Lecture 1 Basic fluid equations

The material in sections 1.1, 1.2 and 1.3 can be found in many texts on hydrodynamics or fluid mechanics. Good treatments can be found in references [1.1] and [1.2]. A different approach is followed by [1.3].

1.1 The material derivative

$\partial / \partial t$ will denote the rate of change of some physical quantity with respect to time *at a fixed position in space*.

D / Dt (the **material derivative**) will denote the rate of change of some quantity with respect to time but travelling along with the fluid.

Let f be any quantity (e.g. temperature). Then

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f, \qquad (1.1)$$

where $\mathbf{u}(\mathbf{r},t)$ is the velocity of the fluid at position $\mathbf{r}$ and time t.

Exercise 1.1 Convince yourself that if the temperature of the fluid varies with position, but that the temperature of any particular parcel of fluid does not change with time, then the rate of change of temperature with time as seen by an observer at a fixed point in space is $-\mathbf{u} \cdot \nabla f$, in agreement with equation (1.1) when $Df / Dt = 0$. Conversely, convince yourself also that equation (1.1) gives the correct result for the change as seen by the fluid when $\partial f / \partial t = 0$. If you need help, press .

1.2 The continuity equation

Consider a volume V, which is fixed in space. The total mass of fluid in V is $\int_V \rho \, dV$, where $\rho(\mathbf{r},t)$ is the density of the fluid. The time derivative of the mass in V is the mass flux into V across its surface S, *i.e.*

$$\frac{d}{dt}\int_V \rho \, dV = -\int_S (\rho \mathbf{u}) \cdot \mathbf{n} \, dS \;, \qquad (1.2)$$

where $\mathbf{n}$ is the outward normal to the surface S. Hence, using the divergence theorem, we obtain

$$\int_V \frac{\partial \rho}{\partial t} dV = -\int_S \rho \mathbf{u} \cdot \mathbf{n} \, dS = -\int_V \nabla \cdot (\rho \mathbf{u}) \, dV. \qquad (1.3)$$

Since this holds for arbitrary V, it follows that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \qquad (1.4)$$

This is the continuity (or mass conservation) equation. Using equation (1.1) this can also be written as

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0. \qquad (1.5)$$

1.3 The momentum equation

One can similarly derive a momentum equation, or equation of motion, for the fluid by considering the rate of change of the total momentum of the fluid inside a volume V. It turns out to be easiest to consider a volume *moving with the fluid*, so that no fluid is flowing across its surface into or out of V. The momentum of the fluid in V is $\int_V \rho \mathbf{u} \, dV$, and the rate of change of this momentum is equal to the net force acting on the fluid in volume V. These are of two kinds. First there are body forces, such as gravity, which act on the particles inside V: their net effect is a force

$$\int_V \rho \mathbf{f} \, dV,$$

where $\mathbf{f}$ is the body force per unit mass. (Nb force per unit mass has dimensions of acceleration.) E.g., $\mathbf{f}$ could be the gravitational acceleration $\mathbf{g}$. The second kind of forces acting are surface forces - forces exerted on the surface S of V by the surrounding fluid. In an *inviscid* fluid, such as we mostly generally be considering, the surface force acts normally to the surface and its net effect is

$$-\int_S p\mathbf{n}\,dS,$$

p being the pressure. Equating force to change of momentum we obtain

$$\frac{d}{dt}\int_V \rho\mathbf{u}\,dV = -\int_S p\mathbf{n}\,dS + \int_V \rho\mathbf{f}\,dV. \qquad (1.6)$$

Since $\rho\,dV$, the mass of a fluid element, is invariant following the motion,

$$\frac{d}{dt}\int_V \rho\mathbf{u}\,dV = \int_V \rho\frac{D\mathbf{u}}{Dt}\,dV \qquad (1.7)$$

and hence, applying the divergence theorem to the surface integral in equation (1.6), we obtain

$$\int_V \rho\frac{D\mathbf{u}}{Dt}\,dV = \int_V (-\nabla p + \rho\mathbf{f})\,dV. \qquad (1.8)$$

Since this holds for arbitary material volume V, it follows that

$$\rho\frac{D\mathbf{u}}{Dt} \equiv \rho\left(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \rho\mathbf{f}. \qquad (1.9)$$

This is the momentum equation for an inviscid fluid.

