ELSEVIER

Discrete Mathematics 182 (1998) 153–167

DISCRETE MATHEMATICS

Kronecker products of paths and cycles: Decomposition, factorization and bi-pancyclicity

Pranava K. Jha*

Department of Computer Engineering, Delhi Institute of Technology, Kashmere Gate, Delhi 110 006, India

Received 11 September 1995; received in revised form 25 May 1996; accepted 15 May 1997

Abstract

Let $G \times H$ denote the Kronecker product of graphs G and H. Principal results are as follows: (a) If m is even and $n \equiv 0 \pmod 4$, then one component of $P_{m+1} \times P_{n+1}$, and each component of each of $C_m \times P_{n+1}$, $P_{m+1} \times C_n$ and $C_m \times C_n$ are edge decomposable into cycles of uniform length rs, where r and s are suitable divisors of m and n, respectively, (b) if m and n are both even, then each component of each of $C_m \times P_{n+1}$, $P_{m+1} \times C_n$ and $C_m \times C_n$ is edge-decomposable into cycles of uniform length ms, where s is a suitable divisor of n, (c) $C_{2i+1} \times C_{2j+1}$ is factorizable into shortest odd cycles, (d) each component $C_{4i} \times C_{4j}$ is factorizable into four-cycles, and (e) each component of $C_m \times C_{4j}$ admits of a bi-pancyclic ordering.

AMS Classification: 05C38; 05C45; 05C70

Keywords: Kronecker product; Path; Cycle; Decomposition; Factorization; Bi-pancyclicity

1. Introduction and preliminaries

The central message of this paper is that a connected component of each of $P_m \times P_n$, $C_m \times P_n$ and $C_m \times C_n$ has a rich cycle structure. Consequently, each of these graphs is amenable to applications in areas such as VLSI layout, computer and communication networks, management of multiprocessors, and X-ray crystallography.

By a graph is meant a finite, simple and undirected graph. Unless indicated otherwise, graphs are connected and contain at least two vertices. The *Kronecker product* $G \times H$ of graphs $G = (V, E)$ and $H = (W, F)$ is defined as follows: $V(G \times H) = V \times W$ and $E(G \times H) = \{\{(u,x),(v,y)\}: \{u,v\} \in E \text{ and } \{x,y\} \in F\}$. Note that $|V(G \times H)| = |V||W|$ and $|E(G \times H)| = 2|E||F|$.

Let C_m and P_n, respectively, denote a cycle on m vertices and a path on n vertices, where $V(C_k) = V(P_k) = \{0, \ldots, k-1\}$ and where adjacencies are defined in the natural

* E-mail: pkj@dit.ernet.in.

PII S0012-365X(97)00138-6

way. If $S=\{v_1,\dots,v_r\}$ is a vertex subset of G, then $\langle S\rangle$ or $\langle v_1,\dots,v_r\rangle$ represents the subgraph induced by S.

Definition 1. A decomposition $\mathscr{D}$ of a graph G consists of subgraphs $G_1,\dots,G_r$ which constitute a partition of the edge set of G.

Definition 2. A factorization $\mathscr{F}$ of a graph G consists of spanning subgraphs $F_1,\dots,F_r$ which constitute a partition of the edge set of G. The spanning subgraphs $F_1,\dots,F_r$ are called factors of G.

$\mathscr{F}$ may be viewed as an edge-coloring of G using $r=|\mathscr{F}|$ colors, where F_i consists of all the edges of color i, $1\leqslant i\leqslant r$. A factorization of G in which each factor is regular of degree k is called a k-factorization, and G is said to be k-factorizable.

Decomposition/factorization of product graphs has a rich literature. For example, see [1,3,4,8,12].

Definition 3. A bipartite graph on an even number of vertices is said to admit of a bi-pancyclic ordering if there is an ordering $v_0,v_1,\dots,v_{2r-1}$ of its vertices such that $\langle v_0,v_1,\dots,v_{2k-1}\rangle$ contains a spanning cycle for all $k\in\{2,\dots,r\}$.

Bi-pancyclicity is essentially a restriction of the concept of pancyclicity (which asks whether a given graph contains cycles of all possible length) to bipartite graphs, all of whose cycles are necessarily even. This topic has received attention for a long time. Bondy [2], Ramachandran and Parvathy [9], Schmeichel and Mitchem [10], and Teichert [11] are representatives.

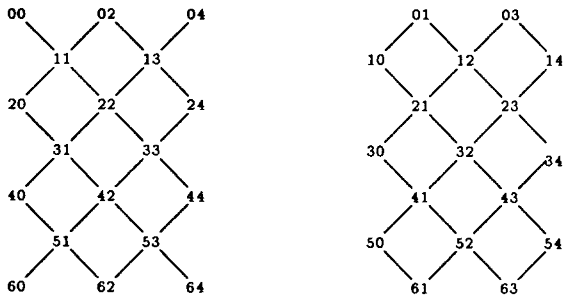
Each of $C_{2i+1}\times P_n$ and $C_{2i+1}\times C_n$ is a connected graph while each of $P_m\times P_n$, $C_{2i}\times P_n$ and $C_{2i}\times C_{2j}$ consists of two connected components. Further, (a) the two components of $P_m\times P_n$ are isomorphic if and only if mn is even, and (b) the two components of $C_{2i}\times P_n$ (resp. $C_{2i}\times C_{2j}$) are isomorphic. It is also useful to note that vertices (p,q) and (r,s) of $P_m\times P_n$ or $C_{2i}\times P_n$ or $C_{2i}\times C_{2j}$ belong to the same component if and only if $p+q$ and $r+s$ are of the same parity. Based on this observation, a component of $P_m\times P_n$ or $C_{2i}\times P_n$ or $C_{2i}\times C_{2j}$ will be called an *even component* (resp. *odd component*) if vertices (p,q) of that component are such that $p+q$ is even (resp. odd).

Graphs $P_7\times P_5$ and $C_6\times P_5$ appear in Figs. 1 and 2, respectively. For the sake of clarity, a vertex (p,q) has been shown as pq.

Among $P_m\times P_n$, $C_m\times P_n$ and $C_m\times C_n$, the graph $C_{2i+1}\times C_{2j+1}$ is nonbipartite while all others are bipartite.

