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Edge Exchanges in Hamiltonian Decompositions of Kronecker-Product Graphs

P. K. JHA, N. AGNIHOTRI* AND R. KUMAR†
Department of Computer Engineering
Delhi Institute of Technology: Delhi
Kashmere Gate, Delhi 110 006, India
pkj@dit.ernet.in

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Abstract—Let G be a connected graph on n vertices, and let α, β, γ and δ be edge-disjoint cycles in G such that

(i) α, β (respectively, γ, δ) are vertex-disjoint and

(ii) $|\alpha| + |\beta| = |\gamma| + |\delta| = n$,

where $|\alpha|$ denotes the length of α. We say that α, β, γ and δ yield two edge-disjoint Hamiltonian cycles by edge exchanges if the four cycles respectively contain edges e, f, g and h such that each of $(\alpha - \{e\}) \bigcup (\beta - \{f\}) \bigcup \{g, h\}$ and $(\gamma - \{g\}) \bigcup (\delta - \{h\}) \bigcup \{e, f\}$ constitutes a Hamiltonian cycle in G. We show that if G is a nonbipartite, Hamiltonian decomposable graph on an even number of vertices which satisfies certain conditions, then Kronecker product of G and K_2 as well as Kronecker product of G and an even cycle admits a Hamiltonian decomposition by means of appropriate edge exchanges among smaller cycles in the product graph.

Keywords—Kronecker product, Hamiltonian decomposition, Edge exchange, Alternate four-cycle.

1. INTRODUCTION

Whether a product of the *Hamiltonian decomposable graphs* (henceforth called *H-decomposable graphs*) is itself H-decomposable has been an object of study for a long time. For example, Barayani and Szasz [1] showed that this problem has an affirmative answer with respect to the lexicographic product. For certain other graph products, similar (not so exact) results were reported by Alspach, Bermond and Sotteau [2], Bosak [3] and Zhou [4]. Among other things, graph products offer an intuitive and systematic means of constructing H-decomposable graphs from smaller such graphs.

H-decomposable graphs possess a highly regular structure, and are amenable to several applications. We would like to mention here applications in the areas of fault-tolerant networks [5] and block designs [6].

In this paper, we continue the study of constructing H-decomposable graphs by means of *Kronecker product* (or $\times$-product) of two or more H-decomposable graphs. For graphs $G = (V, E)$ and $H = (W, F)$, the Kronecker product of G and H is denoted by $G \times H$, where $V(G \times H) = V \times W$ and $E(G \times H) = \{\{(u, x), (v, y)\} \mid \{u, v\} \in E$ and $\{x, y\} \in F\}$. It is easy to see that

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*Present address: Department of Biology & Agricultural Engineering, University of Georgia, Athens, GA 30605, U.S.A.
†Present address: C-DoT, 39 Main Pusa Road, New Delhi 110 005, India.

$|V(G \times H)| = |V| \cdot |W|$ and $|E(G \times H)| = 2 \cdot |E| \cdot |F|$. This product (which is commutative and associative up to isomorphism) is variously known as *direct product* [3], *categorical product* [7], *tensor product* [8] and *graph conjunction* [9]. It is considered to be one of the most important of all graph products. Several applications and characteristics appear in [7,8,10–14].

For H-decomposition of a Kronecker product of H-decomposable graphs, the following result was obtained by one of the authors [15]:

(i) If the number of factor graphs of even order is at most one, then the Kronecker product is H-decomposable; and
(ii) if the number of factor graphs which are bipartite is at least two and the remaining factor graphs (if any) are all of odd order, then the Kronecker product consists of isomorphic components each of which is H-decomposable.

(Here all factor graphs are themselves H-decomposable.) The result for a Kronecker product of two cycles one of which is of odd order was earlier established by Bosak [3] and Zhou [4].

The central problem that we have addressed is whether a Kronecker product of a nonbipartite graph of even order and an even cycle is H-decomposable. Since this product is distributive with respect to edge-disjoint union of graphs, it is easy to see that a clear solution to the foregoing problem coupled with the above-stated result would lead to a complete characterization for H-decomposition of a Kronecker product of finitely many H-decomposable graphs. While an absolute answer is still elusive, we report a partially affirmative answer. An interesting aspect of our scheme is that *Hamiltonian cycles* (henceforth called *H-cycles*) are obtainable by means of appropriate *edge exchanges* among smaller cycles in the product graph.