Taking into account the viscous forces would add the right-hand side of the momentum equation with an additional term $\mu\left(\nabla^2\mathbf{u} + \frac{1}{3}\nabla(\nabla\cdot\mathbf{u})\right)$, where μ is dynamic viscosity - a result which we quote here without derivation. Note that the viscous forces can play a key role in some astrophysical applications which we will not consider in these lectures (in particular, accretion discs).

1.4 Newtonian gravity

A mass m' at position $\mathbf{r}'$ exerts on any other mass m at position $\mathbf{r}$ an attractive force $\mathbf{F} = m\mathbf{g}(\mathbf{r})$; the gravitational acceleration $\mathbf{g}(\mathbf{r})$ can be written as the gradient of a potential function, $\mathbf{g} = -\nabla\psi$, where

$$\psi = -\frac{Gm'}{|\mathbf{r} - \mathbf{r}'|}. \qquad (1.10)$$

Let S be a spherical surface of radius $|\mathbf{r} - \mathbf{r}'|$ centered at $\mathbf{r}'$. With $|\nabla\psi| = Gm'/|\mathbf{r} - \mathbf{r}'|^2$, we have

$$\int_S \mathbf{n} \cdot \nabla\psi \, dS = 4\pi G m', \qquad (1.11)$$

the result which does not depend on $|\mathbf{r} - \mathbf{r}'|$. It can also be verifyed directly that the gravitational potential of our point mass (equation 1.10) satisfyes $\nabla \cdot \nabla\psi \equiv \nabla^2\psi = 0$ (the Laplace equation) everywhere exept of just one point, $\mathbf{r} = \mathbf{r}'$ . Using the divergence theorem, we observe that the surface S in the equation (1.11) can in fact be any (not necessary spherical) surface surrounding m'.

The gravitational field due to a fluid can be written as a potential, namely the sum of the potentials due to all the fluid elements. Summing over all the fluid elements inside S, and applying the divergence theorem once more, we get

$$\int_V \nabla^2\psi \, dV = 4\pi G \int_V \rho \, dV, \qquad (1.12)$$

where V is volume inside S. Since V is arbitrary, this equation can be rewritten as a partial differential equation, Poisson's equation:

$$\nabla^2\psi = 4\pi G\rho \; . \qquad (1.13)$$

1.5. The mechanical and thermal energy equations

If one takes Newton's third law, $\mathbf{F} = m\mathbf{a} = m(d\mathbf{v}/dt)$ and multiplies by velocity $\mathbf{v}$, one obtains that rate of work of the forces, $\mathbf{F}\mathbf{v}$, is equal to the rate of change of kinetic energy, $d\left(\frac{1}{2}mv^2\right)/dt$.

Similarly, taking the dot product of the equation of motion for a fluid, (1.9), with the fluid velocity $\mathbf{u}$ yields

$$\frac{D}{Dt}\left(\frac{1}{2}\mathbf{u}^2\right) = -\frac{1}{\rho}\mathbf{u}\cdot\nabla p + \mathbf{u}\cdot\mathbf{f}. \qquad (1.14)$$

Equation (1.14) says that the rate of change of the kinetic energy of a unit mass of fluid is equal to the rate at which work is done on the fluid by pressure and body forces. This is sometimes called the mechanical energy equation.

Exercise 1.2. Prove equation (1.14).