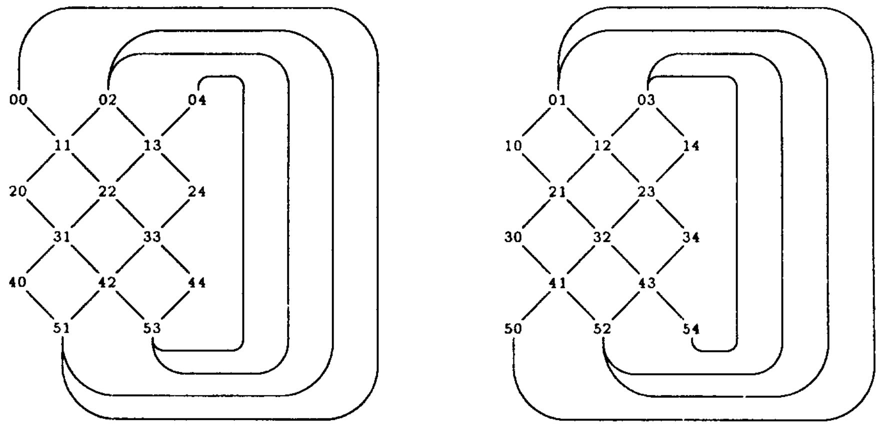
The number of vertices in the even component of $P_m\times P_n$ is $\lceil mn/2\rceil$ while that in the odd component is $\lfloor mn/2\rfloor$. Vertices of this graph which are of degree one or two will be called *border vertices*. (All of the remaining vertices are of degree four.) Some additional remarks are as follows:

- $P_m\times P_n$ contains a total of $2(m+n-2)$ border vertices; out of these, four are of degree one while others are of degree two.



Even Component Odd Component

Fig. 1. Graph $P_7 \times P_5$.



Even Component Odd Component

Fig. 2. Graph $C_6 \times P_5$.

- The border vertices are equally divided between the two components. Further, if m and n are both odd, then all four pendant vertices appear in the even component, otherwise, they are equally divided between the two components.

Note that cycle C_m is obtainable from path P_{m+1} by identifying the terminal vertices. The following proposition makes certain analogous statements.

Proposition 1.1. (1) *The even (resp. odd) component of $C_{2i} \times P_n$ is obtainable from the even (resp. odd) component of $P_{2i+1} \times P_n$ by identifying the pair of border vertices $(0,k)$ and $(2i,k)$ for all even k (resp. odd k) between* 0 *and $n-1$.*

(2) *The even (resp. odd) component of $C_{2i} \times C_{2j}$ is obtainable from the even (resp. odd) component of $P_{2i+1} \times C_{2j}$ by identifying the pair of border vertices $(0,k)$ and $(2i,k)$ for all even k (resp. odd k) between* 0 *and $2j-1$.*

Graphs $P_7 \times P_5$ and $C_6 \times P_5$, which appear in Figs. 1 and 2, illustrate the statement of Proposition 1.1(1). The following result will be useful in the sequel.

Lemma 1.2. *If m is even, $m/2$ is odd and G is a bipartite graph, then each component of $C_m \times G$ is isomorphic to $C_{m/2} \times G$* [6].

A classical result about factorization of graphs is Petersen's theorem that every regular graph of even degree is 2-factorizable [7]. It follows that $C_m \times C_n$ is 2-factorizable. However, one may modify the classical problem by imposing bounds on the number and size of the components of the factors. Indeed, decomposition/factorization into cycles of uniform length has a strong appeal.

It is demonstrated in Section 2 that $C_m \times C_n$ admits of certain 2-factorizations in which each factor consists of cycles of (uniform) length rs, where r and s are suitable divisors of m and n, respectively. Additional results of Section 2 include (a) decompositions of a component of each of $P_m \times P_n$, $C_m \times P_n$ and $C_m \times C_n$ into cycles and paths, and (b) decomposition of these graphs into four-cycles.

It is shown in Section 3 that if m is even and $n \equiv 0 \pmod 4$, then the odd component of $P_{m+1} \times P_{n+1}$ (as well as each component of each of $C_m \times P_{n+1}$ and $P_{m+1} \times C_n$) contains a subgraph on $mn/2$ vertices which has a bi-pancyclic ordering. This leads to a similar ordering of each component of $C_m \times C_{4j}$.

Proposition 1.1 and Lemma 1.2 are frequently invoked in the rest of the paper.

2. Decomposition and factorization

The present section is subdivided into five parts. Section 2.1 builds a cycle decomposition of the odd component of $P_{2i+1} \times P_{4j+1}$, which in turn leads to (a) a similar decomposition of each component of each of $C_m \times P_{4j+1}$ and $P_{2i+1} \times C_{4j}$, and (b) a 2-factorization of each component of $C_m \times C_{4j}$. Analogous results appear in Section 2.2 with respect to $P_{2i+1} \times P_{4j+3}$, $C_m \times P_{4j+3}$ and $C_m \times C_{4j+2}$. Section 2.3 consists of certain decompositions of $P_{2i+1} \times P_{2j}$ and $C_m \times P_n$. That $C_{2i+1} \times C_{2j+1}$ has a factorization into shortest odd cycles appears next. Finally, Section 2.5 deals with four-cycles in these graphs.

