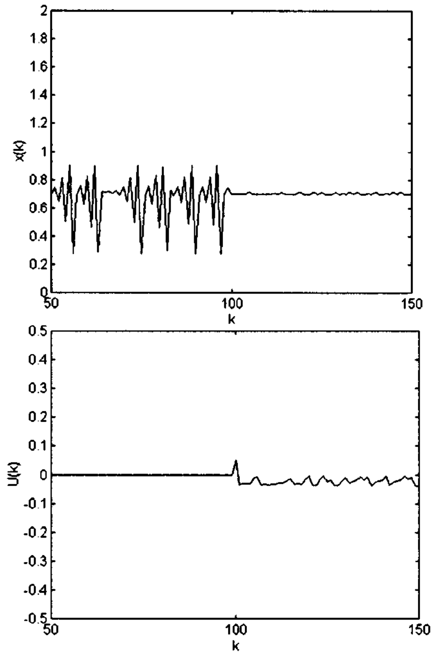

Fig. 6. (a) Time-response of trajectory in chaotic Sin mapping with noise: $X_s = 0.7$; no control when $n \leq 100$; with control when $n > 100$. (b) Control signal $U(k)$.

3) The coefficients in the model are determined in one pass through the data set: no iterative calculation is needed. The scheme is a highly parallel structure leading to efficient numerical computations. It is a fast learning algorithm that converges to the optimal predicting surface as the number of samples becomes large enough. In our approach, usually $20 \sim 30$ data pairs are enough for prediction. On the contrary, many existing standard fuzzy models discussed require very long training time in general, and often converges to a local minimum instead of the global minimum on the tracking error surface.
4) It reduces a complex problem to a least-squares estimation of only a small number of constant parameters. The estimate converges to the conditional mean predicting surface as more and more samples are obtained and used. It also yields a very reasonable prediction using only a few samples.
5) It is particularly suitable for sparse data in a real-time environment, because the predicting surface is instantly defined everywhere, even with just one available sample.
6) The structure of the entire system is simple, so a simulation program is easy to write and use.
7) The control signals are generally quite small due to the fast convergence of the tracking process.

The main disadvantage of this proposed fuzzy predictive modeling technique, without clustering, is that it requires substantial amount of computations (although in parallel) to evaluate new points. There are several ways to overcome this disadvantage; one is to use the clustering method [13], and another is to take advantage of the inherent parallel structure of this fuzzy model (two in combination will provide both high throughput and rapid adaptation).

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Optimal $L(2, 1)$-Labeling of Cartesian Products of Cycles, with an Application to Independent Domination

Pranava K. Jha

***Abstract*—The $L(2, 1)$-labeling of a graph is an abstraction of the problem of assigning (integer) frequencies to radio transmitters, such that transmitters that are “close”, receive different frequencies, and those that are “very close” receive frequencies that are further apart. The least span of frequencies in such a labeling is referred to as the λ-number of the graph. Let n be odd ≥ 5, $k = (n-3)/2$ and let $m_0, \cdots, m_{k-1}, m_k$ each be a multiple of n. It is shown that $\lambda(C_{m_0} \square \cdots \square C_{m_{k-1}})$ is equal to the theoretical minimum of $n - 1$, where C_r denotes a cycle of length r and “$\square$” denotes the Cartesian product of graphs. The scheme works for a vertex partition of $C_{m_0} \square \cdots \square C_{m_{k-1}} \square C_{m_k}$ into smallest (independent) dominating sets.**

***Index Terms*—Cartesian product, cycle, frequency assignment, graph theory, independent dominating set, $L(2, 1)$-labeling.**

I. Introduction

Consider the problem of assigning frequencies to the radio transmitters at various nodes in a region. The transmitters that are close must receive frequencies that are sufficiently apart, for otherwise they may be at the risk of interfering with each other. This problem was first placed on a graph-theoretical footing by Hale [1] in 1980. Subsequently, Roberts [2] proposed a variation, in which distinction is made between transmitters that are “close,” and those that are “very close.”

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The author is with is with the Faculty of Information Science and Technology, Multimedia University, 75450 Melaka, Malaysia (e-mail: pkjha@mmu.edu.my.).

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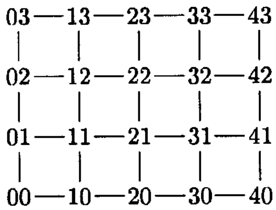

Fig. 1. Graph $P_5 \Box P_4$.