An equation for the total energy - kinetic and internal thermal energy - can be derived in the same manner as was the momentum equation in Section 1.3. Let the internal energy per unit mass of fluid be U. Then the rate of change of kinetic plus internal energy of a material volume (*i.e.* one moving with the fluid) must be equal to the rate of work done on the fluid by surface and body forces, plus the rate at which heat is added to the fluid. Heat can be added in two ways: one is by its being generated at a rate ε per unit mass within the fluid volume (*e.g.* by nuclear reactions), while the second is by the flux of heat $\mathbf{F}$ into the volume from the surroundings (*e.g.* by radiation). Thus

$$\frac{d}{dt}\int_V \left(\frac{1}{2}\mathbf{u}^2 + U\right)\rho\, dV = \int_S \mathbf{u}\cdot(-p\mathbf{n})dS$$

$$+\int_V \mathbf{u}\cdot\mathbf{f}\,\rho\, dV + \int_V \varepsilon\rho\, dV + -\int_S \mathbf{F}\cdot\mathbf{n}\, dS. \qquad (1.15)$$

In the same way as for the momentum equation, one rewrites all the surface integrals in this equation as volume integrals, using the divergence theorem. The resulting equation holds for an arbitrary volume V and so one deduces that

$$\rho\left(\frac{D}{Dt}\left(\tfrac{1}{2}\mathbf{u}^2\right) + \frac{DU}{Dt}\right)$$

$$= -\nabla\cdot(p\mathbf{u}) + \rho\mathbf{u}\cdot\mathbf{f} + \rho\varepsilon - \nabla\cdot\mathbf{F}. \qquad (1.16)$$

One can derive an equation for the thermal energy alone by dividing (1.16) by the density and then subtracting the kinetic energy equation (1.14):

$$\frac{DU}{Dt} = \frac{p}{\rho^2}\frac{D\rho}{Dt} + \varepsilon - \frac{1}{\rho}\nabla\cdot\mathbf{F}. \qquad (1.17)$$

Note that the divergence of $\mathbf{u}$ has been replaced by $-\rho^{-1}D\rho/Dt$ using the continuity equation (1.5).

Noting that the volume per unit mass is just the reciprocal of the density, *i.e.* $V = \rho^{-1}$, we recognise the thermal energy equation (1.17) as a statement of the first law of thermodynamics:

$$dU = -p\, dV + \delta Q\,, \qquad (1.18)$$

that is, the change in the internal energy is equal to the work $-p\, dV$ done (on the fluid) plus the heat added. Note that $p,\ V,\ U$ are properties of the fluid (in fact they are thermodynamic state variables) and we denote changes in them with the symbol d. In contrast, there is no such property as the heat content and so we cannot speak of the change of heat content. Instead, we can only speak of the heat added, and we therefore use a different notation, *i.e.* δQ. The second law of thermodynamics states that

$$\delta Q = T\,dS, \qquad (1.19)$$

where S is a thermodynamic state variable, the *specific entropy* (*i.e.* the entropy per unit mass). Combining this with the first law, equation (1.18), yields

$$dU = T\,dS - p\,dV. \qquad (1.20)$$

1.6. Adiabatic approximation. Ideal gases

In practical applications, the thermal energy equation (1.17) can often be simplified. The most important simplification comes from the so-called adiabatic approximation. In the adiabatic approximation, we neglect any heat generation inside the fluid element and any heat exchange with the surroundings, which means setting $\varepsilon = \mathbf{F} = \delta Q = dS = 0$ (the entropy is conserved).

For the adiabatic changes, pressure and density variations in a fluid element are related with each other through the so-called adiabatic exponent

$$\Gamma_1 = \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_S, \qquad (1.21)$$

subscript S designates that the partial derivative is taken at constant entropy. In terms of material derivatives, the thermal energy equation (1.17) is then equivalent to

$$\frac{Dp}{Dt} = \frac{\Gamma_1 p}{\rho}\frac{D\rho}{Dt}. \qquad (1.22)$$

To evaluate the adiabatic exponent Γ_1, we need to know the equation of state. The simplest (and often quite accurate) approximation here is the equation of state of an ideal gas,

$$pV = NkT, \qquad (1.23)$$

where N is total number of particles (molecules or atoms) in volume V, and k is Boltzmann constant. In the ideal gas, interactions between particles are neglected, and the internal inergy is just the sum of the kinetic energies of all the free particles. For a single particle, the kinetic energy is $kT/2$ times the number of degrees of freedom in its free motion. For a monoatomic gas (in which the particles can be considered as point masses) there are three degrees of freedom (three possible directions of translational motion), and we have

$$U = \frac{3}{2}NkT. \qquad (1.24)$$

This simple expression for the internal energy gives the adiabatic exponent of an ideal monoatomic gas as

$$\Gamma_1 = \frac{5}{3}. \qquad (1.25)$$

Exercise 1.3. Fill in the missing steps to derive equation (1.25).