2.1. Graphs $P_{2i+1} \times P_{4j+1}$, $C_m \times P_{4j+1}$, $P_{2i+1} \times C_{4j}$ and $C_m \times C_{4j}$

Lemma 2.1. *If m and n are even $\geqslant 4$ and $n \equiv 0 \pmod 4$ then the odd component of $P_{m+1} \times P_{n+1}$ is decomposable into two equal-length cycles α and β such that*

1. *each vertex of degree four appears on α as well as on β,*
2. *among the border vertices $(2i+1, 0)$ and $(2i+1, n)$, exactly one belongs to α and the other belongs to β, where $0 \leqslant i \leqslant (m-2)/2$, and*
3. *among the border vertices $(0, 2j+1)$ and $(m, 2j+1)$, exactly one belongs to α and the other belongs to β, where $0 \leqslant j \leqslant (n-2)/2$.*

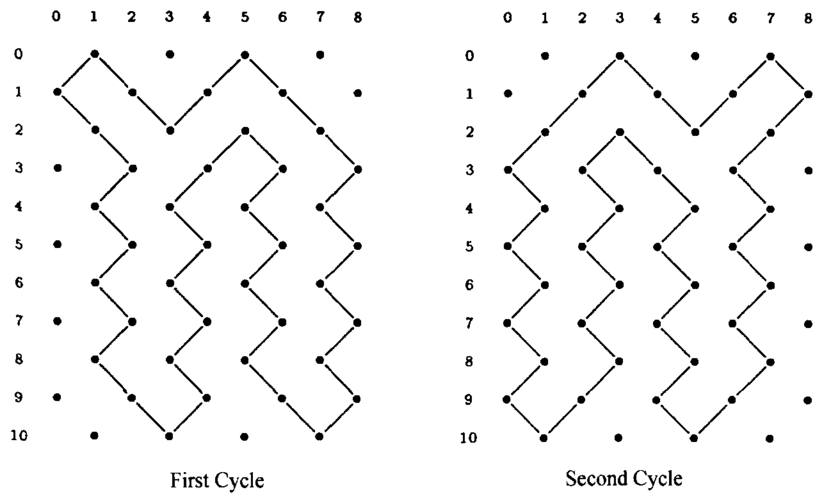


Fig. 3. Cycle decomposition of odd component of $P_{11} \times P_9$.

Proof. Fig. 3 contains a cycle decomposition of the odd component of $P_{11} \times P_9$ ($m = 10$ and $n = 8$). Pattern suggests the proof. □

Remark. The subgraphs (of the odd component of $P_{m+1} \times P_{n+1}$) induced by the two cycles traced in the proof of Lemma 2.1 are isomorphic. Each has $mn/2$ vertices and $mn - (m-1) - (n-1)$ edges.

Theorem 2.2. *Let m, n be even $\geqslant 4$, where $n \equiv 0 (\mathrm{mod}\, 4)$, and let r, s be even $\geqslant 4$ such that $r|m$, $s|n$ and $s \equiv 0 (\mathrm{mod}\, 4)$.*

1. *Each of the following graphs is decomposable into $2(m/r) \cdot (n/s)$ cycles, each of length $rs/2$:*
   (a) *the odd component of $P_{m+1} \times P_{n+1}$, and*
   (b) *each component of each of $C_m \times P_{n+1}$ and $P_{m+1} \times C_n$.*
2. *Each component of $C_m \times C_n$ has a 2-factorization in which each factor consists of $(m/r) \cdot (n/s)$ cycles, all of length $rs/2$.*

Proof. Let m, n, r and s be as stated, and note that (a) P_{m+1} is decomposable into m/r paths, each isomorphic to P_{r+1}, and (b) P_{n+1} is decomposable into n/s paths, each isomorphic to P_{s+1}.

Based on the proof of Lemma 2.1, let $\alpha_{0,0}$ and $\beta_{0,0}$ be the two cycles (each of length $rs/2$) which constitute a decomposition of the odd component of $P_{r+1} \times P_{s+1}$, and let $(a_0, b_0), \ldots, (a_{rs/2-1}, b_{rs/2-1})$ and $(c_0, d_0), \ldots, (c_{rs/2-1}, d_{rs/2-1})$ be the sequences which correspond to $\alpha_{0,0}$ and $\beta_{0,0}$, respectively.

Let $\alpha_{i,j}$ be the cycle given by the sequence

$$(a_0 + ir,\ b_0 + js), \ldots, (a_{rs/2-1} + ir,\ b_{rs/2-1} + js),$$

where $0 \leqslant i \leqslant m/r - 1$ and $0 \leqslant j \leqslant n/s - 1$.

Cycles $\alpha_{i,j}$ are each of length $rs/2$, are mutually vertex-disjoint, and span $mn/2$ vertices of the odd component of $P_{m+1} \times P_{n+1}$, which has a total of $(mn+m+n)/2$ vertices. The following may be noted with respect to $\{\alpha_{i,j}\}$:

- Each vertex of degree four appears on a unique cycle in this collection,
- among the pair of border vertices $(0,2j+1)$ and $(m,2j+1)$, one does not appear on any of these cycles while the other appears on a unique cycle, $0 \leqslant j \leqslant (n-2)/2$, and
- among the pair of border vertices $(2i+1,0)$ and $(2i+1,n)$, one does not appear on any of these cycles while the other appears on a unique cycle, $0 \leqslant i \leqslant (m-2)/2$.

The odd component of $C_m \times P_{n+1}$ is obtainable from the respective component of $P_{m+1} \times P_{n+1}$ as stated in Proposition 1.1(1). This process of construction is such that the cycles $\alpha_{i,j}$ traced in the odd component of $P_{m+1} \times P_{n+1}$ remain 'undisturbed'. An analogous statement holds with respect to the odd component of $P_{m+1} \times C_n$.

The odd component of $C_m \times C_n$ is obtainable from the respective component of $C_m \times P_{n+1}$ as stated in Proposition 1.1(2). Again, the collection of $(m/r)(n/s)$ cycles, derived from $\{\alpha_{i,j}\}$ continues to be one such collection with respect to the odd component of $C_m \times C_n$. Indeed, these cycles constitute a 2-factor of this component.

Next, let $\beta_{i,j}$ be the cycle given by the sequence

$$(c_0 + ir,\ d_0 + js), \ldots, (c_{rs/2-1} + ir,\ d_{rs/2-1} + js),$$

where $0 \leqslant i \leqslant m/r - 1$ and $0 \leqslant j \leqslant n/s - 1$.

Note that $\{\alpha_{i,j}\} \cup \{\beta_{i,j}\}$ constitutes a decomposition of the odd component of $P_{m+1} \times P_{n+1}$. Also, the line of argument presented earlier with respect to the collection $\{\alpha_{i,j}\}$ holds true with respect to $\{\beta_{i,j}\}$ as well.

