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ON PLANARITY OF DIRECT PRODUCT OF MULTIPARTITE COMPLETE GRAPHS

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The planarity of the direct product of two graphs has been widely studied in the past. Surprisingly, the missing part is the product with K_2, which seems to be less predictible. In this piece of work, we characterize which subdivisions of multipartite complete graphs, have their direct product with K_2 planar. This can be seen as a step towards the characterization of all such graphs.

Keywords: Direct product; planarity; complete graph; multipartite complete graph; subdivision.

Mathematics Subject Classification 2000: 05C10, 05C83

1. Introduction

Whether a product of two graphs is *planar* or *non-planar* is a naturally-occurring question, having practical relevance. Accordingly, the topic received attention for a long time. For example, Behzad and Mahmoodian [1] presented a complete characterization for the planarity of the Cartesian product, while Jha and Slutzki [5] stated an analogous result with respect to the strong product. The corresponding problem with respect to the *direct product* (or $\times$-product, which we formally define below) seems to be rather challenging. To that end, Farzan and Waller [3] reported a partial characterization. One case that has not been fully examined in their work is that of the $\times$-product of an arbitrary graph G and K_2. In this paper, we study the case when G is a multipartite complete graph.

A closely-related topic in this kind of study is that of determining whether or not a graph G is a *minor* (defined below) of a product of itself with K_2. The problem is trivial with respect to the Cartesian product and strong product, since G is necessarily a subgraph of each such product. Along these lines, Jha and Slutzki [5] conjectured that every graph G is a minor of $G \times K_2$. However, Bottreau and Metivier [2] came up with a counterexample to the conjecture. In particular, they presented a graph G, obtainable from $K_{3,3}$ through appropriate edge subdivisions, such that G is non-planar, yet $G \times K_2$ is planar (see Fig. 1). Interestingly enough, there is another graph H, obtainable from K_5, with a similar property (see Fig. 2). Among other things, these counterexamples reinforce the common belief that dealing with the direct product is rather difficult.

When we speak of a graph, we mean a finite, simple and undirected graph. Given two graphs G and H, the direct product of these graphs is the graph $G \times H$ on the vertex set $V(G) \times V(H)$, where a vertex (u, a) is adjacent to a vertex (v, b) if and only if u is adjacent to v in G and a to b in H. This product is variously known as Kronecker product, tensor product, cardinal product, cross product and categorical product. For details on this product, see [4], and for any undefined terms, see a standard text on graph theory.

A complete r-partite graph $K_{p1,p2,\ldots,pr}$ is a graph with a vertex set $V = V_1 \cup V_2 \cup \cdots \cup V_r$ of $p_1 + p_2 + \cdots + p_r$ vertices, where V_i are nonempty disjoint sets, $|V_i| = p_i$ for $1 \leq i \leq r$, such that two vertices in V are adjacent if and only if they belong to different V_i.

The central issue in the structural characterization of a planar graph is that of a graph minor. To that end, an *edge-extraction* on a graph $G = (V, E)$ consists of

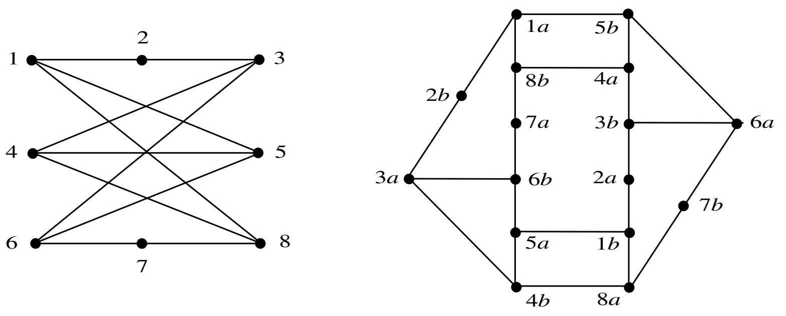


Fig. 1. A non-planar graph G and a planar embedding of $G \times K_2$.

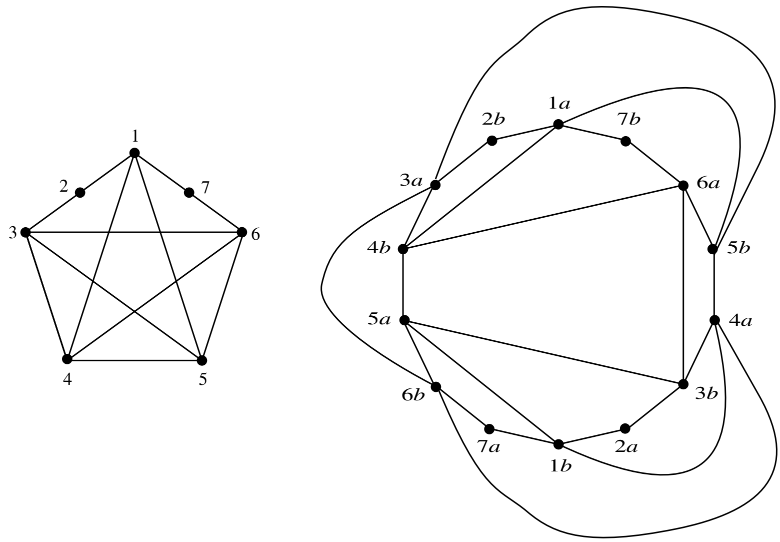


Fig. 2. A non-planar graph H and a planar embedding of $H \times K_2$.

removing an edge e resulting in a graph $G - e$, where $V(G-e) = V$ and $E(G-e) = E - \{e\}$. Further, an *edge-contraction* operation produces a graph from G by removing an edge $\{u, v\}$ of G and identifying (or "merging") u and v, thus creating a new single vertex where the latter inherits all of the adjacencies of the merged vertices, without introducing loops or multiple edges.

Definition 1.1. A graph H is a minor of a graph G if and only if H is obtainable from G by a finite sequence of edge-extraction and edge-contraction operations.

Here is the most useful and most celebrated result on graph planarity.

Theorem 1.2 (Kuratowski and Wagner [6, 7]). *A graph is planar if and only if it has no minor isomorphic to $K_{3,3}$ or K_5.*

Let G be a graph and let G' be the graph obtained from G by removing each vertex of degree one. Then G' is said to be the *1-contraction* of G. We now state Farzan and Waller's result on the planarity of the $\times$-product.

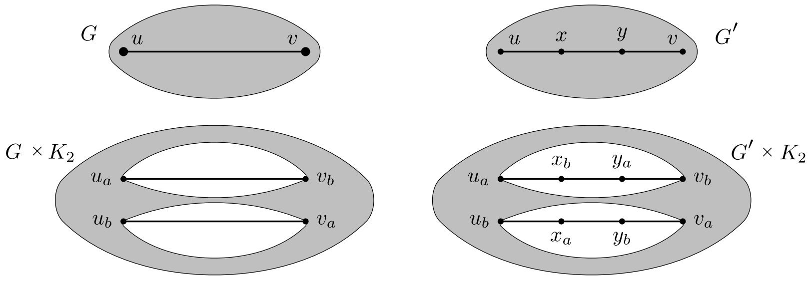
Theorem 1.3 ([3]).

