Lecture 6

Nuclear energy generation

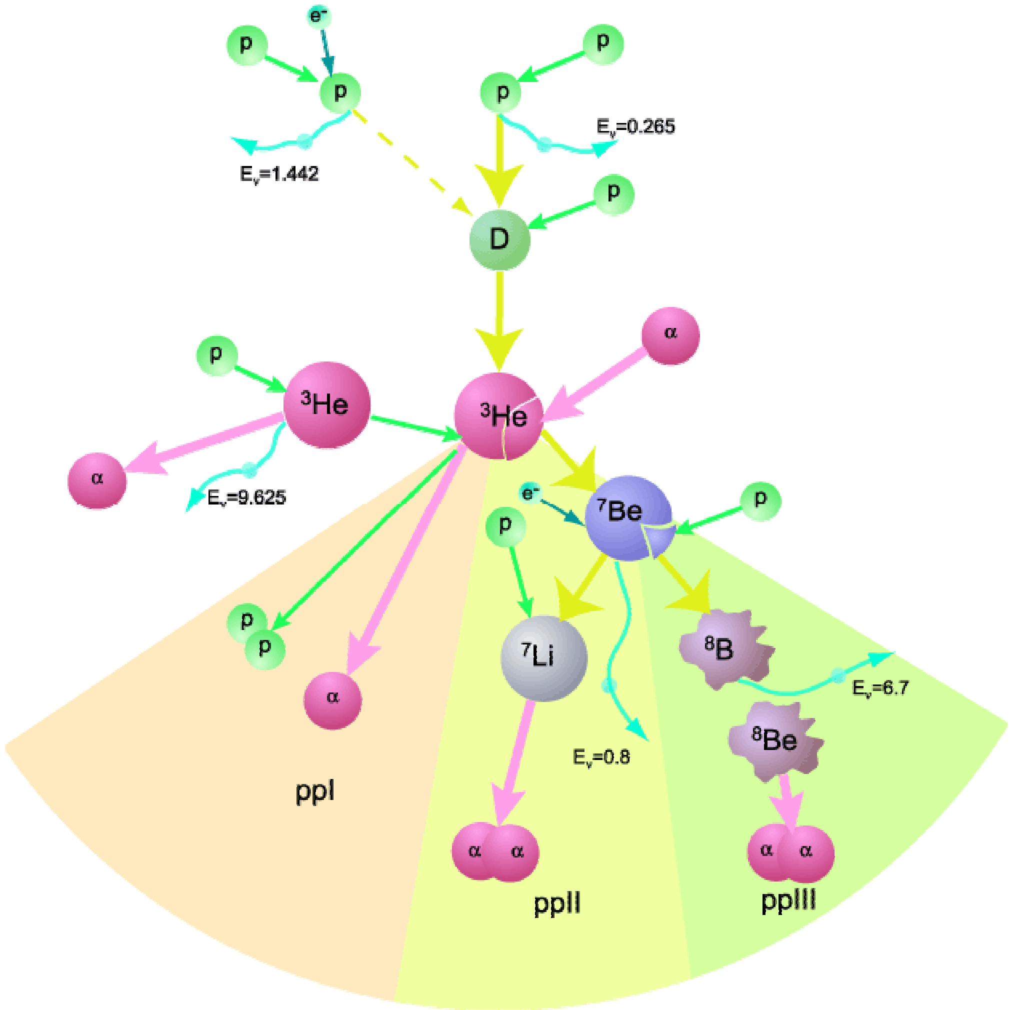

Figure 6.1. The p-p chain.

During most of their lifetimes, stars derive the energy which they radiate from nuclear reactions. The gradual change in chemical composition as the reactions proceed determine the evolution of the stars. Hence, to follow the life history of a star, it is important to understand the properties of the nuclear reactions.

The first goal is to compute the rate ε of energy generation per unit mass, which determines the stellar luminosity L as

$$L = \int_M \varepsilon\, dm = \int_V \varepsilon \rho\, dv = \int_0^R 4\pi r^2 \varepsilon \rho\, dr.$$

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An additional consequence of the nuclear processes is a gradual change of the chemical composition, which controls the evolution of the star. Hence we must also determine the rate of change of the abundances. The computation can be separated into three parts: the cross section for a reaction between a pair of nuclei, which is determined predominantly by the properties of the nuclei; the amount of energy generated per reaction, which again is a property of the nuclei; and the total reaction rate which, besides the cross section, also depends on the statistics of the motion of the nuclei.