LITERATURE

[1.1] Batchelor, G. K., 1967. *An Introduction to Fluid Dynamics* (C.U.P.: Cambridge)

[1.2] Landau, L. D., Lifshitz, E. M., 1959. *Fluid Mechanics* (Pergamon Press: Oxford)

[1.3] Shu, F. H., 1992. *The Physics of Astrophysics Vol. II. Gas Dynamics* (University Science Books: Mill Valley, CA)

ASTROPHYSICAL FLUID DYNAMICS

M. J. Thompson, S. V. Vorontsov

Astronomy Unit, School of Mathematical Sciences, Queen Mary & Westfield College, Mile End Road, London E1 4NS

Contents

Lecture 2 Simple models of astrophysical fluids and their motions

In the previous lecture we established the momentum equation (1.9), the continuity equation (1.5), Poisson's equation(1.13) and the energy equation (1.17). Assuming that the only body forces are due to self-gravity, so that $\mathbf{f} = -\nabla\psi$ in equation (1.9), these equations are:

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p - \rho \nabla \psi , \qquad (2.1)$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0, \qquad (2.2)$$

$$\nabla^2 \psi = 4\pi G \rho , \qquad (2.3)$$

$$\frac{DU}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = \varepsilon - \frac{1}{\rho}\nabla \cdot \mathbf{F}. \qquad (2.4)$$

Note that these contain seven dependent variables, namely ρ, the three components of $\mathbf{u}$, p, ψ and U. The three components of (2.1), together with (2.2) - (2.4), provide six equations, and a seventh is the equation of state (*e.g.* that for an ideal gas) which provides a relation between any three thermodynamic state variables, so that (for example) the internal energy U and temperature T can be written in terms of p and ρ. (ε and $\mathbf{F}$ are assumed to be known functions of the other variables). Thus one might hope in principle to solve these equations, given suitable boundary conditions. In practice this set of equations is intractable to exact solution, and one must resort to numerical solutions. Even these can be extremely problematic so that, for example, understanding turbulent flows is still a very challenging research area. Moreover, an analytic solution to a somewhat idealized problem may teach one much more than a numerical solution. One useful idealization is where we assume that the fluid velocity and all time derivatives are zero. These are called equilibrium solutions and describe a steady state. Although a true steady state may be rare in reality, the time-scale over which an astrophysical system evolves may be very long, so that at any particular time the state of many astrophysical fluid bodies may be well represented by an equilibrium model. Even when the dynamical behaviour of the body is important, it can often be described in terms of small departures from an equilibrium state. Hence in this lecture we start by looking at some equilibrium models and then derive equations describing small perturbations about an equilibrium state.

2.1 Hydrostatic equilibrium for a self-gravitating body

If we suppose that $\mathbf{u} = 0$ everywhere, and that all quantities are independent of time, then equation (2.1) becomes

$$\nabla p + \rho \nabla \psi = 0\,; \tag{2.5}$$

the continuity equation becomes trivial; and equation (2.3) is unchanged. A fluid satisfying equation (2.5) is said to be in hydrostatic equilibrium. If it is self-gravitating (so that ψ is determined by the density distribution within the fluid), then equation (2.3) must also be satisfied.

Putting $\mathbf{u} = 0$ and $\partial / \partial t = 0$ in equation (2.4), we obtain that the heat sources given by ε must be exactly balanced by the heat flux term $\rho^{-1}\nabla \cdot \mathbf{F}$. If this holds, then the fluid is also said to be in thermal equilibrium. Since we have not yet considered what the heat sources might be, nor the details of the heat flux, we shall neglect considerations of thermal equilibrium at this point. Further reading material on the topics of this section may be found in [2.1], [2.2], [2.4] and [2.5].