(1) *Let G and H be connected graphs with more than four vertices each. Then $G \times H$ is planar if and only if one of the following holds*:

   (a) *one of G and H is a path and the other is* 1*-contractable to a path or a cycle.*

   (b) *one of G and H is a cycle and the other is* 1*-contractable to a path.*

(2) *Each of $G \times K_4$ and $G \times (K_4 - e)$ is planar if and only if G is isomorphic to K_2.*



(a) for G (b) for G'

Fig. 3. A planar representation of $G \times K_2$ gives a planar representation of $G' \times K_2$, and conversely.

(3) *$G \times (K_3 + x)$ is planar if and only if G is a path.*
(4) *$G \times K_{1,3}$ is planar if and only if G is a path or a cycle.*
(5) *$G \times C_4$ is planar if and only if G is a tree.*
(6) *$G \times C_3$ is planar if and only if G is a path or 1-contractable to a path.*

2. Preliminary Results

In the following, we only study the direct product of a graph with K_2. We denote the vertices of K_2 by a and b, and if u is a vertex of G, we denote by u_a and u_b respectively the vertices (u, a) and (u, b) of $V(G \times K_2)$. In the graph $G \times K_2$, there is no edge between two vertices u_a and v_a (resp. u_b and v_b). Therefore, the graph is bipartite, one partition containing all the vertices u_a, and the other all the vertices u_b.

Lemma 2.1. *Let G be a graph, and G' a graph obtained by subdividing one of its edges twice. Then $G \times K_2$ is planar if and only if $G' \times K_2$ is planar.*

Proof. The proof is almost contained in Fig. 3. Since $G' \times K_2$ is obtained from $G \times K_2$ by subdividing exactly twice the edges (u_a, v_b) and (u_b, v_a), if $G \times K_2$ is planar, we obtain a natural planar representation of $G' \times K_2$, and conversely. □

Thanks to this lemma, we only have to consider the parity of the number of subdividing vertices for each edge. From now on, we consider subdivisions with 0 or 1 subdividing vertex on each edge of the graph, called 0-1 *subdivisions.*

Let G' be a 0-1 subdivision of a graph G. We call a *subdividing vertex* of G' a vertex added to G during a subdivision, the original vertices of G in G' being called *principal vertices.* Further, we call a *subdivided edge* the path between two principal vertices formed by a subdividing vertex and its two incident edges, by opposition to *non-subdivided* edges originally in G. Two principal vertices in G' are said *directly linked* if there is a non-subdivided edge joining them, and *undirectly linked* if there is a subdivided edge.

We use the same denominations for the corresponding vertices and edges in $G' \times K_2$. To understand what underlies all this study, it is important to notice that in $G' \times K_2$, a non-subdivided edge links two principal vertices u_a and v_b, while a subdivided edge links two principal vertices u_a and v_a or u_b and v_b (the subdividing vertex being some x_b or x_a, respectively).

Proposition 2.2. *Let* $n \geq 0$ *be an integer and* G *a* 0-1 *subdivision of* $K_{1,n}$, $K_{1,1,n}$ *or* $K_{2,n}$. $G \times K_2$ *is a planar graph.*

Proof. $K_{1,n}$ is a tree, thus every subdivision of it is a bipartite planar graph. When multiplied by K_2 we obtain two copies of this planar tree.

Any 0-1 subdivision of $K_{2,n}$ (respectively $K_{1,1,n}$) can be seen as two vertices connected by n (resp. $n+1$) disjoint paths. Figure 4 shows how by use of Lemma 2.1, the planarity of $G \times K_2$ is equivalent to the planarity of the graph $C_3 \times K_2$. $C_3 \times K_2$ is the cycle C_6 on 6 vertices, clearly planar, which proves the proposition. □

We now introduce an operation on 0-1 subdivisions of graphs to simplify the later studies.

Definition 2.3. Given a 0-1 subdivision G' of a graph G and u a principal vertex of G', the switch of G' around u, denoted $S(G', u)$, is the graph obtained from G' by switching the status of every edge incident to u (non-subdivided edges become subdivided and conversely).

Proposition 2.4. *Let* G' *be a* 0-1 *subdivision of some graph* G. *For any principal vertex* $u \in G'$, $S(G', u) \times K_2$ *is planar if and only if* $G' \times K_2$ *is planar.*

Proof. Let us denote $v^1, \ldots, v^k$ principal vertices of G' directly linked to u (i.e. by non-subdivided edges), and $w^1, \ldots, w^l$ principal vertices of G' undirectly linked to u (i.e. by subdivided edges). Then in $S(G', u)$, u is directly linked to $w^1, \ldots, w^l$ and undirectly linked to $v^1, \ldots, v^k$. A planar representation of $S(G', u) \times K_2$ can be obtained from $G' \times K_2$ by interchanging u_a and u_b as shown in Fig. 5 (the subdividing vertices may easily be modified correspondingly). Planarity is clearly conserved.

Two such switches resulting in the identity, the equivalence is proved. □

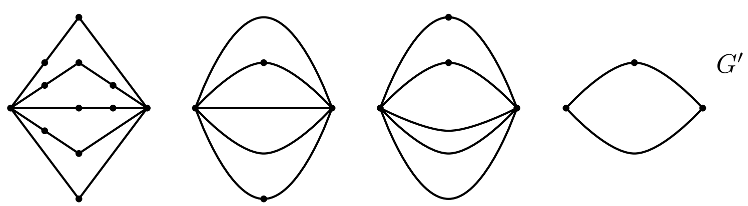


Fig. 4. Using Lemma 2.1 to simplify $K_{1,1,4}$ into a C_3.

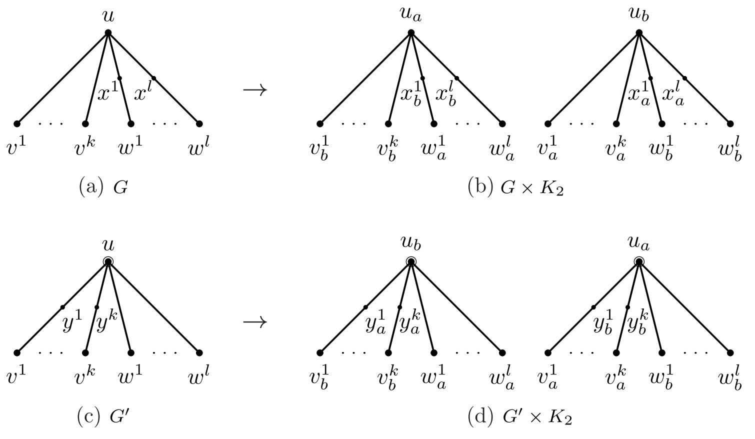


Fig. 5. Proof of Proposition 2.4.