In this Lecture, we will not go into details of these computations, which constitute an extensive and ongoing research effort. We will only describe qualitatively the most important thermonuclear reactions in stars, and consider some simple estimates for the energy generation rate ε which result from these computations.

6.1. Barrier penetration

The reaction between nuclei is caused by the *strong force* acting between nucleons (protons and neutrons). The range of the strong force is essentially limited to the extent of the nucleus; hence, for a reaction to occur the nuclei must be brought so close together that they essentially tauch, and this requires that the Coulomb repulsion between them must be overcome. Hence the potential for a reaction is as indicated in Figure 6.2.

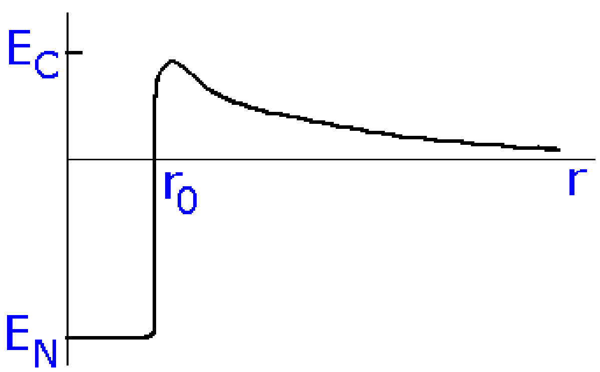

Figure 6.2. Schematic potential energy between two nuclei. For $r<r_0$ the attractive nuclear forces dominate; for $r>r_0$ Coulomb repulsion dominates.

The magnitude of the difficulty in achieving a reaction may be appreciated by noting that the height of the Coulomb barrier at the surface of the nucleus, corresponding to a typical radius of $r_0 \approx 10^{-13}\,\mathrm{cm}$, is

$$E_C \approx \frac{Z_1 Z_2 e^2}{r_0} \approx Z_1 Z_2\ \mathrm{MeV},$$

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(6.2)

where Z1 and Z2 are the atomic numbers of the nuclei taking part in the reaction, and e is the electron charge. This is consistent with the fact that typical nuclear energies are in the MeV range (1 eV = 1.60×10^{-19} J). In contrast, the average kinetic energy of the nuclei is

$$\langle E_{kin} \rangle = \frac{3}{2}kT \approx 130\left(\frac{T}{10^6 K}\right) \mathrm{eV}$$

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(6.3)

(cf equation 3.3). Since typical temperatures in the cores of hydrogen-burning stars are $1\text{-}2\times10^7$K, the average kinetic energy is roughly three orders of magnitude smaller than the energy required to overcome the potential barrier. Even taking into account the distribution of energies, essentially no reactions would be possible within the framework of classical mechanics.

What makes reactions, and hence ultimately our existence, possible is that according to quantum mechanics, there is a finite probability that the nuclei may tunnel through the barrier and react. Even so, the extent of the barrier means that the probability that the nuclei penetrate the potential barrier is small. Thus, in fact the nuclear burning in stellar interiors is generally a very slow process.

The energy produced in the reactions is released as kinetic energy of the particles which result, as well as in some cases in the form of γ photons. This energy is redistributed among the other constituents of the gas through collisions, and through absorption of the photons. As a result of the assumption of thermodynamic equilibrium in the gas, the details of this redistribution process is irrelevant; all that matters is the total amount of heat that is added to the gas. An exception to this general statement results in the cases where *neutrinos* are emitted in the reaction. Because of their extremely small interaction cross section, in almost all cases these do not react with other particles in the gas but escape directly from the star.

The total energy generation rate is obviously the sum over all possible reactions. Similarly, to evaluate the total rate of change of the abundance of a given element, we must take into account both the reactions that destroy and the reactions that create the element. Thus the computation of ε and the evolution of the chemical composition require careful consideration of the possible reaction network.