Certain aspects of our construction are too technical to warrant a description in purely intuitive terms, and hence we spell out the method of attack in general terms. Let G be a nonbipartite graph of even order, say m, such that G is decomposable into two H-cycles, and let n be an even integer ≥ 4. Since the Kronecker product of two even cycles consists of two isomorphic components each of which is decomposable into two H-cycles, it follows that the graph $G \times C_n$ (which is connected) is decomposable into eight cycles each of length $mn/2$. Now, if these eight cycles are "interwoven" in a certain fashion, then it may be possible to do some kind of "dovetailing" by means of appropriate edge exchanges among them to yield four edge-disjoint H-cycles in the graph $G \times C_n$. We present a sufficient condition in terms of the structure of the graph G which facilitates this construction. We further show that for every even $m \geq 6$, there exist graphs on m vertices which are easily constructible and which satisfy that condition. We also present a result on H-decomposition of a Kronecker product of a graph having that characteristic and the graph K_2.

The remainder of this paper is organized as follows. Basic definitions and preliminary results appear in Section 2 while main results appear in Section 3. In Section 4, we show that many graphs G which are responsive to edge exchanges in each of $G \times K_2$ and $G \times C_n$ are easily constructible from the complete graph whose order is same as that of G. Finally in Section 5, we offer concluding remarks.

2. PRELIMINARIES

By a graph we mean a finite, simple and undirected graph having at least two vertices. Graphs are also connected, unless indicated otherwise.

By decomposition of a graph G, we ordinarily mean an edge-decomposition of G into certain subgraphs. A graph is said to admit a cycle decomposition (respectively, H-decomposition) if and only if its edge set may be partitioned into cycles (respectively, Hamiltonian cycles or H-cycles). For example, a complete graph on an odd number of vertices is H-decomposable. The general problem of determining whether or not a graph contains an H-cycle is NP-complete, and so is the problem of determining whether or not a graph G is decomposable into subgraphs isomorphic to a given graph H [16]. For any undefined terms, see [17].

For $m \geq 3$, let C_m denote the cycle on m vertices, where $V(C_m) = \{0, \ldots, m-1\}$ and where adjacencies are defined in the natural way. The following theorem states certain relevant characteristics of Kronecker-product graphs.

THEOREM 2.1. *Let G and H be graphs.*

1. *If G and H are both bipartite, then $G \times H$ consists of two components, otherwise $G \times H$ is connected [14].*
2. *$G \times H$ is bipartite if and only if G or H is bipartite [18].*
3. *Kronecker product of graphs is distributive with respect to edge-disjoint union of graphs.* ∎

The next result refines part (1) of Theorem 2.1.

LEMMA 2.2. *If G and H are bipartite graphs, then vertices (u,x) and (v,y) of the graph $G \times H$ belong to the same component if and only if $d_G(u,v)$ and $d_H(x,y)$ are of the same parity.* ∎

Indeed, if $G = (V_0 \bigcup V_1, E)$ and $H = (W_0 \bigcup W_1, F)$ are bipartite graphs, then $(V_0 \times W_0) \bigcup (V_1 \times W_1)$ and $(V_0 \times W_1) \bigcup (V_1 \times W_0)$, respectively, correspond to vertex sets of the two components of the graph $G \times H$ [15]. By Theorem 2.1(3), it is interesting to note that if G and H are bipartite graphs which, respectively, appear as subgraphs of (not necessarily bipartite) graphs G' and H', then the two components of $G \times H$ appear as vertex-disjoint subgraphs in $G' \times H'$. We will effectively make use of this observation in subsequent discussions.

It is easy to see that if G is an H-decomposable graph on an odd number of vertices, then $G \times K_2$ is H-decomposable and that if G is an H-decomposable bipartite graph (in which case G must have an even number of vertices), then $G \times K_2$ consists of two (vertex-disjoint) copies of G. On the other hand, if G is an H-decomposable, nonbipartite graph on an even number of vertices, then it is not immediately clear whether $G \times K_2$ (which is connected) is H-decomposable. We obtain a partially affirmative answer to this question in the next section.

The following result deals with H-decomposition of a $\times$-product of finitely many H-decomposable graphs.