Finally note that m and n being both even, the odd component of each of $C_m \times P_{n+1}$, $P_{m+1} \times C_n$ and $C_m \times C_n$ is isomorphic to the corresponding even component. □

Remark. The subgraph (of the odd component of $P_{m+1} \times P_{n+1}$) induced by the vertices on each cycle $\alpha_{i,j}$ (or $\beta_{i,j}$) traced in the proof of Theorem 2.2 has $rs/2$ vertices and $rs-(r-1)-(s-1)$ edges.

2.2. Graphs $P_{2i+1} \times P_{4j+3}$, $C_m \times P_{4j+3}$ and $C_m \times C_{4j+2}$

Lemma 2.3. *Let m and n be even $\geqslant 2$, where $n \equiv 2 \pmod 4$. The odd component of $P_{m+1} \times P_{n+1}$ is decomposable into two equal-length paths γ and δ such that*

1. *each vertex of degree four appears on γ as well as on δ,*
2. *the terminal vertices of each of γ and δ are $(0,1)$ and $(m,1)$,*
3. *among the border vertices $(2i+1,0)$ and $(2i+1,n)$, exactly one appears on γ and the other appears on δ, where $0 \leqslant i \leqslant m/2$, and*
4. *among the border vertices $(0,2j+1)$ and $(m,2j+1)$, exactly one appears on γ and the other appears on δ, where $1 \leqslant j \leqslant n/2$.*

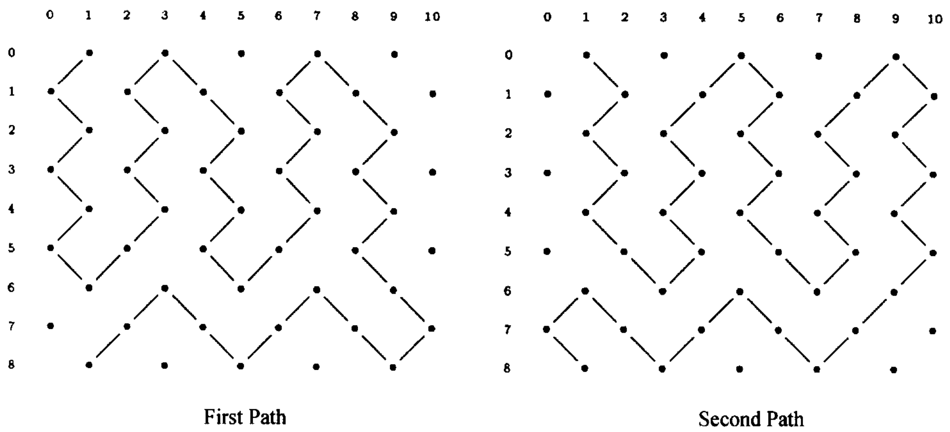


Fig. 4. Path decomposition of the odd component of $P_9 \times P_{11}$.

Proof. Fig. 4 contains a decomposition of the odd component of $P_9 \times P_{11}$ ($m = 8$ and $n = 10$). Proof is implicit in the pattern. □

Remark. Each of the subgraphs (of the odd component of $P_{m+1} \times P_{n+1}$) induced by the paths traced in the proof of Lemma 2.3 has $mn/2 + 1$ vertices and $mn - (m-1) - (n-2)$ edges.

Theorem 2.4. *Let m, n be even $\geqslant 4$, and let $s \geqslant 2$ be such that $s \equiv 2 \pmod 4$ and $s|n$.*

1. *The odd component of $P_{m+1} \times P_{n+1}$ is decomposable into $2n/s$ paths, all of length $ms/2$.*
2. *Each component of $C_m \times P_{n+1}$ is decomposable into $2n/s$ cycles, all of length $ms/2$.*
3. *Each component of $C_m \times C_n$ has a 2-factorization in which each factor consists of n/s cycles, all of length $ms/2$.*

Proof. The argument is somewhat similar to that in the proof of Theorem 2.2. Based on the proof of Lemma 2.3, let γ_0 and δ_0 be two paths (each of length $ms/2$) which constitute a decomposition of the odd component of $P_{m+1} \times P_{s+1}$, and let

$$(a_0, b_0), \ldots, (a_{ms/2}, b_{ms/2})$$

and

$$(c_0, d_0), \ldots, (c_{ms/2}, d_{ms/2})$$

be the sequences, which correspond to γ_0 and δ_0, respectively.

Let γ_j be the path given by the sequence

$$(a_0, b_0 + js), \ldots, (a_{ms/2},\ b_{ms/2} + js),$$

where $0 \leqslant j \leqslant n/s - 1$. Paths in the collection $\{\gamma_j\}$ are each of length $ms/2$, are mutually vertex-disjoint, and span $mn/2 + n/s$ vertices in the odd component of $P_{m+1} \times P_{n+1}$. The following observations are relevant with respect to $\{\gamma_j\}$:

1. Each vertex of degree four appears on a unique path in this collection,
2. Among the pair of border vertices $(2i+1, 0)$ and $(2i+1, n)$, one does not appear on any of these paths while the other appears on a unique path, $0 \leqslant i \leqslant (m-2)/2$, and
3. Among the pairs of border vertices $(0, 2j+1)$ and $(m, 2j+1)$, the pair of $(0, ks+1)$ and $(m, ks+1)$ appear as terminal vertices of γ_k while all of the remaining pairs have the characteristic similar to that mentioned in (2) above, where $0 \leqslant j \leqslant (n-2)/2$ and $0 \leqslant k \leqslant n/s - 1$.

The odd component of $C_m \times P_{n+1}$ is obtainable from the respective component of $P_{m+1} \times P_{n+1}$ as stated in Proposition 1.1(1). In this process, each path γ_k gets transformed into a cycle, say γ'_k, of the same length. The resulting collection $\{\gamma'_k\}$ consists of n/s vertex-disjoint cycles. Similarly, the odd component of $C_m \times C_n$ is obtainable from that of $C_m \times P_{n+1}$ as stated in Proposition 1.1(2), and the collection $\{\gamma'_k\}$ now corresponds to a 2-factor of this component.