2.1.1 Spherically symmetric case

Mass inside a sphere of radius r, centred on the origin, is

$$m(r) = \int_0^r 4\pi r^2 \rho(r)\, dr. \tag{2.6}$$

Gravitational potential ψ is only a function of r; integrating the Poisson's equation (2.3) over the spherical volume gives

$$\nabla \psi = \frac{Gm}{r^2}\,\hat{\mathbf{r}}\,, \tag{2.7}$$

$\hat{\mathbf{r}}$ being a unit vector in the radial direction - a result which could well be anticipated. (NB $\nabla \psi$ is minus the gravitational acceleration $\mathbf{g}$.) Also, by equation (2.5),

$$\nabla p = -\frac{Gm\rho}{r^2}\,\hat{\mathbf{r}}\,. \tag{2.8}$$

The vector ∇p points towards the origin, so the pressure decreases as r increases.

One can only make further progress by assuming some relation between pressure and density. Suppose then that the fluid is an ideal gas, so

$$p = \frac{NkT}{V} = \frac{\Re \rho T}{\mu} \equiv a^2 \rho, \qquad (2.9)$$

where $\Re$ is the universal gas constant and μ is the molecular weight. a is known as the isothermal sound speed. Suppose further that the temperature T, as well as μ, are both constants throughout the fluid, so a is also a constant. Then equation (2.8) becomes

$$a^2 \frac{d\rho}{dr} = -\frac{Gm\rho}{r^2},$$

which implies that

$$\frac{d}{dr}\left(\frac{r^2 a^2}{\rho}\frac{d\rho}{dr}\right) = -4\pi G r^2 \rho. \qquad (2.10)$$

Seeking a solution of the form $\rho = Ar^n$, where A and n are constants, gives

$$\rho = \frac{a^2}{2\pi G r^2}, \qquad p = \frac{a^4}{2\pi G r^2}. \qquad (2.11)$$

This is the singular self-gravitating isothermal sphere solution. It is not physically realistic at $r = 0$, where p and ρ are singular, but nonetheless it is a useful analytical model solution. Of course, in a real nondegenerate star, for example, the interior is not isothermal: the temperature increases with depth, which in turn means that the pressure increases and the star is prevented from collapsing in upon itself without recourse to infinite pressure and density at the centre.

Exercise 2.1. Verify that (2.11) is a solution of (2.10). Verify also that $a^2 = p/\rho$ has dimensions of velocity squared. 

2.1.2 Plane-parallel layer under constant gravity

In modelling the atmosphere and outer layers of a star, the spherical geometry can often be ignored, so that such a region can be approximated as a plane-parallel layer. Moreover, in the rarified outer layers of a star the gravitational acceleration $\mathbf{g}$ may be approximated as a constant vector. Thus, in Cartesian coordinates x, y, z we have a region in which everything is a function of z alone and (taking $\hat{\mathbf{z}}$ pointing downwards), $\mathbf{g} = g\hat{\mathbf{z}}$, where g is constant. Hence (2.5) becomes

$$\frac{dp}{dz} = g\rho(z). \qquad (2.12)$$

Since self-gravity is being ignored, equation (2.3) is not used.

In the *isothermal* case ($p/\rho = a^2$ constant), equation (2.12) can be integrated to give

$$\rho = \rho_0 \exp(gz/a^2) \qquad (2.13)$$

where the constant ρ_0 is the density at $z = 0$. The density scale height H is defined by

$$H = \left|\frac{1}{\rho}\frac{d\rho}{dz}\right|^{-1}. \qquad (2.14)$$

Hence, in this case, $H = a^2/g$ and is constant. Thus $\rho = \rho_0 \exp(z/H)$.

Exercise 2.2. Derive the solution (2.13).