THEOREM 2.3. *[15] Let $G_1, \ldots, G_r$ be H-decomposable graphs and let k be the number of even integers among $|V(G_1)|, \ldots, |V(G_r)|$.*

1. *If $k \leq 1$, then the graph $G_1 \times \cdots \times G_r$ is H-decomposable.*
2. *If $k \geq 2$ and the corresponding graphs are bipartite, then $G_1 \times \cdots \times G_r$ (is disconnected and) consists of isomorphic components, each of which is H-decomposable.* ∎

Here again, if G is a nonbipartite, H-decomposable graph on an even number of vertices and n is even, then it is not immediately clear whether $G \times C_n$ is H-decomposable. We obtain a partially affirmative answer to the foregoing problem in the next section.

We now present the definition of an H-decomposable graph containing an *alternate four-cycle*.

DEFINITION 1. *Let G be a graph such that*

- *$|V(G)|$ is even ≥ 6;*
- *G is decomposable into two H-cycles;*
- *G contains a four-cycle $a-b-c-d-a$ (say) where edges $\{a,b\}$, $\{c,d\}$ belong to one of the two H-cycles while edges $\{b,c\}$, $\{d,a\}$ belong to the other H-cycle; and*
- *vertices a and c (or b and d) are at an odd distance along each of the two (even) H-cycles.*

The cycle $a-b-c-d-a$ is said to be an alternate four-cycle in G. ∎

It is straightforward to check that a graph which satisfies Definition 1 is nonbipartite. The converse is not true. We next show that for every even $n \geq 6$, there exists a graph on n vertices which conforms to Definition 1. Let $n = 2k$, $k \geq 3$. We specify a graph A_n as follows:

$V(A_n) = \{0, 1, \ldots, n-1\}$, and $E(A_n) = E_1 \bigcup E_2$, where E_1 consists of the edges $0 - 1 - 2 - \cdots - (n-1) - 0$, while E_2 consists of the following edges: $0 - (k-1) - (k+1) - (k-3) - (k+3) - \cdots - 1 - (2k-1) - 2 - (2k-2) - 4 - (2k-4) - \cdots - k - 0$ if k is even, and $0-(k-1)-(k+1)-(k-3)-(k+3)-\cdots-2-(2k-2)-1-(2k-1)-3-(2k-3)-\cdots-k-0$ if k is odd. Note that $E_1 \bigcap E_2 = \emptyset$, and each of E_1 and E_2 constitutes an H-cycle in A_n. Further, the cycle $1 - 2 - (2k-2) - (2k-1) - 1$ is an alternate four-cycle in A_n for the following reasons:

(i) $\{1, 2\}, \{2k-2, 2k-1\} \in E_1$ and $\{1, 2k-1\}$, $\{2, 2k-2\} \in E_2$; and
(ii) the distance between 1 and $2k-2$ along E_1 is three, and the distance between 1 and $2k-2$ along E_2 is either one or three.

Graphs A_8 and A_{10} appear in Figure 1.

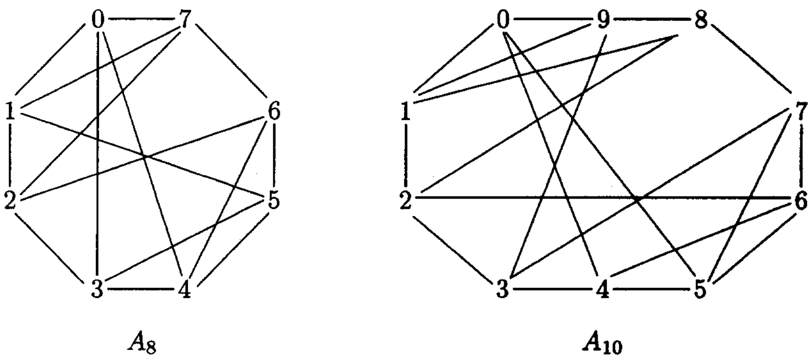

Figure 1. Graphs A_8 and A_{10}.

It will follow from discussions in Section 4 that for every even $n \geq 6$, the graph K_n admits a decomposition into certain spanning subgraphs, most of which are isomorphic to A_n.

DEFINITION 2. *Let $\mathcal{ALT}$ denote a class of graphs, each member G of which is such that $|V(G)|$ is even ≥ 6 and G is decomposable into spanning subgraphs $G_1, \ldots, G_r$, all of which satisfy Definition 1.* ∎

Thus $G \in \mathcal{ALT}$ if and only if

(i) $|V(G)|$ is even ≥ 6;
(ii) G is decomposable into an even number, say r, of H-cycles; and
(iii) there is a partition of these r H-cycles into $r/2$ pairs such that there is an alternate four-cycle between each such pair.