When a 0-1 subdivision G' of a graph G may be obtained by a sequence of switches from another 0-1 subdivision $G^\dagger$, we say that the two subdivisions are *switch-equivalent*. It is straightforward to verify that switch-equivalence is an equivalence relation. Moreover, as a consequence of Proposition 2.4, we know that if G' and $G^\dagger$ are switch-equivalent 0-1 subdivisions of a graph G, then G' is planar if and only if $G^\dagger$ is planar.

In the following, we often need to draw some very dense graphs. To avoid very heavy drawings, we use an unusual convention for these drawings: we do not draw the edges, but we draw the subdivided edges with plain lines and the non edges by dashed lines. To sum up, dashed lines represent non existing edges, plain lines represent subdivided edges and invisible lines represent non-subdivided edges.

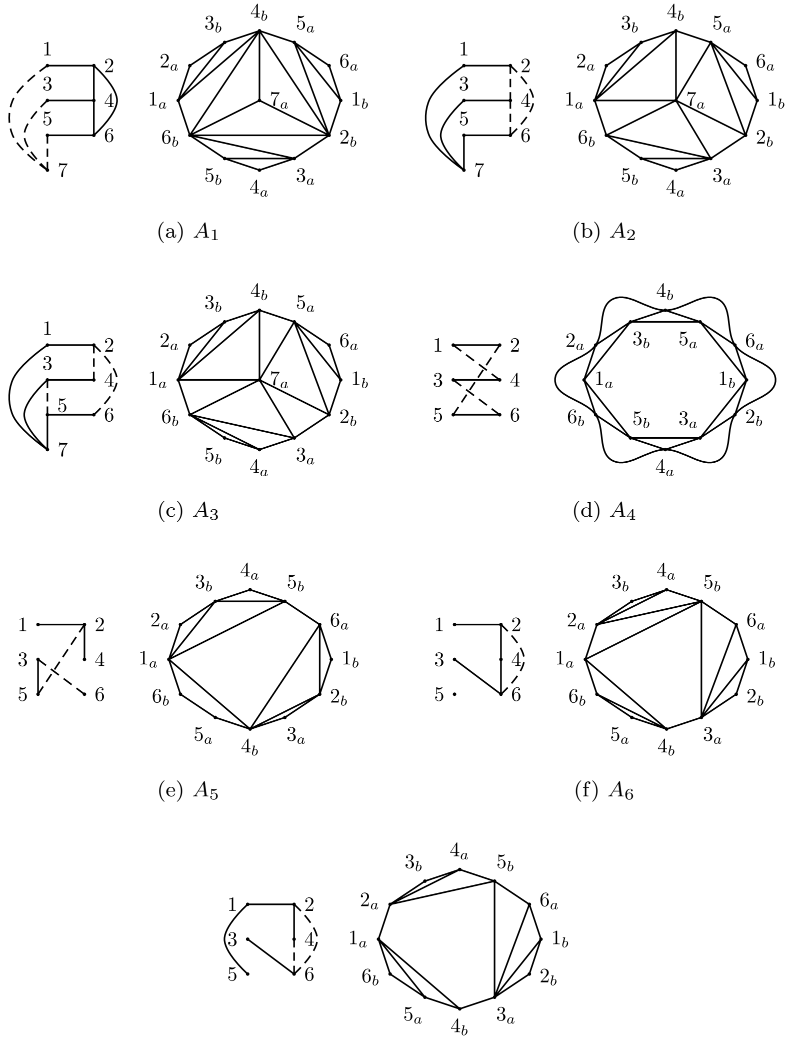
The only exception to this rule is Fig. 6, where the factor on the left hand side of the drawings use this convention while the product on the right hand side is a usual drawing of a graph.

3. Dense Planar Patterns

In this section, we propose a list of dense graphs that are planar when multiplied by K_2 (see Fig. 6). In the proof of the main theorem, we often refer to these graphs to prove that some graph is planar.

Lemma 3.1. *All the graphs A_1 to A_7 have a planar representation of their direct product with K_2.*

To prove planarity, we propose in Fig. 6 a planar representation of each product. The vertex 7_b is not drawn for A_1, A_2 and A_3, and some edges are also missing in all the products except for A_4. These products are symmetric, that is everything inside the external face of our drawings should be copied outside the face,



(a) A_1 (b) A_2 (c) A_3 (d) A_4 (e) A_5 (f) A_6 (g) A_7

Fig. 6. Planar patterns.

interchanging the $_a$ and $_b$s. Obviously, doing so does not change planarity. The whole graph is drawn for A_4 since there is no such symmetry.

4. Non-Planar Patterns

We propose in Fig. 7 a list of some 0-1 subdivisions of graphs that have their product with K_2 non planar. If one is induced in any 0-1 subdivision of a graph G, we can state that G is non planar. To prove non planarity, we just precise the vertices that are contracted in order to get a K_5 minor or a $K_{3,3}$ minor.

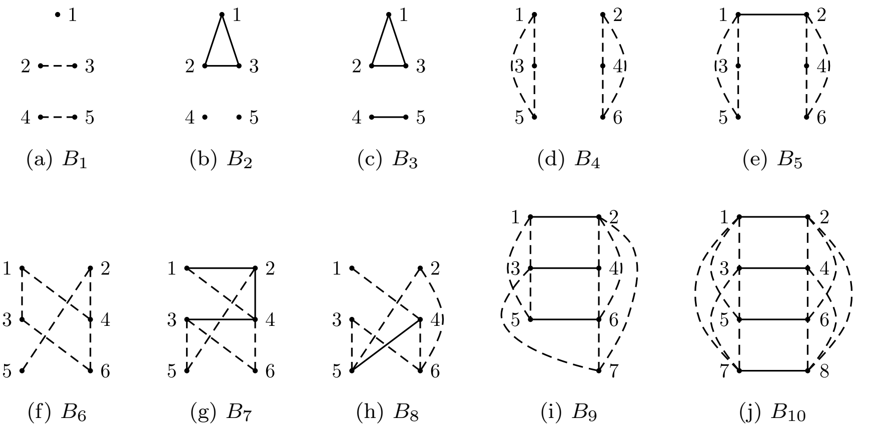


Fig. 7. Non-planar patterns.

Recall that dashed lines represent non edges, plain lines represent subdivided edges and invisible lines represent non-subdivided edges.

Proof.