Next, let δ_j be the path given by the sequence

$$(c_0, d_0 + js), \ldots, (c_{ms/2},\ d_{ms/2} + js),$$

where $0 \leqslant j \leqslant n/s - 1$. Note that $\{\gamma_j\} \cup \{\delta_j\}$ forms a decomposition of the odd component of $P_{m+1} \times P_{n+1}$. Also, the line of argument presented earlier with respect to γ_j applies to δ_j as well. Additional details are routine. □

2.3. *Graphs $P_{2i+1} \times P_{2j}$ and $C_m \times P_n$*

Consider a component of the graph $P_{2i+1} \times P_{2j}$. It has exactly two pendant vertices, and hence, cycle decomposition of this graph is not possible. Lemma 2.5 below shows that this graph has a decomposition into a cycle and a path.

Lemma 2.5. *Let m and s be even $\geqslant 4$.*

1. *If $s \equiv 0 \pmod 4$, then the even component of $P_{m+1} \times P_s$ has a decomposition into a cycle C of length $ms/2 - 2$ and a path P of length $ms/2 - m + 2$.*
2. *If $s \equiv 2 \pmod 4$, then the even component of $P_{m+1} \times P_s$ has a decomposition into a path P of length $ms/2 - 2$ and a cycle C of length $ms/2 - m + 2$.*

In each case, cycle C and path P satisfy the following conditions:

- *Each vertex of degree four appears on C as well as on P,*
- *the terminal vertices of P are $(0,0)$ and $(m,0)$ which are the pendant vertices of this graph, and*
- *among the border vertices $(0,2j)$ and $(m,2j)$, exactly one belongs to C and the other belongs to P, where $1 \leqslant j \leqslant (s-2)/2$.*

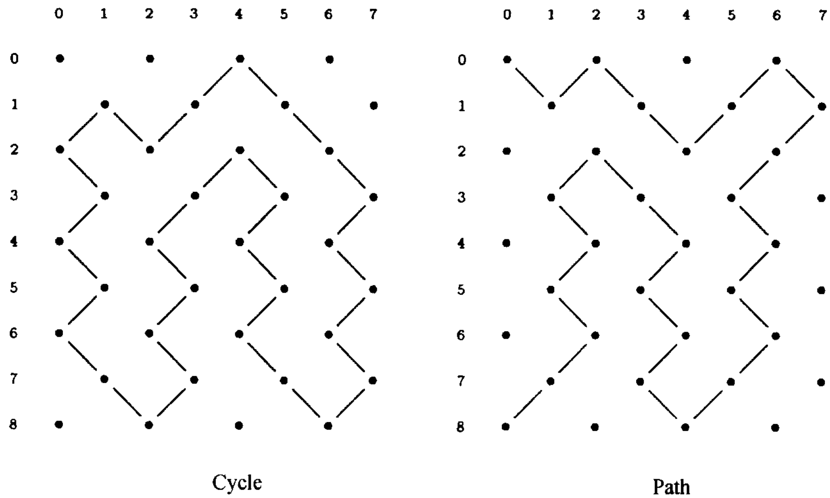


Fig. 5. Decomposition of the even component of $P_9 \times P_8$.

Proof. Argument for (1) is implicit in the decomposition of the even component of $P_9 \times P_8$ ($m = 8$ and $s = 8$) that appears in Fig. 5. Construction for (2) is similar. □

Corollary 2.6. *Let $m, n \geqslant 4$, where m is even, and let $s \geqslant 4$ be even such that $(s-1) \mid (n-1)$. Each component of $C_m \times P_n$ has a decomposition into a total of $2(n-1)/(s-1)$ cycles, where half of the cycles are of length $ms/2 - 2$ each, and the remaining cycles are of length $ms/2 - m + 2$ each.*

An application of Lemma 1.2 to Corollary 2.6 leads to a similar result with respect to $C_{2i+1} \times P_{2j}$.

2.4. Shortest odd cycles in $C_{2i+1} \times C_{2j+1}$

It is known that a shortest odd cycle in (the nonbipartite graph) $C_{2i+1} \times C_{2j+1}$ is of length $\max\{2i+1, 2j+1\}$. The present subsection consists of a theorem dealing with shortest odd cycles in this graph.

Theorem 2.7. *If m and n are both odd, $m \geqslant n$, then $C_m \times C_n$ has a 2-factorization in which each factor consists of n shortest odd cycles.*

Proof. Let σ_0 denote the shortest odd cycle of $C_m \times C_n$ given by the sequence $(0, b_0)$, $(1, b_1), \ldots, (m-1, b_{m-1})$, where $b_i = i$ for $0 \leqslant i \leqslant n-1$, and $b_i = (i+1) \bmod 2$ for $n \leqslant i \leqslant m-1$. For $i \in \{1, \ldots, n-1\}$, consider the sequence $(0, b_0 + i), (1, b_1 + i), \ldots, (m-1, b_{m-1} + i)$, where the sum $b_j + i$ is modulo n. This sequence corresponds to a shortest odd cycle, say σ_i, in $C_m \times C_n$. Further, the resulting cycles $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$ constitute a 2-factor of $C_m \times C_n$.

Next consider the sequence $(0,c_0),(1,c_1),\dots,(m-1,c_{m-1})$, where (i) $c_0=0$, (ii) for $1\leqslant j\leqslant n-1$, $c_j=n-j$, and (iii) for $n\leqslant j\leqslant m-1$, $c_j=0$, if j is odd, and $c_j=n-1$, if j is even. This sequence corresponds to a shortest odd cycle, say τ_0. Additional $n-1$ shortest odd cycles $\tau_1,\dots,\tau_{n-1}$ are obtainable from τ_0 in exactly the same way as $\sigma_1,\dots,\sigma_{n-1}$ have been obtained from σ_0. The cycles $\tau_0,\dots,\tau_{n-1}$ constitute another 2-factor of $C_m\times C_n$. Also, $\{\sigma_i\}\cup\{\tau_i\}$ forms a decomposition of $C_m\times C_n$. □

2.5. Four-cycles in $P_m\times P_n$, $C_m\times P_n$ and $C_m\times C_n$

In certain cases, there is a decomposition/factorization into four-cycles. This is being treated separately, since it is not covered by results of the previous subsections.