Note that A_n is in $\mathcal{ALT}$. We will show in Section 4 that $\mathcal{ALT}$ contains several other easily constructible graphs.

We next define edge exchanges in graphs. Let G be a connected graph on n vertices, and let α, β, γ and δ be edge-disjoint cycles in G such that

(i) α, β (respectively, γ, δ) are vertex-disjoint, and
(ii) $|\alpha| + |\beta| = |\gamma| + |\delta| = n$.

We say that α, β, γ and δ yield two edge-disjoint H-cycles by edge exchanges if the four cycles respectively contain edges e, f, g and h such that each of $(\alpha - \{e\}) \bigcup (\beta - \{f\}) \bigcup \{g, h\}$ and $(\gamma - \{g\}) \bigcup (\delta - \{h\}) \bigcup \{e, f\}$ constitutes an H-cycle in G.

3. MAIN RESULTS

We first present a theorem which deals with H-decomposition of $G \times K_2$, where G is a graph as in Definition 1.

THEOREM 3.1. *If G is a graph on an even number of vertices such that G is decomposable into two H-cycles and G contains an alternate four-cycle, then $G \times K_2$ admits an H-decomposition.*

PROOF. Let G be as stated, and let H_1 and H_2 be two edge-disjoint H-cycles in G. Let $a-b-c-d-a$ be an alternate four-cycle in (the nonbipartite graph) G, where edges $\{a,b\},\{c,d\}$ belong to H_1 and edges $\{b,c\},\{d,a\}$ belong to H_2. Further suppose that $|V(G)| = m$ (even), and that 0 and 1 are the two (adjacent) vertices of K_2.

By Theorem 2.1(3), the graph $G \times K_2$ admits a decomposition into $H_1 \times K_2$ and $H_2 \times K_2$. Further, $H_1 \times K_2$ (respectively, $H_2 \times K_2$) is decomposable into cycles α and β (respectively, γ and δ), each of length m. Thus, α, β, γ and δ constitute a cycle decomposition of $G \times K_2$.

By Lemma 2.2, we may assume that edges $e = \{(a,0),(b,1)\}$ and $f = \{(c,0),(d,1)\}$, respectively, belong to α and β while edges $g = \{(a,0),(d,1)\}$ and $h = \{(c,0),(b,1)\}$, respectively, belong to γ and δ. Note that each of $\{\alpha,\beta\}$ and $\{\gamma,\delta\}$ constitute a vertex decomposition of $G \times K_2$.

Finally observe that the following sets of edges constitute an H-decomposition of $G \times K_2$: $(\alpha - \{e\}) \bigcup (\beta - \{f\}) \bigcup \{g,h\}$ and $(\gamma - \{g\}) \bigcup (\delta - \{h\}) \bigcup \{e,f\}$. ∎

Note that the four edges which take part in the "exchange" process in the proof of Theorem 3.1 are $\{(a,0),(b,1)\}$, $\{(b,1),(c,0)\}$, $\{(c,0),(d,1)\}$ and $\{(d,1),(a,0)\}$, which constitute a four-cycle in $G \times K_2$. Alternatively, we could employ the edges $\{(a,1),(b,0)\}$, $\{(b,0),(c,1)\}$, $\{(c,1),(d,0)\}$ and $\{(d,0),(a,1)\}$ for that purpose.

Let G be a graph on an even number of vertices such that G admits an H-decomposition into two H-cycles. We claim that H-decomposition of $G \times K_2$ by means of edge exchanges (as in the proof of Theorem 2.1) is possible if and only if G contains an alternate four-cycle. Whereas the "if" part of the claim is implicit in the proof of that theorem, the "only if" follows easily.

By Theorem 2.1(3), there is a natural generalization of Theorem 3.1 as follows: if $G \in \mathcal{ALT}$, then $G \times K_2$ admits an H-decomposition. (Recall Definitions 1 and 2).

We now proceed to obtain a result (analogous to that of Theorem 3.1) for H-decomposition of $G \times C_n$ where G is as in Definition 1 and n is even. The following lemma is a special case of Theorem 2.3. We state and prove it because our subsequent arguments heavily rely on it.