- $B_1 : (1_b)(5_b)(4_b)[2_a][3_a][1_a, 2_b, 3_b, 4_a, 5_a]$ induce a $K_{3,3}$ minor.
- $B_2 : (1_a)(2_a)(3_a)(4_b)(5_a, 3_b, 4_a, 5_b)$ induce a K_5 minor.
- B_3 : it is a bipartite graph so its product with K_2 is two isomorphic copies of itself. Since B_3 is not planar, $B_3 \times K_2$ is not planar.
- B_4 : it is a bipartite graph so its product with K_2 is two isomorphic copies of itself. Since B_4 is not planar, $B_4 \times K_2$ is not planar.
- $B_5 : (2_a)(4_a)(6_a)[3_b][5_b][1_a, 1_b, 2_b, 3_a, 4_b, 5_a, 6_b]$ induce a $K_{3,3}$ minor.
- $B_6 : (1_a)(3_a)(6_a)[2_b][5_b][2_a, 3_b, 5_a, 4_b, 6_b]$ induce a $K_{3,3}$ minor.
- $B_7 : (1_b)(2_b)(3_a, 4_a)(6_a, 5_b)(1_a, 2_a, 3_b, 4_b, 6_b, 5_a)$ induce a K_5 minor.
- $B_8 : (4_a, 5_a)(3_b, 2_a)(6_b, 1_a)(1_b, 6_a, 5_b)(2_b, 3_a, 4_b)$ induce a K_5 minor.
- $B_9 : (1_b)(5_b)(4_b)[7_a][4_a, 7_b, 5_a][2_a, 1_a, 6_b, 3_a, 2_b]$ induce a $K_{3,3}$ minor.
- $B_{10} : (1_a, 4_a, 5_b, 6_b)(1_a)(2_a, 3_b)(8_b, 7_b)(4_b, 5_a, 6_a)$ induce a K_5 minor. □

5. Main Theorem

Theorem 5.1. *Given G a subdivision of a multipartite complete graph, its product with K_2 is planar if and only if the corresponding* 0-1 *subdivision has no partial subgraph switch-equivalent to one of the B_is.*

To prove this theorem, we enumerate all the multipartite complete graphs and their 0-1 subdivisions. To reduce the study, we use some arguments related to switches (and Proposition 2.4) to reduce the number of subdivisions we have to study. Then we represent each of the subdivisions left in a figure with a caption that tells about the planarity of the product of this subdivision with K_2, as well as its proof. There are three types of captions:

- P(bipartite): In that case, since the graph is bipartite, its product with K_2 consists of two disjoint copies of the graph. Since the graph is also planar, its product is planar.
- P(A_i): This means the product is planar. The labels on the vertices give a mapping from the subdivision to a subgraph of A_i.
- NP(B_i): This caption means the product of this subdivision with K_2 is non-planar. The labels on the vertices give an isomorphism from a partial subgraph of the subdivision to a B_i.

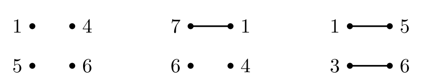
In the two later cases, sometimes the mapping is not from this 0-1 subdivision, but from another switch-equivalent subdivision. Whenever so, some vertices are circled with a thin line, they correspond to the vertices we need to switch to get the switch-equivalent subdivision that maps to A_i or B_i.

Subdivisions with less than 4 principal vertices. We prove that the product of subdivisions of K_4 with K_2 are planar. Since every principal vertex has 3 neighbors, if a principal vertex u has two or more incident edges subdivided, we can decrease the number of subdivided edges by operating a switch of G around u. Thus, we can restrict our study to subdivisions where every principal vertex has at most one subdivided incident edge. Any 0-1 subdivision of K_4 can be reduced by a sequence of switches to one of the three graphs in Fig. 8.

Since all the 0-1 subdivisions of K_4 are planar when multiplied by K_2, it is also true for every 0-1 subdivision of any graph of order less or equal than 4.

Subdivisions with 5 principal vertices. Subdivisions of $K_{1,4}$, $K_{2,3}$ and $K_{1,1,3}$ are already proved to have a planar product with K_2 in Proposition 2.2. We thus have to study the subdivisions of $K_{1,2,2}$, $K_{1,1,1,2}$ and K_5.

Let us start with the subdivisions of $K_{1,2,2}$. Any subdivision is switch equivalent to some subdivision whose vertex in the stable set of order 1 has at most two subdivided incident edges and whose other vertices have at most one subdivided incident edge. The corresponding 0-1 subdivisions are listed on Fig. 9. Notice that in case (b) for example, the non edge between 4 and 5 is mapped to an edge in A_5. This is allowed since this is a mapping to a partial subgraph of A_5. Recall that circled vertices (e.g. in (f)) need to be switched before applying the mapping.



(a) P (A_5) (b) P (A_3) (c) P (A_7)

Fig. 8. Study of K_4.

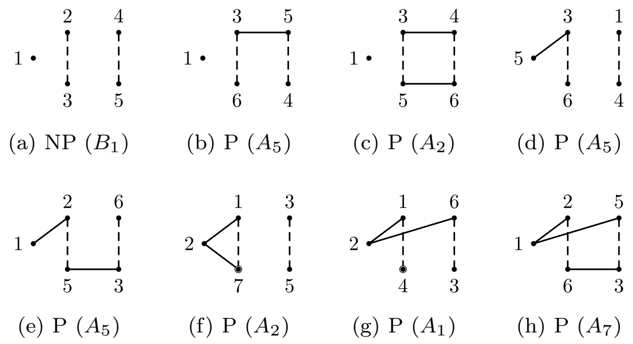


Fig. 9. Study for $K_{1,2,2}$.

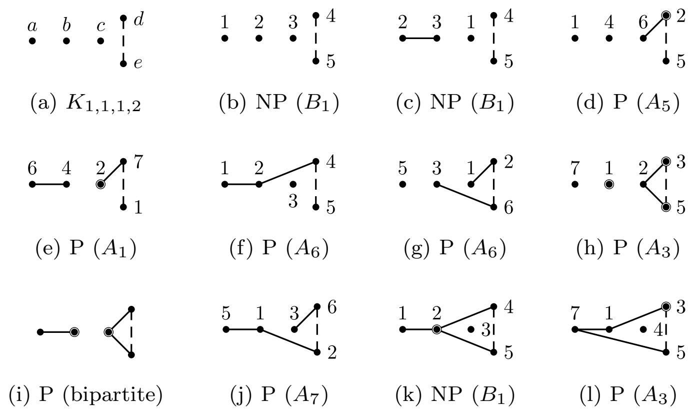
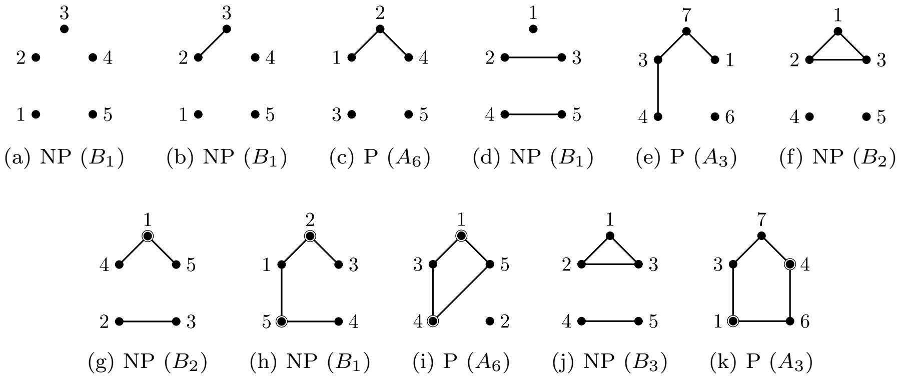


Fig. 10. Study of $K_{1,1,1,2}$.