This enabled Griggs and Yeh [3] to formulate the $L(2,\,1)$-labeling of graphs, that has since been an object of extensive research [4]–[7].

Formally, an $L(2,\,1)$-labeling of a graph G is an assignment f of nonnegative integers to the vertices of G such that

$$|f(u) - f(v)| \geq \begin{cases} 2, & \text{if } d(u,\,v) = 1 \\ 1, & \text{if } d(u,\,v) = 2. \end{cases}$$

The difference between the largest label and the smallest label assigned by f is called the span of f, and the minimum span over all $L(2,\,1)$-labelings of G is called the λ-*number* of G, denoted by $\lambda(G)$. The general problem of determining $\lambda(G)$ is NP-hard [4]. Indeed, if the graph is known to be a tree, then there is an efficient solution [6].

By "graph", it is meant a finite, simple, undirected, and connected graph. The *Cartesian product* $G \Box H$ of graphs $G = (V,\,E)$ and $H = (W,\,F)$ is defined as follows: $V(G \Box H) = V \times W$ and $E(G \Box H) = \{\{(u,\,x),\,(v,\,y)\}$: either $u = v$ and $\{x,\,y\} \in F$ or $x = y$ and $\{u,\,v\} \in E\}$. This product (that is commutative and associative in a natural way) is one of the most important graph products, with potential applications. For example, the n-cube Q_n is easily seen to be the $\Box$-product of n copies of K_2. It is known that 1) $G \Box H$ is connected iff G and H are connected, and 2) $G \Box H$ is bipartite iff G and H are bipartite.

A vertex subset S of a graph G is said to be an *independent set* if elements of S are mutually nonadjacent in G. If every x not in S is adjacent to at least one element of S, then S is said to be a *dominating set*. An *independent dominating set* has an obvious definition. The general problem of obtaining an (independent) dominating set of smallest size is NP-hard [8].

There is a clear connection between independent sets and λ-numbering of a graph. In particular, if $\lambda(G) = n$, then there is a partition of $V(G)$ into at most $n+1$ independent sets.

For $m \geq 3$ and $n \geq 2$, let C_m denote the *cycle* on m vertices, and let P_n denote the *path* on n vertices, where $V(C_k) = V(P_k) = \{0,\,\cdots,\,k-1\}$ and where adjacencies are defined in the natural way. The graph $P_5 \Box P_4$ appears in Fig. 1. For simplicity, a vertex $(p,\,q)$ has been shown as pq. It is easy to see that the graph $C_m \Box C_n$ is obtainable from $P_m \Box P_n$ by introducing the edges $\{(0,\,j),\,(m-1,\,j)\}, 0 \leq j \leq n-1$, and the edges $\{(i,\,0),\,(i,\,n-1)\}, 0 \leq i \leq m-1$.

Lemma 1.1: (Griggs and Yeh [3]) Let G be a graph with maximum degree $\Delta \geq 2$. If G contains three vertices of degree Δ such that one of them is adjacent to the other two, then $\lambda(G) \geq \Delta + 2$. ■

The lower bound of Lemma 1.1 is achievable in certain cases. This is particularly so with respect to $P_{m_0} \Box \cdots \Box P_{m_{k-1}}$, where each $m_i \geq 3$, and either 1) $m_j \geq 5$ for some j, or 2) $m_p = m_q = 4$ for some $p,\,q$ where $p \neq q$ [5].

Let n be odd ≥ 5, $k = (n-3)/2$ and let $m_0,\,\cdots,\,m_{k-1},\,m_k$ each be a multiple of n. The central result of this paper is that $\lambda(C_{m_0} \Box \cdots \Box C_{m_{k-1}})$ is equal to the theoretical minimum of $n-1$. The method of attack leads to a vertex partition of $C_{m_0} \Box \cdots \Box C_{m_{k-1}} \Box C_{m_k}$ into smallest (independent) dominating sets. Among other things, the latter result constitutes a simple proof of a theorem of Klavžar and Seifter [9] relating to the domination number of $C_{m_0} \Box \cdots \Box C_{m_{k-1}} \Box C_{m_k}$. Finally, it is shown that the graphs $C_{m_0} \Box \cdots \Box C_{m_{k-1}}$ and $C_{m_0} \Box \cdots \Box C_{m_{k-1}} \Box C_{m_k}$ admit optimal labeling with respect to certain other distance-two parameters also.