6.2. Hydrogen burning

During most of their life, stars derive their energy from the fusion of hydrogen into helium. This reaction may be written schematically as

$$4\,{}^{1}H \rightarrow {}^{4}He + 2e^{+} + 2\nu_{e}.$$

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$$4\,{}^{1}H \rightarrow {}^{4}He + 2e^{+} + 2\nu_{e}. \qquad (6.4)$$

The release of the two positrons is required to maintain charge balance, and results from the conversion of two protons into neutrons. Due to the requirement of lepton number conservation, the emission of two anti-leptons (the positrons) must be balanced by the emission of two leptons, the two electron neutrinos.

It is obvious that the reaction does not take place as indicated in equation (6.4): the probability that four protons come together and react at one point is entirely negligible. Instead the reaction may proceed through a number of different paths, which we discuss in more detail below. Regardless of these details, we can determine the total amount of energy liberated in the reaction (6.4), using the equivalence between mass and energy, from the difference in mass between the particles entering on the two sides of the reaction. The result is

$$Q_{tot} = 26.73\,\text{MeV} \qquad \text{for hydrogen burning.}$$

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In calculating accurately the energy generation rate ε, we must subtract the neutrino energy, since neutrinos escape directly from the star. This correction, however, depends on the detailed reactions in which the neutrinos are produced.

In principle, all possible reactions between the constituents of the gas must be considered, to determine which reactions dominate. The outcome is that there are two basically different ways (each with some variations) in which the overall reaction (6.4) may be accomplished: one (the PP-chains) which directly involves fusion of protons, the second (the CNO-cycle) in which the fusion occurs through a sequence of reactions involving C, N and O, which effectively act as catalysts.

The PP-chains

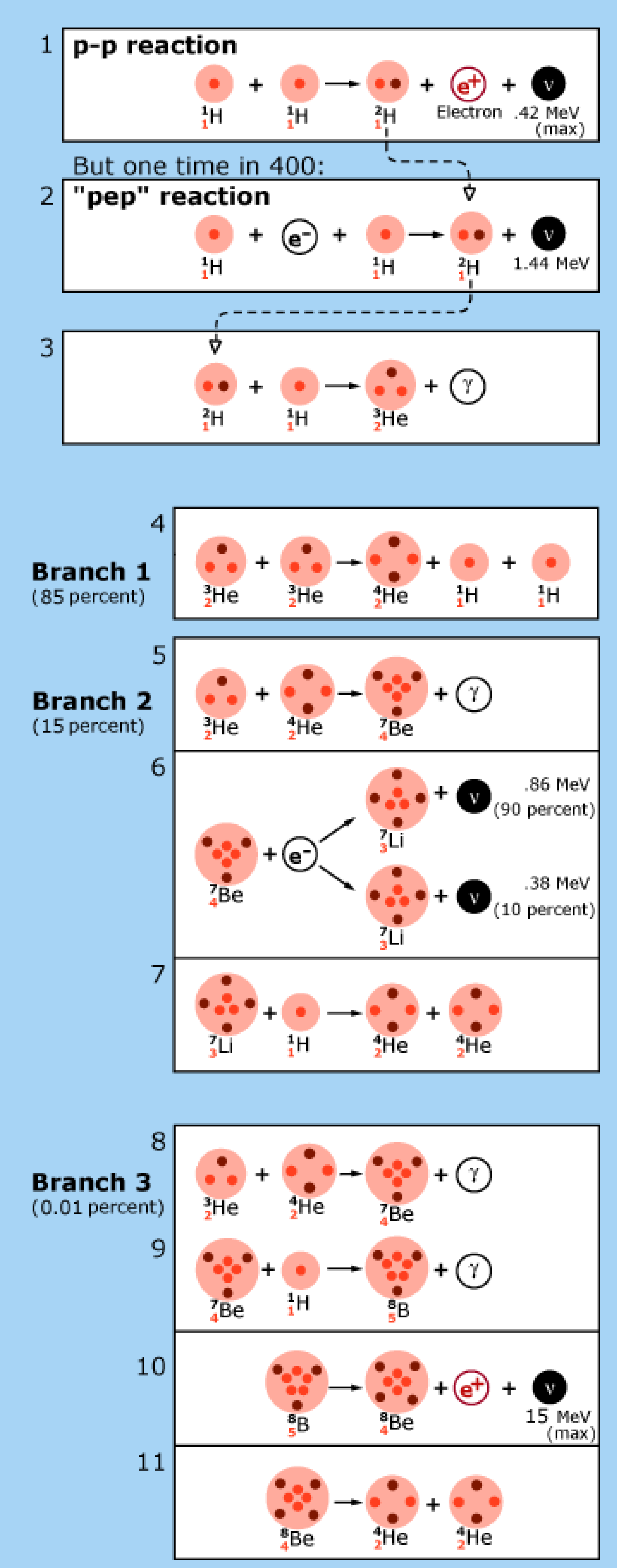
1
p-p reaction
1H
1H
2H
Electron
.42 MeV (max)
But one time in 400:
2
"pep" reaction
e-
1H
1H
2H
1.44 MeV
3
2H
1H
3He
4
Branch 1
(85 percent)
3He
3He
4He
1H
1H
5
Branch 2
(15 percent)
3He
4He
7Be
6
7Be
e-
7Li
.86 MeV (90 percent)
7Li
.38 MeV (10 percent)
7
7Li
1H
4He
4He
8
Branch 3
(0.01 percent)
3He
4He
7Be
9
7Be
1H
8B
10
8B
8Be
15 MeV (max)
11
8Be
4He
4He