Theorem 2.8. *Let m,n be odd $\geqslant 3$, and let*

$$p=\lfloor (m+1)/4\rfloor\lfloor (n+1)/4\rfloor+\lfloor (m-1)/4\rfloor\lfloor (n-1)/4\rfloor,$$

$$q=\lfloor (m+1)/4\rfloor\lfloor (n-1)/4\rfloor+\lfloor (m-1)/4\rfloor\lfloor (n+1)/4\rfloor.$$

The odd component of $P_m\times P_n$ admits of a decomposition into four-cycles $\alpha_1,\dots,\alpha_p$, $\beta_1,\dots,\beta_q$ such that $\alpha_1,\dots,\alpha_p$ (resp. $\beta_1,\dots,\beta_q$) are mutually vertex-disjoint.

Proof. Let G denote the odd component of $P_m\times P_n$. Further, let S be the set

$$\{(4r,4s+1)\colon\ 0\leqslant r\leqslant\lfloor (m-3)/4\rfloor,\ 0\leqslant s\leqslant\lfloor (n-3)/4\rfloor\}$$
$$\cup\{(4r+2,4s+3)\colon\ 0\leqslant r\leqslant\lfloor (m-5)/4\rfloor,\ 0\leqslant s\leqslant\lfloor (n-5)/4\rfloor\},$$

and let S' be the set

$$\{(4r,4s+3)\colon\ 0\leqslant r\leqslant\lfloor (m-3)/4\rfloor,\ 0\leqslant s\leqslant\lfloor (m-5)/4\rfloor\}$$
$$\cup\{(4r+2,4s+1)\colon\ 0\leqslant r\leqslant\lfloor (m-5)/4\rfloor,\ 0\leqslant s\leqslant\lfloor (n-3)/4\rfloor\}.$$

It is easy to check that for each (a,b) in S, the set

$$\{(a,b),(a+1,b-1),(a+2,b),(a+1,b+1)\}$$

induces a four-cycle, say G_{ab}, in G, and if $(a,b)\neq(a',b')$, then G_{ab} and $G_{a'b'}$ are mutually vertex-disjoint. Analogous statement holds with respect to the set S'.

Note further that $S\cap S'=\emptyset$, and the four-cycles corresponding to $S\cup S'$ are mutually edge-disjoint. To conclude the proof, observe that $|S|=p$, $|S'|=q$ and $p+q=(m-1)(n-1)/4$, which is one-fourth of the number of edges in G. □

It is useful to note that if m and n are both odd, then the largest number of vertex-disjoint four-cycles in the odd component of $P_m\times P_n$ is exactly

$$\lfloor (m+1)/4\rfloor\lfloor (n+1)/4\rfloor+\lfloor (m-1)/4\rfloor\lfloor (n-1)/4\rfloor.$$

Because of the result of Theorem 2.8, it suffices to show that the number of vertex-disjoint four-cycles cannot exceed this figure. This can be argued by appropriately partitioning the set of border vertices into four subsets S_0, S_1, S_2 and S_3, and by showing that for each i, at least $\lfloor |S_i|/2 \rfloor$ vertices cannot participate in any collection of vertex-disjoint four-cycles of this graph.

Corollary 2.9. *Each component of each of $C_m \times P_{2j+1}$ and $C_m \times C_{2j}$ admits of a decomposition into four-cycles.*

The next result is analogous to Theorem 2.8.

Theorem 2.10. *Let m be even $\geqslant 4$, $n \geqslant 3$, and let $p = \lfloor m/4 \rfloor \lfloor (n-1)/2 \rfloor$.*
1. *Each component of $C_m \times P_n$ contains edge-disjoint four-cycles $\alpha_1, \ldots, \alpha_p$, $\beta_1, \ldots, \beta_p$ such that $\alpha_1, \ldots, \alpha_p$ (resp. $\beta_1, \ldots, \beta_p$) are mutually vertex-disjoint.*
2. *The largest number of vertex-disjoint four-cycles in each component of $C_m \times P_n$ is at most $\lfloor (mn/8) - \lfloor m/4 \rfloor /2 \rfloor$.*

Proof. Consider the odd component of $C_m \times P_n$, where m is even. Let S be the vertex subset

$$\{(4r, 4s+1)\colon 0 \leqslant r \leqslant \lfloor m/4 \rfloor - 1,\ 0 \leqslant s \leqslant \lfloor (n+1)/4 \rfloor - 1\}$$
$$\cup \{(4r+2, 4s+3)\colon 0 \leqslant r \leqslant \lfloor m/4 \rfloor - 1,\ 0 \leqslant s \leqslant \lfloor (n-1)/4 \rfloor - 1\},$$

and let S' be the vertex subset

$$\{(4r+2, 4s+1)\colon 0 \leqslant r \leqslant \lfloor m/4 \rfloor - 1,\ 0 \leqslant s \leqslant \lfloor (n+1)/4 \rfloor - 1\}$$
$$\cup \{(4r, 4s+3)\colon 0 \leqslant r \leqslant \lfloor m/4 \rfloor - 1, 0 \leqslant s \leqslant \lfloor (n-1)/4 \rfloor - 1\}.$$

Analogous to the proof of Theorem 2.8, the set S (resp. S') corresponds to a collection of vertex-disjoint four-cycles $\alpha_1, \ldots, \alpha_p$ (resp. $\beta_1, \ldots, \beta_p$) such that $\alpha_1, \ldots, \alpha_p$, $\beta_1, \ldots, \beta_p$ are mutually edge-disjoint, where $p = \lfloor m/4 \rfloor \lfloor (n-1)/2 \rfloor$.

For the upper bound on the number of vertex-disjoint four-cycles, the argument is somewhat similar to that preceding Corollary 2.9. □

Note that when $m \equiv 0 \pmod 4$ and n is odd, the lower bound and the upper bound on the number of vertex-disjoint four-cycles, appearing in the statement of Theorem 2.10, coincide and hence yield the exact value of $m(n-1)/8$.

Remark. By Lemma 1.2, if m is odd and $n \geqslant 3$, then a result with respect to (the connected graph) $C_m \times P_n$ holds simply by replacing '$\lfloor m/4 \rfloor \lfloor (n-1)/2 \rfloor$' by '$\lfloor m/2 \rfloor \lfloor (n-1)/2 \rfloor$,' and '$\lfloor (mn/8) - \lfloor m/4 \rfloor /2 \rfloor$' by '$\lfloor (mn/4) - \lfloor m/2 \rfloor /2 \rfloor$' in the statement of Theorem 2.10.