LEMMA 3.2. *If m and n are even integers, then the graph $C_m \times C_n$ consists of two isomorphic components, each of which is H-decomposable.*

PROOF. [15] Let m, n be even ≥ 4. There is a natural bipartition of $V(C_m)$ into the following sets: $V_0 = \{0,2,\ldots,m-2\}$ and $V_1 = \{1,3,\ldots,m-1\}$. Let W_0 and W_1 correspond to analogous bipartition of $V(C_n)$. Now consider the component of $C_m \times C_n$ on the vertex subset $(V_0 \times W_0) \bigcup (V_1 \times W_1)$. The following sequences $w_0,\ldots,w_{(mn/2)-1}$ and $x_0,\ldots,x_{(mn/2)-1}$ of vertices correspond to an H-decomposition of that component: $w_{mi+j} = (j,a), x_{mi+j} = (j,b)$ where $0 \leq i \leq (n/2)-1, 0 \leq j \leq m-1$, $a = 2 \cdot i + (j \bmod 2)$ and $b = (-a) \bmod n$. The other component of $C_m \times C_n$ is on vertex subset $(V_0 \times W_1) \bigcup (V_1 \times W_0)$. The following sequences $y_0,\ldots,y_{(mn/2)-1}$ and $z_0,\ldots,z_{(mn/2)-1}$ of vertices correspond to an H-decomposition of that component: $y_{mi+j} = (j,c)$, $z_{mi+j} = (j,d)$ where i and j are as above, and $c = ((n - 2 \cdot i) + (j+1) \bmod 2) \bmod n$ and $d = (2-c) \bmod n$.

That the two components of $C_m \times C_n$ are isomorphic follows from a simple observation. ∎

Certain remarks on the construction in the proof of Lemma 3.2 are in order. Let m and n be even. Note that there are exactly m edges of the form $\{(a,0),(b,1)\}$ in each component of $C_m \times C_n$. Out of these m edges in the first component, exactly $m-1$ appear in the H-cycle corresponding to the sequence $w_0,\ldots,w_{(mn/2)-1}$, while the remaining edge appears in the H-cycle corresponding to $x_0,\ldots,x_{(mn/2)-1}$. An analogous statement holds for the two H-cycles in the second component. Observe also that edge $\{(a,0),(b,1)\}$ appears in one component of $C_m \times C_n$ if and only if edge $\{(a,1),(b,0)\}$ appears in the other component. Based on this statement and general symmetry existing in the twin components of $C_m \times C_n$, we have the following corollary.

COROLLARY 3.3. *Let m, n be even ≥ 4, and let $\{i, j\}$ be an arbitrary but fixed edge of C_n. For an edge $\{(a, i), (b, j)\}$ in a particular component of $C_m \times C_n$, there exists a decomposition of that component into two H-cycles α and β, such that α (respectively, β) includes (respectively, excludes) $\{(a, i), (b, j)\}$ and excludes (respectively, includes) the remaining $m - 1$ edges of that type.* ∎

We are now ready to state and prove our central result.

THEOREM 3.4. *Let G be a graph on an even number of vertices such that G is decomposable into two H-cycles and G contains an alternate four-cycle, and let n be even. The graph $G \times C_n$ admits an H-decomposition.*

PROOF. Let G and n be as stated, where $|V(G)| = m$ (even). Let G be decomposable into H-cycles H_1 and H_2, and let $p - q - r - s - p$ be an alternate four-cycle between H_1 and H_2 where $\{p, q\}, \{r, s\} \in V(H_1)$ and $\{q, r\}, \{s, p\} \in V(H_2)$. The (sub)graph $H_1 \times C_n$ (respectively, $H_2 \times C_n$) consists of two components, say, X_1, X_2 (respectively, Y_1, Y_2), where each of X_1, X_2, Y_1 and Y_2 is isomorphic to a component of $C_m \times C_n$.

The following eight edges correspond to $\times$-product of the cycle $p - q - r - s - p$ of G and the edge $0 - 1$ of C_n, and will play an important role in our construction: $a = \{(p, 0), (q, 1)\}$, $b = \{(r, 1), (s, 0)\}$, $c = \{(p, 1), (q, 0)\}$, $d = \{(r, 0), (s, 1)\}$, $e = \{(q, 1), (r, 0)\}$, $f = \{(p, 1), (s, 0)\}$, $g = \{(q, 0), (r, 1)\}$, and $h = \{(p, 0), (s, 1)\}$. Since $\{p, q\}, \{r, s\} \in H_1$, it follows that $a, b, c, d \in E(H_1 \times C_n)$. Similarly $e, f, g, h \in E(H_2 \times C_n)$.