Figure 6.3. The PP-chain. The relative frequencies of the three branches refer to the recent Sun.

The first reaction in the PP-chain (reactions numbered by 1 and 2 in Figure 6.3) is the collision of two protons, which leads to the formation of ^{2}H (deuterium). Despite having the lowest possible Coulomb barrier, this reaction is by far the slowest in the PP-chain. The reason is that this reaction involves the conversion of a proton into a neutron through the effect of weak interaction; this leads to an extremely small cross-section factor. Hence the combined rate of energy generation is controlled by this reaction. The remaining reactions are *in equilibrium*, in the sense that equal amounts of ^{2}H and ^{3}He are produced and destroyed.

After formation of 3He, the PP-chain can proceed in three separate branches (Figures 6.1 and 6.3). The branching ratios between the different parts of the PP-chain depend on the balance between the competing reactions, and hence on the temperature. Under the conditions in the solar core the PP-I chain dominates, and the PP-III chain makes a very small contribution to the energy generation. On the other hand, the PP-III chain is very important for attempts to detect solar neutrinos: due to their high energies, the neutrinos from this chain dominate the measurements in the ^{37}Cl detector, and only the PP-III neutrinos can be seen by the detector based on neutrino scattering in water. Since the electron capture in ^{7}B depends weakly on temperature, the branching ratio between the PP-II and the PP-III chains, and hence the flux of PP-III neutrinos, in principle provide a very sensitive measurement of the temperature in the solar core.

As for the PP-I chain, the subsequent reactions in the PP-II and PP-III chains may be assumed to be in equilibrium; hence the reaction rates in these chains are determined by the rate of production of ^{3}He, and therefore again by the very first reaction in the PP-chain. The combined energy generation rate from the PP chains depends somewhat on the branching between the different chains, but as a first approximation these complications may be neglected. Calculation of the reaction rates then lead to a simple estimate of the energy generation rate in the PP chain, appropriate for typical stellar conditions, as

$$\varepsilon_{pp} = 2.6 \times 10^{-37} X^2 \rho T^{4.5} \text{ J/s/kg},$$

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where X is hydrogen mass fraction. A more general expression is

$$\varepsilon_{pp} = \varepsilon_0 X^2 \rho T^{\alpha} , \quad 3.5 < \alpha < 6,$$

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(6.7)

with some constant value of ε_0 .