Notice also that the subdivision in Fig. 9(c) is switch equivalent to the one in Fig. 9(g) (switch for example vertices labeled 3 and 6) and that the subdivision in Fig. 9(h) is switch-equivalent to the one in Fig. 9(e) (switch vertices labeled 1 and 3). Thus, we have the following remark.

Remark 5.2. All the classes of switch-equivalent 0-1 subdivisions of $K_{1,2,2}$ are generated by the elements of Figs. 9(a)–9(g).

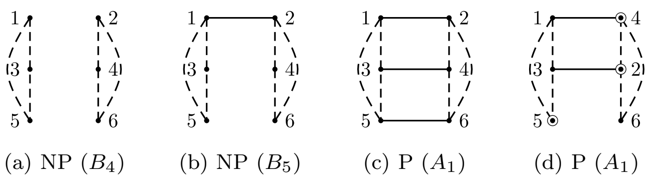
To treat the subdivisions of $K_{1,1,1,2}$ (see Fig. 10(a)), we can consider that the subgraph induced by a, b and c has at most one subdivided edge (otherwise we can switch the vertex with degree 2). Then we can switch vertices d and e so that they have at most one subdivided incident edge. Therefore we only need to check the subdivisions listed in Fig. 10. Notice that on case (b) or (c) for example, there is no edge between 2 and 3 in B_1. Since we look for an isomorphism from a subgraph of the subdivision to B_1, the subdivision may contain an edge or a subdivided edge linking 2 and 3.



(a) NP (B_1) (b) NP (B_1) (c) P (A_6) (d) NP (B_1) (e) P (A_3) (f) NP (B_2)

(g) NP (B_2) (h) NP (B_1) (i) P (A_6) (j) NP (B_3) (k) P (A_3)

Fig. 11. Study of K_5.



(a) NP (B_4) (b) NP (B_5) (c) P (A_1) (d) P (A_1)

Fig. 12. Study of $K_{3,3}$.

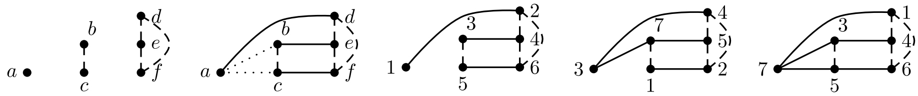
We now have to study the subdivisions of K_5. The switching argument allows us to consider only subdivisions with at most two subdivided edges incident to each edge. We obtain the enumeration represented in Fig. 11.

Subdivisions with 6 principal vertices. Subdivisions of $K_{1,5}$, $K_{2,4}$ and $K_{1,1,4}$ are already known to have a planar product with K_2 by Proposition 2.2. Let us study the 0-1 subdivisions of $K_{3,3}$. By switch-equivalence, we may consider only subdivisions with at most one subdivided edge incident to each vertex. They are listed in Fig. 12.

Since the subdivision in Fig. 12(d) is switch-equivalent to the one in Fig. 12(c) (with the same switches represented on the figure), the following remark holds.

Remark 5.3. Given a 0-1 subdivision of $K_{3,3}$, its product with K_2 is planar if and only if the subdivision is switch-equivalent to the subdivision in Fig. 12(c).

Consider now the 0-1 subdivisions of $K_{1,2,3}$. $K_{1,2,3}$ contains $K_{3,3}$ as a subgraph (see in Fig. 13(a), the partition being $\{a, b, c\}$ and $\{d, e, f\}$). By the previous remark, we know that whenever the 0-1 subdivision of this $K_{3,3}$ subgraph is not switch equivalent to the subdivision of Fig. 12(c), the graph is not planar. We therefore consider only the other situation, drawn on Fig. 13(b), for which we only need to state whether the (dotted) edges *ab* and *ac* are subdivided or not. As proved in Fig. 13, these subdivisions are all planar.



(a) $K_{1,2,3}$ (b) (c) P (A_1) (d) P (A_2) (e) P (A_2)

Fig. 13. Study of $K_{1,2,3}$.

We now study the 0-1 subdivisions of $K_{2,2,2}$ (see Fig. 14). Consider the induced $K_{1,2,2}$ on the set of vertices $\{a, b, c, d, e\}$. If the 0-1 subdivision of this subgraph has a non-planar product with K_2 (switch equivalent to Fig. 9(a)), then the corresponding subdivision of $K_{2,2,2}$ also has a non-planar product with K_2. We therefore study only the cases when this subdivision is planar. By Remark 5.2, it is then switch equivalent to one of the subdivision in Figs. 9(b)–9(g). Moreover, we may assume that the remaining vertex f has at most two subdivided incident edges, and if it has two, we can force it to be adjacent to the vertex d by switching around f. The remaining subdivision are studied in Figs. 15–19.

Let us consider 0-1 subdivisions of $K_{1,1,1,3}$. First remark that $K_{3,3}$ is a subgraph of $K_{1,1,1,3}$ (in Fig. 20, consider the sets $\{a, b, c\}$ and $\{d, e, f\}$). Therefore, like we did for $K_{2,2,2}$, Remark 5.3 allows us to consider only subdivisions enumerated in Fig. 20.

Let us consider the 0-1 subdivisions of $K_{1,1,2,2}$ that are planar when multiplied by K_2. As it contains a $K_{3,3}$ as a partial subgraph, there is a sequence of switches leading to one of the graphs in Figs. 21(a) and 21(b).

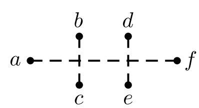
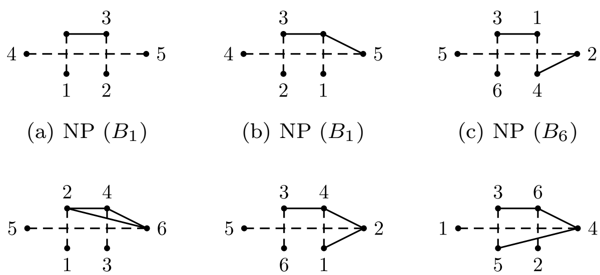


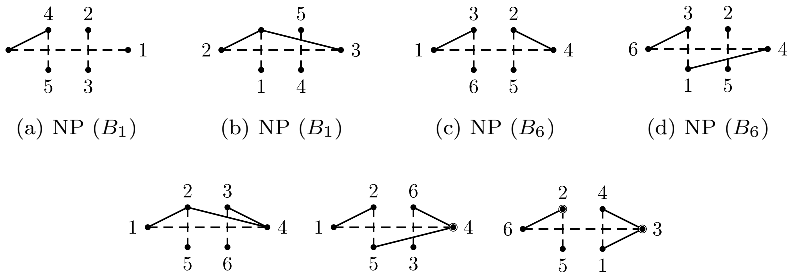
Fig. 14. Study for $K_{2,2,2}$.