Remark: For $m_0,\,\cdots,\,m_{k-1} \geq 3$, the graph $C_{m_0} \Box \cdots \Box C_{m_{k-1}}$ is edge-decomposable into k Hamiltonian cycles [10], and hence it has high fault tolerance with respect to node failure and edge failure. Also, the shortest distance between vertices $(u_0,\,\cdots,\,u_{k-1})$ and $(w_0,\,\cdots,\,w_{k-1})$ of this graph is given by the simple formula of $\sum_{i=0}^{k-1} d(u_i,\,w_i)$, where $d(u_i,\,w_i)$ denotes the shortest distance between u_i and w_i in C_{m_i}, $0 \leq i \leq k-1$. Accordingly, this graph is amenable to application as an interconnection network in computer science and telecommunication engineering. In particular, the toroidal network used in multiprocessor architecture is representable as the Cartesian product of two cycles.

II. Main Result

Theorem 2.1: Let n be odd ≥ 5, and let $k = (n-3)/2$. If $m_0,\,\cdots,\,m_{k-1}$ are each a multiple of n, then $\lambda(C_{m_0} \Box \cdots \Box C_{m_{k-1}}) = n-1$.

Proof: Let G denote $C_{m_0} \Box \cdots \Box C_{m_{k-1}}$, and note that G is a regular graph of degree $2k = n-3$. By Lemma 1.1, $\lambda(G) \geq n-1$, so it suffices to present an $L(2,\,1)$-labeling of G using the labels $0,\,\cdots,\,n-1$.

Let a vertex $v = (v_0,\,\cdots,\,v_{k-1})$ of G be assigned the integer

$$f(v) = \left[\sum_{i=0}^{k-1} 2(i+1) \cdot v_i\right] \bmod n,$$

where $0 \leq v_i \leq m_i - 1$. The assignment is clearly well-defined. Also, it corresponds to a desired $L(2,\,1)$-labeling for the case when $n = 5$. In what follows, let n be odd ≥ 7, and let $u = (u_0,\,\cdots,\,u_{k-1})$ and $w = (w_0,\,\cdots,\,w_{k-1})$ be two distinct vertices of G.

Case 1: $d(u,\,w) = 1$: There exists some i such that 1) $|u_i - w_i| = 1$ or $m_i - 1$, and 2) $u_j = w_j$, $j \neq i$. If $|u_i - w_i| = 1$, then the label assigned to one of u and w is of the form $N \bmod n$, and that to the other is of the form $[N + 2(i+1)] \bmod n$, where N is some nonnegative integer. Since $2(i+1)$ is between 2 and $n-3$, $|f(u) - f(w)|$ is equal to $2(i+1)$ or $n - 2(i+1)$, each of which is between 2 and $n-2$.

If $|u_i - w_i| = m_i - 1$, then assume without loss of generality that $u_i = 0$ and $w_i = m_i - 1 = pn - 1$ (say). Here $f(u)$ may be written as $N \bmod n$ while $f(w)$ may be written as $[N + 2(i+1) \cdot (pn-1)] \bmod n = [N - 2(i+1)] \bmod n$. Again, $|f(u) - f(w)|$ is between 2 and $n-2$.

Case 2: $d(u,\,w) = 2$: Here the k-tuple representations of u and w will differ by precisely 2 in one position, or by precisely 1 in two positions. If the former condition holds, then 1) $|u_i - w_i| = 2$ or $m_i - 2$, and 2) $u_j = w_j$, $j \neq i$. In case $|u_i - w_i| = 2$, one of $f(u)$ and $f(w)$ will be of the form $N \bmod n$ while the other will be of the form $[N + 4(i+1)] \bmod n$. Since n is odd ≥ 7 and $1 \leq i+1 \leq (n-3)/2$, it is easy to see that $f(u) \neq f(w)$. A similar conclusion is reached for the case when $|u_i - w_i| = m_i - 2 = pn - 2$ (say).

Next examine the case when u and w differ in precisely two positions, say i and j, such that 1) $u_i,\,w_i$ are adjacent in C_{m_i}, 2) $u_j,\,w_j$ are adjacent in C_{m_j}, where $i \neq j$, and 3) $u_l = w_l$, $l \neq i,\,j$. It suffices to consider the following four subcases.

Subcase 2.1: $w_i = u_i + 1$ and $w_j = u_j + 1$: Here $f(u)$ may be written as $N \bmod n$, and $f(w)$ may be written as $[N + 2(i+1) + 2(j+1)] \bmod n = [N + 2(i+j+2)] \bmod n$. Since n is odd and $2 \leq i+j+2 \leq n-3$, it follows that $f(u) \neq f(w)$.