Using equations (6.1) and (6.7), and assuming hydrogen abundance X uniform in stellar interior, the total luminosity then estimates as

$$L = 4\pi\varepsilon_0 X^2 \int_0^R T^{\alpha} \rho^2 r^2 \, dr.$$

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(6.8)

When the star is approximated by a polytrope of index n , we have $\rho=\rho_c\theta^n$ (equation 5.5), $T=T_c\theta$ (equation 5.20), $r=(R/\xi_1)\xi$ (equations 5.7, 5.10), and hence

$$L = 4\pi\varepsilon_0 X^2 T_c^{\alpha} \rho_c^2 \left(\frac{R}{\xi_1} \right)^3 \int_0^{\xi_1} \theta^{2n+\alpha} \xi^2 \, d\xi.$$

$$L = 4\pi\varepsilon_0 X^2 T_c^{\alpha} \rho_c^2 \left(\frac{R}{\xi_1} \right)^3 \int_0^{\xi_1} \theta^{2n+\alpha} \xi^2 \, d\xi.$$

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(6.9)

With $T_c \propto M\mu/R$ specified by equation (5.21), and $\rho_c \propto M/R^3$ specified by equation (5.16), we obtain a simple scaling relation

$$L = A_n X^2 \mu^\alpha \frac{M^{2+\alpha}}{R^{3+\alpha}},$$

$$L = A_n X^2 \mu^\alpha \frac{M^{2+\alpha}}{R^{3+\alpha}}, L = A_n X^2 \mu^\alpha \frac{M^{2+\alpha}}{R^{3+\alpha}}, \qquad (6.10)$$

where μ is mean molecular weight, and A_n depends on the polytropic index n only.

Exercise 6.1. Fill in the missing steps in deriving equation (6.10).

CNO cycle

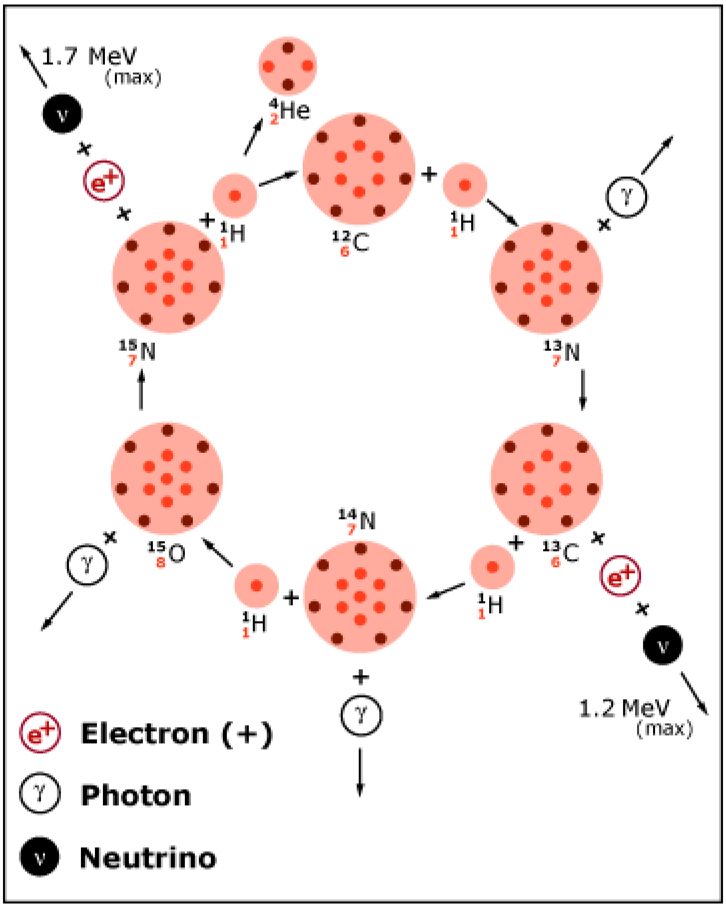

Figure 6.4. The CNO cycle.

A second sequence of reactions is possible in stars that contain C, N and O. This is shown in Figure 6.4. The reactions start with ^{12}C and proceed through a sequence of proton captures, interrupted by positron decay (with emission of neutrinos) to convert protons into neutrons; ^{12}C is produced at the end of the sequence of reactions, and therefore acts as a catalyst.