(a) NP (B_1) (b) NP (B_1) (c) NP (B_6)

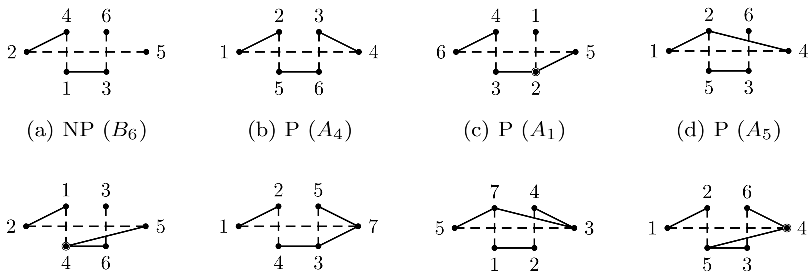
(d) P (A_1) (e) NP (B_7) (f) NP (B_8)

Fig. 15. Study for 9(b).



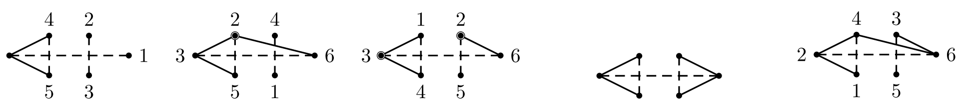
(a) NP (B_1) (b) NP (B_1) (c) NP (B_6) (d) NP (B_6) (e) NP (B_7) (f) NP (B_7) (g) P (A_5)

Fig. 16. Study for 9(d).



(a) NP (B_6) (b) P (A_4) (c) P (A_1) (d) P (A_5) (e) NP (B_7) (f) P (A_3) (g) P (A_3) (h) NP (B_7)

Fig. 17. Study for 9(e).



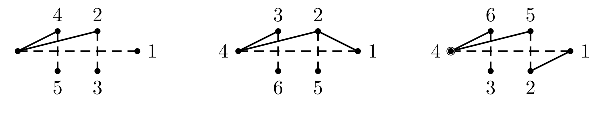
(a) NP (B_1) (b) P (A_5) (c) P (A_5) (d) P (planar and bipartite) (e) P (A_6)

Fig. 18. Study for 9(f).

We now consider 0-1 subdivisions of $K_{1,1,1,1,2}$. They are necessarily obtained from a good subdivision of $K_{1,1,2,2}$ (Figs. 22(c), 23(c) and 23(d)), by adding an edge either subdivided or not. So the enumeration in Fig. 24 is complete.

Let us continue with 0-1 subdivisions of K_6. It contains a subdivision of $K_{1,1,1,1,2}$ which must be planar when multiplied by K_2. Thus, there exists a sequence of switches resulting in one of the subgraphs in Figs. 24(a) and 24(b). We just need to decide the status of the deleted edge (see Fig. 25).

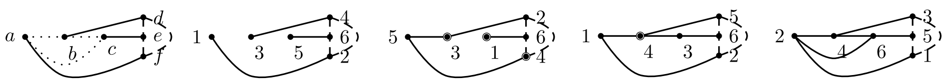
Graphs with 7 principal vertices. Subdivisions of $K_{1,6}$, $K_{2,5}$ and $K_{1,1,5}$ are already known to be planar by Proposition 2.2. Let us study the 0-1 subdivisions of $K_{3,4}$. We can consider the induced $K_{3,3}$ in order to enumerate these graphs.



(a) NP (B_1) (b) NP (B_7) (c) NP (B_7)

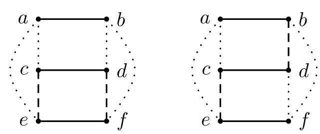
(d) NP (B_6) (e) P (A_6) (f) P (A_6)

Fig. 19. Study for 9(g).



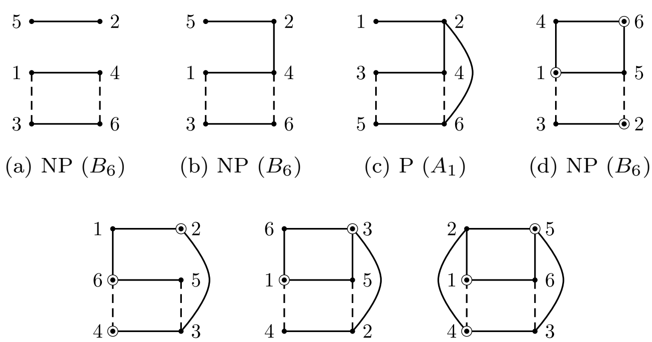
(a) Unknown edges (b) P (A_2) (c) P (A_7) (d) P (A_6) (e) P (A_1)

Fig. 20. Study of $K_{1,1,1,3}$.



(a) (b)

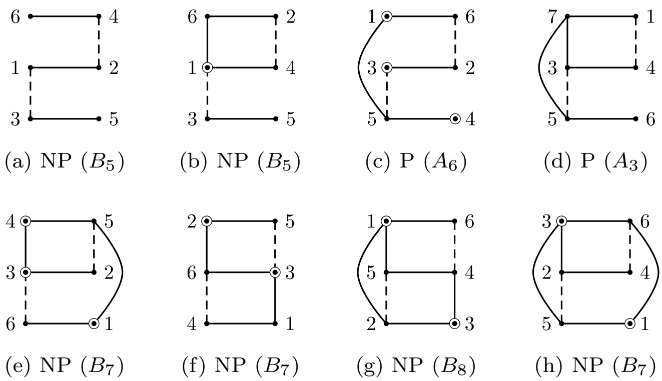
Fig. 21. $K_{1,1,2,2}$ contains a $K_{3,3}$.



(a) NP (B_6) (b) NP (B_6) (c) P (A_1) (d) NP (B_6)

(e) NP (B_8) (f) NP (B_7) (g) NP (B_6)

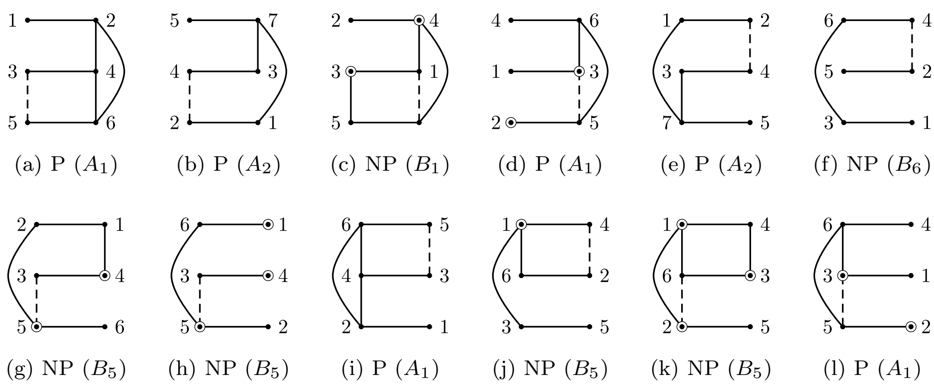
Fig. 22. Study of 21(a).