Subcase 2.2: $w_i = u_i + 1$ and $w_j = u_j - 1$: Here $f(u)$ may be written as $N \bmod n$, and $f(w)$ may be written as $[N + 2(i+1) - 2(j+1)] \bmod n = [N + 2(i-j)] \bmod n$. Since $i \neq j$ and $|i-j| \leq \max\{i,\,j\} \leq (n-5)/2$, it follows that $f(u) \neq f(w)$.

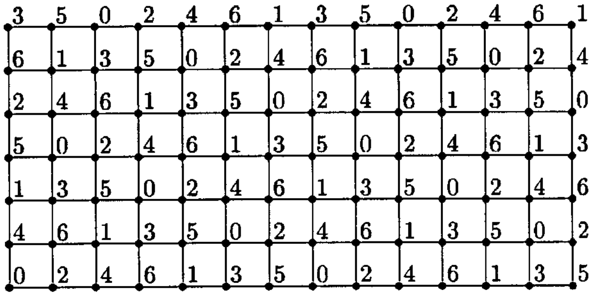

Fig. 2. $L(2, 1)$-labeling of $P_{14}\square P_7$ toward that of $C_{14}\square C_7$.

Subcase 2.3: ($u_i = 0, w_i = m_i - 1$) and ($u_j = 0, w_j = m_j - 1$): Here $f(u)$ may be written as $N \bmod n$, and $f(w)$ may be written as $[N + 2(i+1)(m_i - 1) + 2(j+1)(m_j - 1)] \bmod n = [N - 2(i+1) - 2(j+1)] \bmod n$, since each of m_i and m_j is a multiple of n. Now, n is odd and $2 \le i + j + 2 \le n - 3$, hence $f(u) \ne f(w)$.

Subcase 2.4: ($u_i = 0, w_i = m_i - 1$) and ($u_j = m_j - 1, w_j = 0$): Here $f(u)$ may be written as $[N + 2(j+1)(m_j - 1)]\, mod\, n = [N - 2(j+1)] \bmod n$, and $f(w)$ may be written as $[N + 2(i+1)(m_i - 1)] \bmod n = [N - 2(i+1)] \bmod n$. Since $i \ne j$ and each of $i+1$ and $j+1$ is between 1 and $(n-3)/2$, it follows that $f(u) \ne f(w)$. ∎

Example: For $n = 7$, the $L(2, 1)$-labeling of $C_{14}\square C_7$ based on the proof of Theorem 2.1 is illustrated in Fig. 2 by means of the labeling of $P_{14}\square P_7$. (Recall the statement preceding Lemma 1.1.)

Corollary 2.2: If a vertex v of $C_{m_0}\square \cdots \square C_{m_{k-1}}$ receives label j in the proof of Theorem 2.1, then those adjacent to v receive labels from $\{0, \cdots, n-1\} \setminus \{j-1, j+1\}$, where $j - 1$ and $j + 1$ are each modulo n.

Proof: It suffices to prove that if a vertex is labeled 0, then none of its neighbors is labeled $n - 1$. Let $v = (v_0, \cdots, v_{k-1})$ be one such vertex in which case labels of the neighbors of v are from

$$\begin{aligned}&\{[2(i+1)] \bmod n: 0 \le i \le k-1\}\\ &\quad \cup \{[-2(i+1)] \bmod n: 0 \le i \le k-1\}\\ &\quad = \{2j: 1 \le j \le (n-3)/2\} \cup \{2j+1: 1 \le i \le (n-3)/2\}\\ &\quad = \{2, \cdots, n-2\}.\end{aligned}$$ ∎

Corollary 2.3: For $0 \le j \le n - 1$, let V_j denote the set of vertices of $C_{m_0}\square \cdots \square C_{m_{k-1}}$ that receive label j in the proof of Theorem 2.1.

1) $V_0, \cdots, V_{n-1}$ form a vertex partition of the graph into equal-size independent sets.
2) Each V_j dominates a total of $(n - 2/n) \cdot |V|$ vertices (including those in V_j itself) where V denotes the vertex set of the graph. ∎

The sets $V_0, \cdots, V_{n-1}$ in Corollary 2.3 are such that 1) elements of each V_j correspond to as many vertex-disjoint $K_{1, n-3}$'s, and 2) for $0 \le i \le (n-3)/2$, elements of each $(V_{2i} \cup V_{2i+1})$ correspond to as many edge-disjoint $K_{1, n-3}$'s.