By Lemma 2.2, we may assume that

(i) $a, b \in E(X_1)$,
(ii) $c, d \in E(X_2)$,
(iii) $e, f \in E(Y_1)$,
(iv) $g, h \in E(Y_2)$.

Note that X_1, X_2 (respectively, Y_1, Y_2) constitute a vertex decomposition of $G \times C_n$.

By results 3.2 and 3.3, we may assume that

(i) X_1 (respectively, X_2) is decomposable into H-cycles α_a and α_b (respectively, α_c and α_d), where α_a, α_b, α_c and α_d, respectively, contain the edges a, b, c and d, and
(ii) Y_1 (respectively, Y_2) is decomposable into H-cycles α_e and α_f (respectively, α_g and α_h), where α_e, α_f, α_g and α_h, respectively, contain the edges e, f, g and h.

Note that $\alpha_a, \ldots, \alpha_h$ (each of which is of length $mn/2$) constitute a cycle decomposition of $G \times C_n$. The successive edge decompositions of $G \times C_n$ mentioned above are shown in Figure 2.

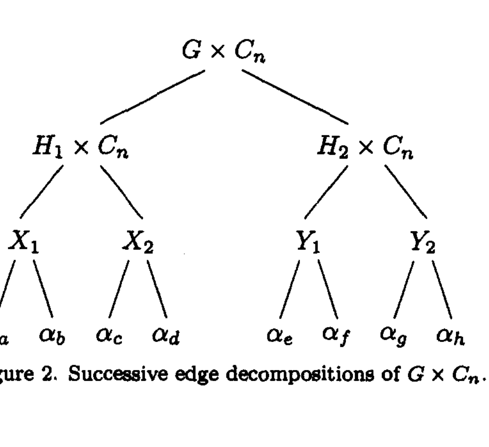

Figure 2. Successive edge decompositions of $G \times C_n$.

The following sets of edges form an H-decomposition of $G \times C_n$:

(1) $(\alpha_a - \{a\}) \bigcup (\alpha_d - \{d\}) \bigcup \{e, h\}$,

(2) $(\alpha_e - \{e\}) \bigcup (\alpha_h - \{h\}) \bigcup \{a, d\}$,

(3) $(\alpha_b - \{b\}) \bigcup (\alpha_c - \{c\}) \bigcup \{f, g\}$,

(4) $(\alpha_f - \{f\}) \bigcup (\alpha_g - \{g\}) \bigcup \{b, c\}$. ∎

By results 2.1(3) and 3.1, if $G \in \mathcal{ALT}$ and H is an H-decomposable graph on an even number of vertices, then $G \times H$ is H-decomposable. Indeed, we may now strengthen the result of Theorem 2.3 as follows.

THEOREM 3.5. *Let $G_1, \dots, G_r$ be H-decomposable graphs and let k be the number of even integers among $|V(G_1)|, \dots, |V(G_r)|$.*

1. *If $k \leq 1$, then the graph $G_1 \times \cdots \times G_r$ is H-decomposable.*
2. *If $k \geq 2$ and the corresponding graphs are bipartite, then $G_1 \times \cdots \times G_r$ (is disconnected and) consists of isomorphic components, each of which is H-decomposable.*
3. *If $k \geq 2$ and at least $k-1$ of the corresponding graphs are in $\mathcal{ALT}$, then $G_1 \times \cdots \times G_r$ is H-decomposable.* ∎

4. MEMBERSHIP PROPERTIES

In this section, we show that the class $\mathcal{ALT}$ contains certain easily constructible and familiar graphs. The following result is attributed to Walecki. (See [2].)