(a) NP (B_5) (b) NP (B_5) (c) P (A_6) (d) P (A_3)

(e) NP (B_7) (f) NP (B_7) (g) NP (B_8) (h) NP (B_7)

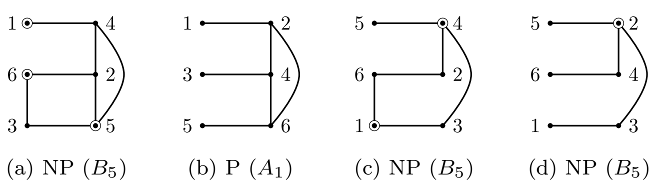
Fig. 23. Study of 21(b).



(a) P (A_1) (b) P (A_2) (c) NP (B_1) (d) P (A_1) (e) P (A_2) (f) NP (B_6)

(g) NP (B_5) (h) NP (B_5) (i) P (A_1) (j) NP (B_5) (k) NP (B_5) (l) P (A_1)

Fig. 24. Study of $K_{1,1,1,1,2}$.

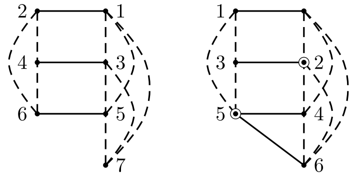


(a) NP (B_5) (b) P (A_1) (c) NP (B_5) (d) NP (B_5)

Fig. 25. Study of K_6.

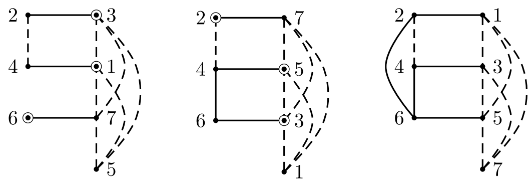
Moreover, the fourth vertex of the 4 -stable can be considered to have at most one incident subdivided edge (see Fig. 26).

Let us study 0-1 subdivision of $K_{1,2,4}$. The induced subdivision of $K_{3,4}$ can be reduced to the graph in Fig. 26(a) by a sequence of switches. These graphs are studied in Fig. 27.



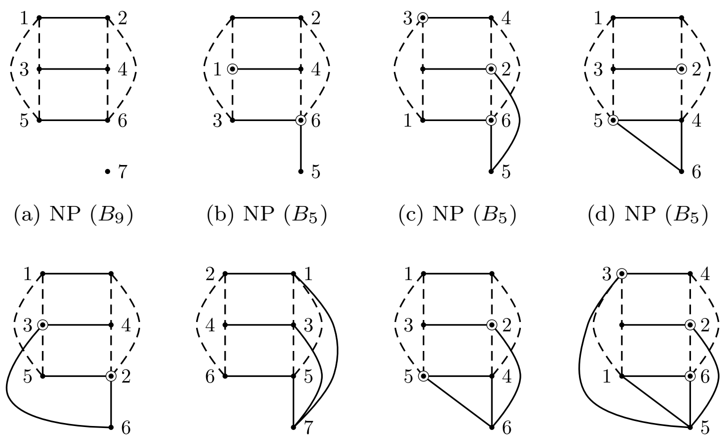
(a) P (A_1) (b) NP (B_5)

Fig. 26. Study of $K_{3,4}$.



(a) P (A_1) (b) P (A_1) (c) P (A_1)

Fig. 27. Study of $K_{1,2,4}$.



(a) NP (B_9) (b) NP (B_5) (c) NP (B_5) (d) NP (B_5)

(e) NP (B_5) (f) P (A_3) (g) NP (B_5) (h) NP (B_5)

Fig. 28. Study of $K_{1,3,3}$.

In order to study subdivisions of $K_{1,3,3}$ we consider the induced $K_{3,3}$ and assume that the remaining vertex has at most 3 incident subdivided edges. The resulting enumeration is in Fig. 28.

To study subdivisions of $K_{2,2,3}$, we consider the included subdivision of $K_{3,4}$, we get the graph drawn in Fig. 29(a). It appears that by setting 0 or 1 unknown edge to subdivided, we get a B_9 as a subgraph, so that it is not planar. The only way to ensure that the graph does not contain B_9 is to set at least two edges as

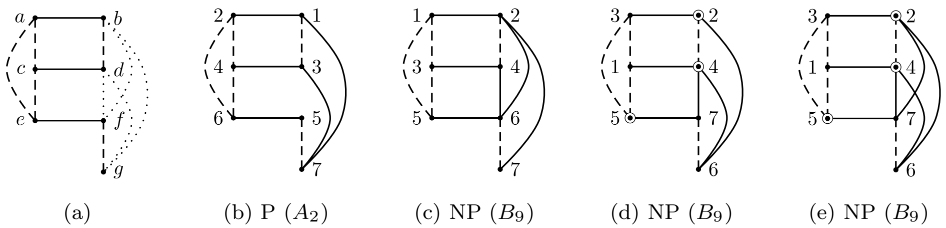


Fig. 29. Study of $K_{2,2,3}$.

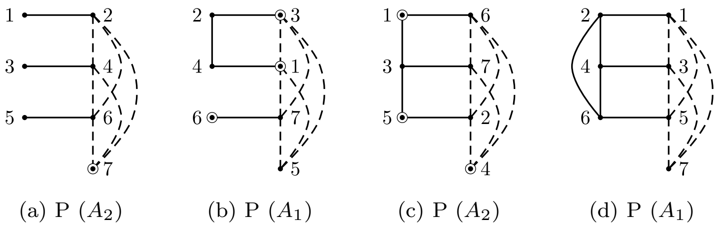


Fig. 30. Study of $K_{1,1,1,4}$.

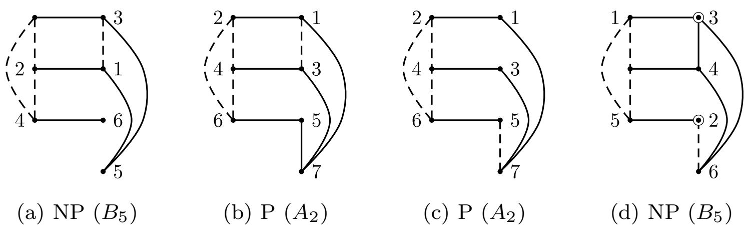


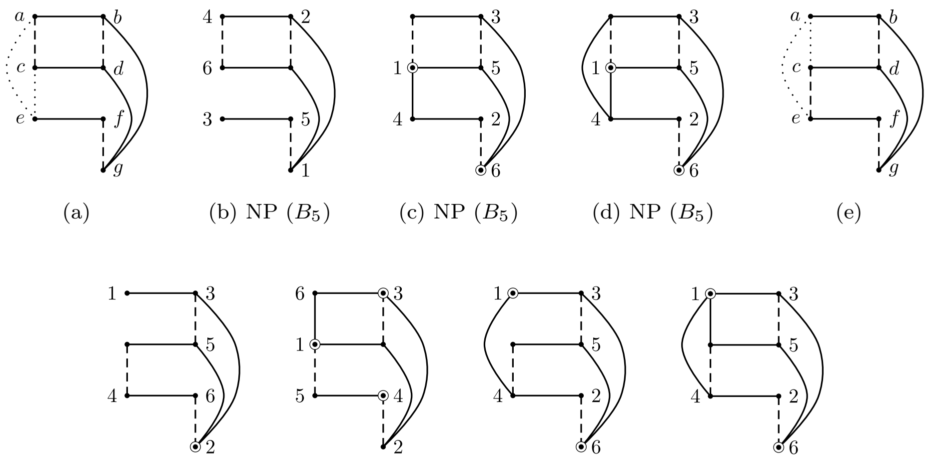
Fig. 31. Study of $K_{1,1,2,3}$.