III. Independent Domination in Products of Cycles

Klavžar and Seifter [9] proved that if n is odd ≥ 3, $p = (n-1)/2$, and $m_0, \cdots, m_{p-1}$ are each a multiple of n, then a smallest dominating set of $C_{m_0}\square \cdots \square C_{m_{p-1}}$ is of cardinality $(1/n)|V|$ where V is the vertex set of the graph. It turns out that a minor change in the statement and proof of Theorem 2.1 leads to the foregoing result in a much simpler way. In fact, there is a vertex partition of the graph into n such sets.

Theorem 3.1: Let n be odd ≥ 3, and let $p = (n-1)/2$. If $m_0, \cdots, m_{p-1}$ are each a multiple of n, then the graph $C_{m_0}\square \cdots \square C_{m_{p-1}}$ admits of a vertex partition into smallest (independent) dominating sets.

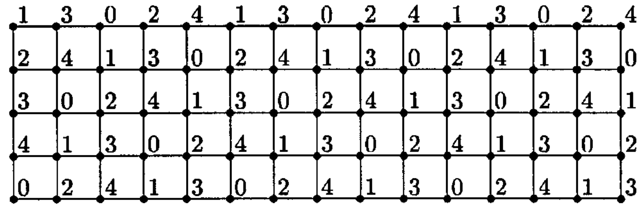

Fig. 3. Labeling of $P_{15}\square P_5$ toward that of $C_{15}\square C_5$ (cf. Theorem 3.1).

Proof: Let G denote $C_{m_0}\square \cdots \square C_{m_{p-1}}$, and note that G is a regular graph of degree $2p = n - 1$. Therefore, an (independent) dominating set of G must include at least $(1/n)$th of the vertices, so it suffices to present a vertex partition of G into equal-size sets $V_0, \cdots, V_{n-1}$ such that the distance between any two distinct elements of each V_j is at least three. The claim is trivially true for $n = 3$. In what follows, let n be odd ≥ 5.

Let a vertex $v = (v_0, \cdots, v_{p-1})$ of G be assigned the integer

$$\phi(v) = \left[\sum_{i=0}^{p-1} 2(i+1) \cdot v_i\right] \bmod n.$$

The reader may check to see that the argument in the proof of Theorem 2.1 [with "$k = (n-3)/2$" replaced by "$p = (n-1)/2$"] works toward the following: If u and w are vertices such that $1 \le d(u, w) \le 2$, then $\phi(u) \ne \phi(w)$.

Let V_j be the set of vertices of G that receive label j, where $0 \le j \le n - 1$. The resulting $V_0, \cdots, V_{n-1}$ constitute a vertex partition of G into smallest (independent) dominating sets. ∎

Example: For $n = 5$, a vertex partition of $C_{15}\square C_5$ based on the proof of Theorem 3.1 is illustrated in Fig. 3 by means of the labeling of $P_{15}\square P_5$.

IV. Related Parameters

It is shown here that the products of cycles (mentioned in the previous sections) admit optimal labelings, with respect to certain other distance-two parameters, also.

- An $L(1, 1)$*-labeling of a graph* G is an assignment of nonnegative integers such that vertices u, v receive different labels if $1 \le d(u, v) \le 2$. The least span of an $L(1, 1)$-labeling of G is denoted by $\lambda_0(G)$ [7].
- A *consecutive (no-hole)* $L(2, 1)$*-labeling of a graph* G is an $L(2, 1)$-labeling such that the labels used are consecutive. The least span of such a labeling (if one exists) is denoted by $\lambda_c(G)$ [11].
- A *circular distance-two labeling of a graph* G is an assignment g of integers $0, \cdots, k-1$ (for some k) to the vertices of G such that

$$|g(u) - g(v)|_k \ge \begin{cases} 2, & \text{if } d(u, v) = 1 \\ 1, & \text{if } d(u, v) = 2 \end{cases}$$

where $|x|_k := \min\{|x|, k - |x|\}$ is the circular difference modulo k. The least k for which G has a circular distance-two labeling is denoted by $\sigma(G)$ [12].

Each of the foregoing parameters is meaningful in its own way. It is not difficult to see that $\lambda_0(G) \le \lambda(G) \le 2\lambda_0(G)$ and $\lambda(G) + 1 \le \sigma(G) \le \lambda(G) + 2$ [7].

Theorem 4.1: Let n be odd, $k = (n-3)/2$, and $m_0, \cdots, m_k$ each a multiple of n.

