j \ge k \ge 1$, then $\lambda_k^j(K_n \times K_2) \le j + (2n-3)k$.

Proof: For $0 \le i \le n-1$, let $f(i, 0) = ik$ and $f(i, 1) = j + (2n-3-i)k$. It is clear that f is a well-defined mapping from $V(K_n \times K_2)$ to $\{0, \ldots, j + (2n-3)k\}$. We claim that it is a valid $L(j,\, k)$-labeling.

Let (u, v) and (x, y) be two distinct vertices of $K_n \times K_2$, where $0 \le u, x \le n-1$ and $0 \le v, y \le 1$. If $v = y$, then the two vertices are distance two apart and (since labels are not repeated) they receive different labels. Without loss of generality, let $v = 0$ and $y = 1$. If $u = x$, then the two vertices are distance three apart, so suppose that $u \ne x$. Now,

$$\begin{aligned} |f(x,1) - f(u,0)| &= |j + (2n-3-x)k - uk| \\ &= |j + (2n-3)k - (u+x)k|. \end{aligned}$$

Since $u \ne x$, we have $0 \le u + x \le 2n-3$. Accordingly, $|f(x,1) - f(u,0)| \ge j$. ∎

It turns out that if $j = 2$ and $k = 1$, then we can do better than Theorem 3.1.

2) Theorem 3.2: If $n \ge 3$, then $\lambda_1^2(K_n \times K_2) = 2(n-1)$.

Proof: Let each of the vertices $(i, 0)$ and $(i, 1)$ of $K_n \times K_2$ receive the label $2i$, where $0 \le i \le n-1$. This is a valid $L(2,1)$-labeling of the graph for the following reasons: 1) for each i, vertices $(i, 0)$ and $(i, 1)$ are distance three apart, so they may as well receive the same label, and 2) any two distinct vertices that are within a distance of two from each other receive labels that differ by at least two. Accordingly, $\lambda_1^2(K_n \times K_2) \le 2(n-1)$.

For the reverse inequality, first note that no label can be used more than twice. Now suppose that a label, say l, is used twice. Then the two vertices with this label must be of the form $(i, 0)$ and $(i, 1)$, and since $N[i, 0]$ and $N[i, 1]$ span the entire vertex set of $K_n \times K_2$, the labels $l-1$ and $l+1$ cannot be used. At this point, it is easy to see from the pigeonhole principle that if only labels 0 through $2n-3$ were used, then some pair of consecutive labels must be used to label at least three vertices, a contradiction. Therefore, $\lambda_1^2(K_n \times K_2) \ge 2(n-1)$. ∎

The latter half of the proof of Theorem 3.2 is a generalization of Jonas's proof [13] (p. 57) of the fact that $\lambda_1^2(Q_3) \ge 6$. Observe that the value of $\lambda_1^2(K_n \times K_2)$ given by Theorem 3.2 is strictly greater than the lower bound suggested by Lemma 1.2.

ACKNOWLEDGMENT

The authors thank the referees for their suggestions that led to an improvement in the presentation of the brief.

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