The conversion of ^{14}N to ^{15}O has the smallest probability amongst the reactions in the cycle; hence, once the cycle operates in equilibrium, this reaction determines the overall reaction rate. Calculation leads to an estimate of the energy generation rate in the CNO cycle, appropriate for typical stellar conditions, as

$$\varepsilon_{CNO} = 7.9 \times 10^{-118} XZ\rho T^{16} \ \mathrm{J/s/kg}.$$

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Here we assumed that the total abundance of CNO elements is a fixed fraction of the total heavy element abundance Z . A more general expression is

$$\varepsilon_{CNO} = \varepsilon_1 XZ\rho T^{\beta}, \quad 12 < \beta < 20.$$

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We see that the energy-generation rate from the CNO cycle is much more temperature-dependent than the energy-generation rate for the PP-chains. Hence the PP-chains dominate at relatively low temperature, whereas the CNO cycle dominates at relatively high temperature. From the estimate of the stellar internal temperature in section 4.2, we therefore expect the CNO cycle to be important in massive stars.

6.3. Later reactions

After the exhaustion of hydrogen, the next reactions that may take place involve ^{4}He. Indeed, given that the ^{3}He + ^{4}He reaction plays a role in the PP-chains, one might have expected that the ^{4}He + ^{4}He reaction could also have set in during hydrogen burning. That this is not the case is due to the fact that there are no stable nuclei with atomic weight eight. Thus ^{4}He-burning has to take place through the so-called *triple-alpha process*:

$$3\ {}^4\mathrm{He} \rightarrow {}^{12}\mathrm{C} + \gamma.$$

$$3\ {}^4\mathrm{He} \rightarrow {}^{12}\mathrm{C} + \gamma 3\ {}^4\mathrm{He} \rightarrow {}^{12}\mathrm{C} + \gamma. \qquad (6.13)$$

In practice the probability that three ^{4}He should come together at precisely the same time is negligible, and hence the reaction actually occur through the generation of ^{8}Be. But since ^{8}Be is unstable, with a mean lifetime of approximately 10^{-14} seconds, the complete reaction requires that the third ^{4}He arrives within a very short time after the initial reaction. Thus the reaction is effectively a three-body process. Typical temperatures where it takes place is 1-2$\times 10^8$K.

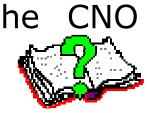
The triple-alpha reaction is followed by successive α-captures by ^{12}C, ^{16}O, and so on.

When helium is exhausted, the next element to react is ^{12}C. This takes place at temperatures around 5-10$\times 10^8$K. Following the ^{12}C exhaustion, the next lowest Coulomb barrier is in the reaction between two ^{16}O. This reaction occurs at temperatures exceeding 10^9K.

A detailed discussion of these reactions, as well as of the reactions involved in the helium burning, was given by Clayton (Clayton D. D., 1968, *Principles of Stellar Evolution and Nucleosynthesis*, McGraw-Hill, Ney York).

Further exercises

Exercise 6.2. For a star in which $X=0.74$ and $Z=0.02$, calculate the temperature at which the rate of energy generation from the PP-chain is the same as from the CNO cycle.

In a second star, $X=0.7$ and $Z=0.001$. What percentage of the energy generation in this star now comes from the CNO cycle, assuming the temperature to be the same as in the first star. 

Solution to Exercise 6.2.

We have the energy generation rate from the pp-chain (equation 6.6) the same as from the CNO cycle (equation 6.11):

$$2.6 \times 10^{-37} X^2 \rho T^{4.5} = 7.9 \times 10^{-118} XZ\rho T^{16},$$

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hence

$$T^{11.5} = \frac{2.6}{7.9} 10^{118-37} \frac{X}{Z} = \frac{2.6 \times 0.74}{7.9 \times 0.02} \times 10^{81},$$
$$T = 1.37 \times 10^{7}\,\mathrm{K}.$$

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$$T = 1.37 \times 10^{7}\,\mathrm{K}.$$

For the second star,

$$\varepsilon_{PP} = 2.6 \times 10^{-37} (0.7)^2 \rho \left(1.37 \times 10^7\right)^{4.5}$$

$$\simeq 1.7 \times 10^{-5} \rho \quad \text{(in SI units),}$$

$$\varepsilon_{CNO} = 7.9 \times 10^{-118} \times 0.001 \times 0.7 \rho \left(1.37 \times 10^7\right)^{1}$$

$$\simeq 8.9 \times 10^{-7} \rho \quad \text{(in SI units),}$$

$$\varepsilon_{PP} = 2.6 \times 10^{-37} (0.7)^2 \rho \left(1.37 \times 10^7\right)^{4.5}$$