The next theorem deals with four-cycles in $C_{2i} \times C_{2j}$. (Proof is omitted.)

Theorem 2.11. *Let m and n be even, and let $p = 2\lfloor m/4 \rfloor \lfloor n/4 \rfloor$. Each component of $C_m \times C_n$ contains mutually edge-disjoint four-cycles $\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_p$ such that $\alpha_1, \dots, \alpha_p$ (resp. $\beta_1, \dots, \beta_p$) are mutually vertex-disjoint.*

It follows that each component of $C_{4i} \times C_{4j}$ has a 2-factorization into four-cycles.

Remark. By Lemma 1.2, if m is odd and n is even, then a result with respect to (the connected graph) $C_m \times C_n$ holds simply by replacing '$2\lfloor m/4 \rfloor \lfloor n/4 \rfloor$' by '$2\lfloor m/2 \rfloor \lfloor n/4 \rfloor$' in the statement of Theorem 2.11.

The final result of this subsection deals with four-cycles in $\times$-product of odd cycles. (Proof is omitted.)

Theorem 2.12. *Let m and n be odd, and $p = (m-1)(n-1)/4$. The graph $C_m \times C_n$ contains mutually edge-disjoint four-cycles $\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_p$ such that $\alpha_1, \dots, \alpha_p$ (resp. $\beta_1, \dots, \beta_p$) are mutually vertex-disjoint.*

Note. The lower bound on the largest number of vertex-disjoint four-cycles appearing in results 2.10, 2.11 and 2.12 appear in [5] also, and was established by the author toward an upper bound on the length of a longest induced cycle.

3. Bi-pancyclicity

The end result of this section is that each component of $C_m \times C_{4j}$ has a bi-pancyclic ordering. The method of attack is similar to that in Subsection 2.1.

Lemma 3.1. *If m is even $\geqslant 4$, then the odd component of $P_{m+1} \times P_5$ contains a subgraph G on $2m$ vertices such that G has a bi-pancyclic ordering. Vertices missed by G are $(0,3)$, $(m,1)$, $(1,4)$, and $(2i+1,0)$, $1 \leqslant i \leqslant (m-2)/2$.*

Proof. Let m be even $\geqslant 4$, and consider the following sequence $x_0, \dots, x_{2m-1}$ of vertices in the odd component of $P_{m+1} \times P_5$:

- $x_0 = (1,0)$, $x_1 = (2,1)$, $x_2 = (1,2)$, $x_3 = (0,1)$,
- $x_4 = (3,2)$, $x_5 = (2,3)$,
- $x_6 = (4,3)$, $x_7 = (3,4)$,
- $x_8 = (4,1)$, $x_9 = (5,2)$,
  $x_{10} = (6,3)$, $x_{11} = (5,4)$,
- $x_{4k} = (2k,1)$, $x_{4k+1} = (2k+1,2)$, $x_{4k+2} = (2k+2,3)$, $x_{4k+3} = (2k+1,4)$,
  where $3 \leqslant k \leqslant (m-2)/2$.

Let G be the (induced) subgraph $\langle x_0, \dots, x_{2m-1} \rangle$. It is claimed that for all $2 \leqslant i \leqslant m$, the subgraph $\langle x_0, \dots, x_{2i-1} \rangle$ contains a spanning cycle. Proof is by induction on i. Clearly $\langle x_0, x_1, x_2, x_3 \rangle$ induces a four-cycle. Also, each of $\langle x_0, \dots, x_5 \rangle$ and $\langle x_0, \dots, x_7 \rangle$ contains a (unique) spanning cycle.

Suppose that C is a spanning cycle of $\langle x_0,\ldots,x_{2i-1}\rangle$ for some $i \geqslant 4$. First assume that i is even. The vertex $x_{2i-2}=(i,3)$ is of degree two in $\langle x_0,\ldots,x_{2i-1}\rangle$. Therefore, the edge $(i-1,2)$–$(i,3)$ must appear on C. Now, $x_{2i}=(i,1)$ and $x_{2i+1}=(i+1,2)$. Cycle C may be extended by replacing the edge $(i-1,2)$–$(i,3)$ by the segment $(i-1,2)$–$(i,1)$–$(i+1,2)$–$(i,3)$. The resulting cycle is of length $|C|+2$, and spans all vertices of $\langle x_0,\ldots,x_{2i+1}\rangle$. Argument is similar for the case when i is odd.

Vertices missed by G are as claimed. □

Theorem 3.2. *If m and n are even $\geqslant 4$ and $n \equiv 0 \pmod 4$, then the odd component of $P_{m+1} \times P_{n+1}$ contains a subgraph G on $mn/2$ vertices such that G has a bi-pancyclic ordering. Vertices missed by G are*

(a) $(0,4s+3)$, $0 \leqslant s \leqslant (n-4)/4$,

(b) $(m,4s+1)$, $0 \leqslant s \leqslant (n-4)/4$,

(c) $(2r+1,0)$, $1 \leqslant r \leqslant (m-2)/2$, *and*

(d) $(1,n)$.

Proof. Let m and n be even $\geqslant 4$, where $n=4k$. Proof is by induction on k, the basis being immediate from Lemma 3.1.

Suppose that for some $n=4k$, the odd component of $P_{m+1} \times P_{n+1}$ contains a subgraph G on $mn/2$ vertices such that G has characteristics mentioned in the statement of the theorem. It will be shown that the odd component of $P_{m+1} \times P_{n+5}$ contains a subgraph on $m(n+4)/2$ vertices with similar properties.

Let $x_0,x_1,\ldots,x_{2m-1}$ be the sequence of vertices presented in the proof of Lemma 3.1, and suppose that $x_i=(a_i,b_i)$. Consider the sequence $y_0,y_1,\ldots,y_{2m-1}$ where $y_i=(a_i,b_i+n)$, $0 \leqslant i \leqslant 2m-1$. Note that $y_0=(1,n)$, $y_1=(2,n+1)$, $y_2=(1,n+2)$ and $y_3=(0,n+1)$. Let

$$G'=\langle V(G)\cup\{y_0,y_1,\ldots,y_{2m-1}\}\rangle.$$

G' is a subgraph of the odd component of $P_{m+1} \times P_{n+5}$ and $|V(G')|=m(n+4)/2$. In what follows, a bi-pancyclic ordering of G is extended to a similar ordering of G'.