subdivided. And if there are exactly two of them, they must be distinct from df and bf (by symmetry) or it would also contain a B_9. Therefore, the enumeration in Fig. 29 is complete.

The graph $K_{1,1,1,4}$ contains a $K_{3,4}$. We thus know that subdivisions of $K_{1,1,1,4}$ similar to the one represented in Fig. 26(b) are not planar. Cases similar to the subdivision in Fig. 26(a) are studied in Fig. 30.

Let us study subdivisions of $K_{1,1,2,3}$ based on the only planar configuration for $K_{2,2,3}$ which can be placed in two different ways in this graph. We get the enumeration in Fig. 31.

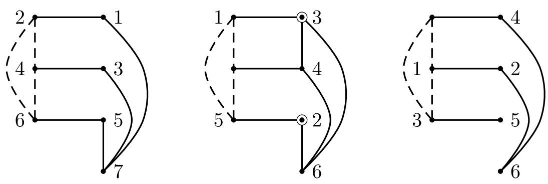
For the study of $K_{1,2,2,2}$ we consider its partial subdivision isomorphic to a subdivision of $K_{2,2,3}$. We conclude that there is a sequence of switches leading to configurations in Figs. 32(a) or 32(e). Thus the enumeration is complete and we can conclude that there is no subdivisions of $K_{1,2,2,2}$ resulting in planar graph when multiplied by K_2.



(a) (b) NP (B_5) (c) NP (B_5) (d) NP (B_5) (e)

(f) NP (B_5) (g) NP (B_5) (h) NP (B_5) (i) NP (B_5)

Fig. 32. Study of $K_{1,2,2,2}$.



(a) P (A_2) (b) NP (B_5) (c) NP (B_5)

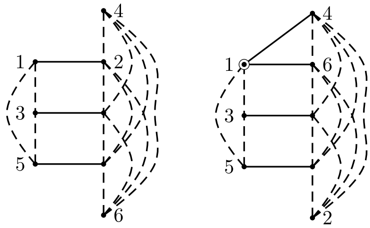
Fig. 33. Study of $K_{1,1,1,1,3}$.

For $K_{1,1,1,1,3}$, we start from planar configurations of $K_{1,1,2,3}$ (see Figs. 31(b) and 31(c)). So that we only have to study three configurations (see Fig. 33).

Any subdivision of $K_{1,1,1,2,2}$ contains a subdivision of $K_{1,2,2,2}$ and thus is a non planar configuration. The same stands for $K_{1,1,1,1,1,2}$ and K_7.

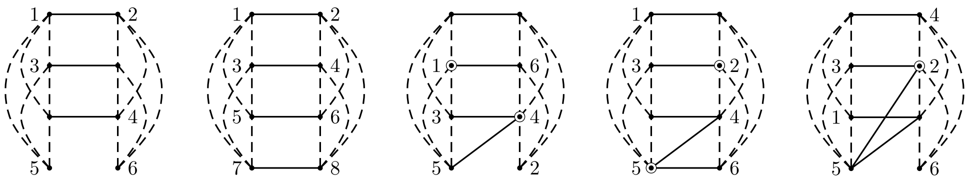
Graphs with 8 principal vertices. Subdivisions of $K_{1,7}$, $K_{2,6}$ and $K_{1,1,6}$ are already known to be planar by Proposition 2.2. Let us study the 0-1 subdivisions of $K_{3,5}$. We can consider the induced $K_{3,4}$ in order to enumerate these graphs. Moreover, the fifth vertex of the 5-stable can be considered to have at most one incident subdivided edge (Fig. 34).

We now check that any subdivision of $K_{4,4}$ is non planar when multiplied by K_2. We consider the subdivision of $K_{3,4}$ included and check that the remaining vertex can be considered to have at most two incident subdivided edges. The study is drawn in Fig. 35.



(a) NP (B_5) (b) NP (B_5)

Fig. 34. Study of $K_{3,5}$.



(a) NP (B_5) (b) NP (B_{10}) (c) NP (B_5) (d) NP (B_5) (e) NP (B_5)

Fig. 35. Study of $K_{4,4}$.

Other multipartite complete graph with 8 vertices contain either a $K_{3,5}$ or a $K_{4,4}$ and thus can never be planar when multiplied by K_2. We observe that the only multipartite complete graph with 8 vertices that admit a planar configuration are $K_{1,7}$, $K_{2,6}$ and $K_{1,1,6}$.

Graphs with $n \geq 9$ principal vertices. Subdivisions of $K_{1,n-1}$, $K_{2,n-2}$ and $K_{1,1,n-2}$ are already known to be planar by Proposition 2.2. Let us consider a 0-1 subdivision of $K_{n_1,n_2,\ldots,n_k}$ ($k \geq 2$ since K_n contain K_8). We assume that the n_i's are listed from smallest to biggest. If there exists a subset of $S \subset \{1,\ldots,n\}$ such that $\sum_{i\in S} n_i = 3$ or 4, then it is clearly non planar since $n \geq 9$ and thus the graph contains a $K_{3,5}$, or a $K_{4,4}$.

Therefore, we can consider there is no such subset. The only way is to have either $n_1 \geq 5$, or $n_1 = 2$ and $n_2 \geq 5$, or $n_1 = n_2 = 1$ and $n_3 \geq 5$.

If $n_1 \geq 5$, then $n_2 \geq n_1 \geq 5$ and thus the subdivisions contain a $K_{3,5}$ and is non planar when multiplied by K_2.

If $n_1 = 2$ and $n_2 \geq 5$, since we suppose it is not $K_{2,n-2}$, we can assume that $k \geq 3$. Thus, $n_3 \geq n_2 \geq 5$. We can exhibit a $K_{3,5}$ and the configuration is non-planar.

Similarly, if $n_1 = n_2 = 1$ and $n_3 \geq 5$, we can assume that $k \geq 4$ ($K_{1,1,n-2}$ already studied). Thus, $n_3 \geq n_2 \geq 5$. We can exhibit a $K_{3,5}$ and the configuration is non-planar.

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