$$\simeq 1.7 \times 10^{-5} \rho \quad \text{(in SI units),}$$

$$\varepsilon_{CNO} = 7.9 \times 10^{-118} \times 0.001 \times 0.7 \rho \left(1.37 \times 10^7\right)^{16}$$

$$\simeq 8.9 \times 10^{-7} \rho \quad \text{(in SI units),}$$

$$\varepsilon_{PP} = 2.6 \times 10^{-37} (0.7)^2 \rho \left(1.37 \times 10^7\right)^{4.5}$$

$$\simeq 1.7 \times 10^{-5} \rho \quad \text{(in SI units),}$$

$$\varepsilon_{CNO} = 7.9 \times 10^{-118} \times 0.001 \times 0.7 \rho \left(1.37 \times 10^7\right)^{16}$$

$$\simeq 8.9 \times 10^{-7} \rho \quad \text{(in SI units),}$$

Hence

$$\frac{\varepsilon_{CNO}}{\varepsilon_{PP} + \varepsilon_{CNO}} \simeq \frac{8.9}{170 + 8.9} \simeq 0.05\,.$$

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Exercise 6.3. In a given star of polytropic structure, due to nuclear burning in the PP chain, X changes in time from 0.8 to 0.4. Assuming n, M and R have not changed, determine the percentage change in the luminosity.

Solution to Exercise 6.3.

Using equation (6.10),

$$L \propto X^2\mu^\alpha \frac{M^{2+\alpha}}{R^{3+\alpha}} \propto X^2\mu^\alpha, \qquad \mu = \frac{4}{3+5X}\,,$$

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$$L \propto X^2\mu^\alpha \frac{M^{2+\alpha}}{R^{3+\alpha}} \propto X^2\mu^\alpha, \qquad \mu = \frac{4}{3+5X}\,,$$

hence

$$\frac{L_1}{L_2} = \left(\frac{X_1}{X_2}\right)^2 \left(\frac{3+5X_2}{3+5X_1}\right)^{\alpha}$$

$$= \left(\frac{0.8}{0.4}\right)^2 \left(\frac{3+2}{3+4}\right)^{\alpha} = 4\left(\frac{5}{7}\right)^{\alpha},$$

$$\frac{L_1}{L_2} = \left(\frac{X_1}{X_2}\right)^2 \left(\frac{3+5X_2}{3+5X_1}\right)^{\alpha} \qquad \frac{L_1}{L_2} = \left(\frac{X_1}{X_2}\right)^2 \left(\frac{3+5X_2}{3+5X_1}\right)^{\alpha}$$

$$= \left(\frac{0.8}{0.4}\right)^2 \left(\frac{3+2}{3+4}\right)^{\alpha} = 4\left(\frac{5}{7}\right)^{\alpha}, \qquad = \left(\frac{0.8}{0.4}\right)^2 \left(\frac{3+2}{3+4}\right)^{\alpha} = 4\left(\frac{5}{7}\right)^{\alpha},$$

and with $\alpha \simeq 4.5$ $\alpha \simeq 4.5 \alpha \simeq 4.5$, we have

$$\frac{L_1}{L_2} \simeq 0.88, \qquad \frac{L_2 - L_1}{L_1} = \frac{L_2}{L_1} - 1 \simeq 0.14\,.$$

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Exercise 6.4. In a set of polytropic stars of the same polytropic index and of the same chemical composition, $\rho_c T_c^{4.5}$ is the same for all of them. Show that if the energy generation is through the PP chain only, then, for these stars, luminosity L is proportional to mass M.

Solution to Exercise 6.4.

From equation (6.9), for polytropic stars of the same polytropic index and of the same chemical composition,

$$L \propto T_c^{\alpha} \rho_c^2 R^3 .$$

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With $\alpha = 4.5$ $\alpha = 4.5 \alpha = 4.5$ and

$$\rho_c T_c^{4.5} = \mathrm{const}$$

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$$L \propto \rho_c R^3 ,$$

$L \propto \rho_c R^3 L \propto \rho_c R^3 ,$

and with $\rho_c \propto M / R^3$ $\rho_c \propto M / R^3$

$\rho_c \propto M / R^3$ (see equation 5.16), we have

$$L \propto M .$$

$L \propto M L \propto M .$