1) $\lambda_0(C_{m_0}\square \cdots \square C_{m_{k-1}}\square C_{m_k}) = n - 1$, where $n \ge 3$.
2) $\lambda_c(C_{m_0}\square \cdots \square C_{m_{k-1}}) = n - 1$, where $n \ge 5$.
3) $\sigma(C_{m_0}\square \cdots \square C_{m_{k-1}}) = n$, where $n \ge 5$.

Proof: For 1), recall the assignment ϕ in the proof of Theorem 3.1. For 2), recall that the labels used in the optimal labeling of $C_{m_0}\Box\cdots\Box C_{m_{k-1}}$ in the proof of Theorem 2.1 are consecutive. For 3), recall the statement and proof of Corollary 2.2. ■

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On Stability of Relaxive Systems Described by Polynomials with Time-Variant Coefficients

Danilo P. Mandic and Jonathon A. Chambers

***Abstract*—The problem of global asymptotic stability (GAS) of a time-variant m-th order difference equation**

$$\boldsymbol{y}(n) = \boldsymbol{a}^T(n)\boldsymbol{y}(n-1) = a_1(n)y(n-1) + \cdots + a_m(n)y(n-m)$$

for $\|\boldsymbol{a}(n)\|_1 < 1$ was addressed in [1], whereas the case $\|\boldsymbol{a}(n)\|_1 = 1$ has been left as an open question. Here, we impose the condition of convexity on the set $\mathcal{C}_0$ of the initial values $\boldsymbol{y}(n) = [y(n-1), \cdots, y(n-m)]^T \in \mathbb{R}^m$ and on the set $\mathcal{A} \in \mathbb{R}^m$ of all allowable values of $\boldsymbol{a}(n) = [a_1(n), \cdots, a_m(n)]^T$, and derive the results from [1] for $a_i \geq 0, i = 1, \cdots, n$, as a pure consequence of convexity of the sets $\mathcal{C}_0$ and $\mathcal{A}$. Based upon convexity and the fixed-point iteration (FPI) technique, further GAS results for both $\|\boldsymbol{a}(n)\|_1 < 1$, and $\|\boldsymbol{a}(n)\|_1 = 1$ are derived. The issues of convergence in norm, and geometric convergence are tackled.

***Index Terms*—Contraction mapping, convergence, fixed-point iteration, global asymptotic stability, linear systems, relaxation.**

I. INTRODUCTION

The issue of global asymptotic stability (GAS) of

$$\boldsymbol{y}(n) = \boldsymbol{a}^T(n)\boldsymbol{y}(n-1) = a_1(n)y(n-1) + \cdots + a_m(n)y(n-m) \tag{1}$$

is important in the theory of linear systems [2]–[4]. Equation (1) represents an autonomous system, which under certain conditions converges. Actually, it is a relaxation equation, which stems from a general linear system

$$\boldsymbol{Y}(n+1) = \boldsymbol{A}(n)\boldsymbol{Y}(n) + \boldsymbol{B}(n)\boldsymbol{u}(n) \tag{2}$$

for the zero exogenous input vector $\boldsymbol{u}(n) = \mathbf{o}, \forall n$ [2], [4]. Equation (1) can be further written in the state-space form as

$$\begin{bmatrix} y(n+1) \\ y(n) \\ \vdots \\ y(n-m+1) \end{bmatrix} = \begin{bmatrix} a_1(n) & a_2(n) & \cdots & a_m(n) \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} y(n) \\ y(n-1) \\ \vdots \\ y(n-m) \end{bmatrix} \tag{3}$$

with $y(n+1) = [1\ 0\ \cdots\ 0]\boldsymbol{Y}(n+1)$. Matrix $\boldsymbol{A}$, where the index "n" is dropped for convenience, is a Frobenius matrix, which is a special form of the companion matrix of the characteristic polynomial [5], [6]. Namely, let us denote the characteristic equation of a general matrix $\boldsymbol{M}$ by $(-1)^n[\lambda^n - p_n\lambda^{n-1} - \cdots - p_0] = 0$, then, the characteristic equation of $\boldsymbol{A}$ (3) is identical to the characteristic equation of $\boldsymbol{M}$, and the matrix $\boldsymbol{A}$ is called the companion matrix of the characteristic polynomial of $\boldsymbol{M}$. Since $\boldsymbol{M}$ and $\boldsymbol{A}$ have the same characteristic polynomial, it

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D. P. Mandic is with the School of Information Systems, University of East Anglia, Norwich, U.K.
J. A. Chambers is with the Department of Electrical and Electronic Engineering, Imperial College of Science, Technology and Medicine, London, U.K.
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