Let $V(G)=\{w_0,w_1,\ldots,w_{mn/2-1}\}$, and let C be a spanning cycle of G. It is claimed that for all $j \in \{1,\ldots,m\}$, the induced subgraph

$$\langle w_0,w_1,\ldots,w_{mn/2-1},y_0,y_1,\ldots,y_{2j-1}\rangle$$

contains a spanning cycle.

First note that the vertex $(3,n)$ appears on G (hence on C) and is of degree two with respect to G. Therefore, the edge $(2,n-1)$–$(3,n)$ is necessarily a part of C. Now, this cycle may be extended to a cycle of length $|C|+2$ by replacing the edge $(2,n-1)$–$(3,n)$ by the segment $(2,n-1)$–y_0–y_1–$(3,n)$. Recall that $y_0=(1,n)$ and $y_1=(2,n+1)$, and hence, this cycle extension is valid. It follows that $\langle w_0,w_1,\ldots,w_{mn/2-1},y_0,y_1\rangle$ contains a spanning cycle. That

$$\langle w_0,w_1,\ldots,w_{mn/2-1},y_0,y_1,\ldots,y_{2j-1}\rangle$$

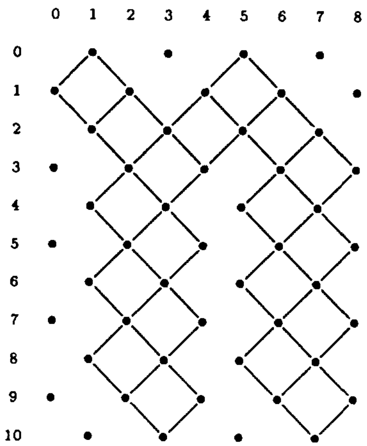


Fig. 6. Illustration of the proof of Theorem 3.2.

contains a spanning cycle for all $j \in \{2,\dots,m\}$ follows by an argument as in the proof of Lemma 3.1.

The reader may further verify that the vertices of $P_{m+1} \times P_{n+5}$ missed by G' are: (a) $(0,4s+3)$, $0 \leqslant s \leqslant n/4$, (b) $(m,4s+1)$, $0 \leqslant s \leqslant n/4$, (c) $(2r+1,0)$, $1 \leqslant r \leqslant (m-2)/2$, and (d) $(1,n+4)$. □

Example. Proof of Theorem 3.2 is illustrated in Fig. 6, where $m=10$ and $n=8$. Only relevant edges have been shown. The isolated vertices are those missed by G.

Letting G be the subgraph of the odd component of $P_{m+1} \times P_{n+1}$ (where $m=2i$ and $n=4j$) as in the statement of Theorem 3.2, it is easy to see that the vertices missed by G are such that an invocation of Proposition 1.1(1–2) to this theorem leads to analogous result with respect to a component of each of $C_m \times P_{n+1}$, $P_{m+1} \times C_n$ and $C_m \times C_n$.

Corollary 3.3. *Let m, n be even $\geqslant 4$, where $n \equiv 0 \pmod 4$.*
1. *Each component of each of $C_m \times P_{n+1}$ and $P_{m+1} \times C_n$ contains a subgraph on $mn/2$ vertices which admits of a bi-pancyclic ordering.*
2. *Each component of $C_m \times C_n$ has a bi-pancyclic ordering.*

An application of Lemma 1.2 to Corollary 3.3(2) yields analogous result with respect to $C_{2i+1} \times C_{4j}$.

Acknowledgements

I am thankful to Nischal Abrol and Apurv Mathur for useful discussions, and to Sandi Klavžar for encouragement and support. I am also grateful to (a) Delhi Institute of Technology, (b) Department of Science and Technology (Government of India), and (c) Council of Scientific and Industrial Research, New Delhi, for financial assistance which enabled me to attend the Third Slovenian International Conference on Graph Theory. Finally, I thank the referee for several constructive comments.

References

[1] B. Alspach, J.-C. Bermond, D. Sotteau, Decompositions into cycles I: Hamilton decompositions, in: G. Hahn et al. (Eds.), Cycles and Rays, Kluwer, The Netherlands, 1990, pp. 9–18.

[2] J.A. Bondy, Pancyclic graphs, J. Combin. Theory Ser B 11 (1971) 80–84.

[3] J. Bosak, Decompositions of Graphs, Kluwer, Dordrecht, 1991.

[4] P.K. Jha, Decompositions of the Kronecker product of a cycle and a path into long cycles and long paths, Ind. J. Pure Appl. Math. 23 (1992) 585–602.

[5] P.K. Jha, On edge-disjoint coils in the Kronecker product of a cycle and a path, J. Combin. Inform. System Sci. 19 (1994) 147–160.

[6] P.K. Jha, S. Klavžar, B. Zmazek, Isomorphic components of Kronecker product of bipartite graphs, 1994, Discuss Math-Graph Theory, to appear.

[7] M. Kouider, G. Sabidussi, Factorisations of 4-regular graphs and Petersen's theorem, J. Combin. Theory, Ser. B 63 (1995) 170–184.

[8] A. Muthusamy, P. Paulraja, Factorization of product graphs into cycles of uniform length, Graphs Combin. 11 (1995) 69–90.

[9] S. Ramachandran, R. Parvathy, Pancyclicity and extendability in strong products, J. Graph Theory 22 (1996) 75–82.

[10] E. Schmeichel, J. Mitchem, Bipartite graphs with cycles of all even length, J. Graph Theory 6 (1982) 429–439.

[11] H.-M. Teichert, On the pancyclicity of some product graphs, Elektron Informationsverarb u Kybernet, J. Inform. Process. Cybernet.-EIK 19 (1983) 345–356.

[12] M. Zhou, Decomposition of some product graphs into 1-factors and Hamiltonian cycles, Ars. Combin. 28 (1989) 258–268.