LEMMA 4.1. *If m is even ≥ 4, then the graph K_n is decomposable into $(n-2)/2$ H-cycles and a perfect matching.*

PROOF. [2] For $n = 4$, the result is clear. For $n = 2k \geq 6$, let C be the cycle $0-1-2-(2k-1)-3-(2k-2)-\cdots-(k-1)-(k+2)-k-(k+1)-0$ and let σ be the permutation $(0)(1\ \ 2\ \ 3\ \cdots\ 2k-2\ \ 2k-1)$. Then, $C, \sigma \circ C, \dots, \sigma^{k-2} \circ C$ are $k-1$ edge-disjoint H-cycles of K_n. The remaining edges $\{0, k\}, \{k-1, k+1\}, \{k-2, k+2\}, \dots, \{1, 2k-1\}$ form a perfect matching. ∎

THEOREM 4.2. *Let n be even ≥ 6.*

1. *If $n = 4i+2$, then K_n admits a decomposition into a perfect matching and i graphs, each isomorphic to A_n.*
2. *If $n = 4i$, then K_n admits a decomposition into a perfect matching, an H-cycle and $i-1$ graphs, each isomorphic to A_n.*

NOTE. Definition of the graph A_n appears in Section 1.

PROOF OF THEOREM 4.2. Let n be as stated, and let $C, \sigma \circ C, \dots, \sigma^{(n/2)-2} \circ C$ be the edge-disjoint H-cycles of K_n as outlined in the proof of Lemma 4.1.

For $n = 4i+2$, we partition the foregoing H-cycles into the following pairs: $\{C, \sigma^i \circ C\}, \{\sigma \circ C, \sigma^{i+1} \circ C\}, \dots, \{\sigma^{i-1} \circ C, \sigma^{2i-1} \circ C\}$. Let $B_0, \dots, B_{i-1}$ be the spanning subgraphs of K_n where B_j has all the edges of the j^{th} pair, i.e., $E(B_j) = E(\sigma^j \circ C) \bigcup E(\sigma^{i+j} \circ C)$, $0 \leq j \leq i-1$. (By σ^0 we mean identity permutation.) Because of symmetry, it is easy to see that graphs $B_0, \dots, B_{i-1}$ are mutually isomorphic. Further, each is isomorphic to A_n. (Verification is left to the reader.)

For $n = 4i$, we consider the following pairs of H-cycles of K: $\{C, \sigma^i \circ C\}, \{\sigma \circ C, \sigma^{i+1} \circ C\}, \dots, \{\sigma^{i-2} \circ C, \sigma^{2i-2} \circ C\}$. Note that $\sigma^{i-1} \circ C$ does not appear in any of these pairs and that each of the remaining H-cycles appears in exactly one pair. The spanning subgraphs of K_n corresponding to these $i-1$ pairs are each isomorphic to A_n. ∎

Let n be even ≥ 6. By Theorem 4.2, the graph K_n contains $\lfloor (n-2)/4 \rfloor$ edge-disjoint (spanning) subgraphs isomorphic to A_n. Clearly $A_n \in \mathcal{ACT}$. Note further that any subgraph of K_n which is obtainable by union of edge sets of two or more of those spanning subgraphs belongs to $\mathcal{ACT}$. In particular, if $n = 4i + 2$, then "K_n minus a perfect matching" is in $\mathcal{ACT}$.

COROLLARY 4.3.

1. *If m and n are both even ≥ 4, and at least one is of the form $4i+2$, then $K_m \times K_n$ contains at least $(m-2)(n-2)/2$ edge-disjoint H-cycles.*
2. *If m and n are both multiples of four, $m \geq 8$ and $m \geq n$, then $K_m \times K_n$ contains at least $(m-4)(n-2)/2$ edge-disjoint H-cycles.* ∎

It is easy to see that

(a) if m and n are both odd, then $K_m \times K_n$ is H-decomposable, and

(b) if m is odd and n is even ≥ 4, then $K_m \times K_n$ contains at least $(m-1)(n-2)/2$ edge-disjoint H-cycles.

5. CONCLUDING REMARKS

Let $G_1, \ldots, G_r$ be H-decomposable graphs. We have addressed the problem of obtaining conditions for H-decomposition of the Kronecker product (or $\times$-product) of $G_1, \ldots, G_r$. While a complete characterization is still elusive, we have obtained a partial characterization which appears in Theorem 3.5.

We have a similar result for Kronecker product of an H-decomposable graph and K_2. Our schemes rely on constructing Hamiltonian cycles by means of suitable edge exchanges among smaller cycles in the product graph. In the process, we have defined a class $\mathcal{ACT}$ of graphs which has interesting membership properties.

The results yield impressive lower bound on the number of edge-disjoint H-cycles in $K_m \times K_n$. Sprague [19] has developed a concept of edge exchanges, which is different from ours, and which deals with Hamiltonian cycles in interval graphs.

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