2.2 Small perturbations about equilibrium

In many interesting instances, such as the oscillations of a Cepheid, the motion of a fluid body can be considered to be small disturbances about an equilibrium state. Suppose that in equilibrium the pressure, density and gravitational potential are given by $p = p_0$, $\rho = \rho_0$, $\Psi = \Psi_0$

(all possibly functions of position, but independent of time) and of course $\mathbf{u} = 0$. Using equations (2.5) and (2.3), the equilibrium quantities satisfy

$$\nabla p_0 = -\rho_0 \nabla \psi_0 \,,$$
$$\nabla^2 \psi_0 = 4\pi G \rho_0 \,. \qquad (2.15)$$

Suppose now that the system undergoes small motions about the equilibrium state, so

$$p = p_0 + p', \quad \rho = \rho_0 + \rho', \quad \psi = \psi_0 + \psi', \quad (2.16)$$

so for example $p'(\mathbf{r}, t) \equiv p(\mathbf{r}, t) - p_0(\mathbf{r}, t)$ is the difference between the actual pressure and its equilibrium value at position $\mathbf{r}$. Substituting these expressions into equations (2.1) - (2.3) yields

$$(\rho_0 + \rho')\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right)$$
$$= -\nabla(p_0 + p') - (\rho_0 + \rho')\nabla(\psi_0 + \psi'),$$
$$\frac{\partial}{\partial t}(\rho_0 + \rho') = -\nabla \cdot ((\rho_0 + \rho')\mathbf{u}), \qquad (2.17)$$
$$\nabla^2(\psi_0 + \psi') = 4\pi G(\rho_0 + \rho').$$

We suppose that the perturbations (the primed quantities and the velocity) are *small*; hence we neglect the products of two or more small quantities, since these will be even smaller. This is known as linearizing, because we only retain equilibrium terms and terms that are linear in small quantities. This simplifies equations (2.17) to:

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla(p_0 + p') - (\rho_0 + \rho')\nabla\psi_0 - \rho_0 \nabla \psi',$$
$$\frac{\partial \rho'}{\partial t} = -\nabla \cdot (\rho_0 \mathbf{u}), \qquad (2.18)$$
$$\nabla^2(\psi_0 + \psi') = 4\pi G(\rho_0 + \rho').$$

Subtracting equations (2.15) leaves a set of equations all the terms of which are linear in small quantities (*e.g.* [2.3]):

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla p' - \rho' \nabla \psi_0 - \rho_0 \nabla \psi', \qquad (2.19a)$$

$$\frac{\partial \rho'}{\partial t} = -\nabla \cdot (\rho_0 \mathbf{u}), \qquad (2.19b)$$

$$\nabla^2 \psi' = 4\pi G \rho'. \qquad (2.19c)$$

Equations (2.19) give 5 equations (counting the vector equation as three) for 6 unknowns ($\mathbf{u}, p', \rho', \psi'$). We need another equation to close the system: that equation comes from energy considerations.

In full generality, we should perturb the energy equation (2.4) in the same manner as equations (2.1) - (2.3). But there are two limiting cases which are sufficiently common to be very useful and are simpler than using the full perturbed equation (2.4) because they don't involve a detailed description of how ε and $\mathbf{F}$ are perturbed.

Adiabatic fluctuations

Let the typical time scale and length scale on which the perturbations vary be T and λ, respectively. If T is much shorter than the timescale on which heat can be exchanged over a distance λ, then we can say that over a timescale T the heat gained or lost by a fluid element is zero: $\delta Q = 0$. In the previos lecture we established the adiabatic relation between the material derivatives if pressure and density (equation 1.22):

$$\frac{Dp}{Dt} = \frac{\Gamma_1 p}{\rho} \frac{D\rho}{Dt}. \qquad (2.20)$$

The linearized form of this equation is

$$\frac{\partial p'}{\partial t} + \mathbf{u} \cdot \nabla p_0 = \frac{\Gamma_1 p_0}{\rho_0} \left(\frac{\partial \rho'}{\partial t} + \mathbf{u} \cdot \nabla \rho_0 \right). \qquad (2.21)$$

(In the last equation, the adiabatic exponent Γ_1 is also an equilibrium quantity because we have linearized, but for clarity the zero subscript has been omitted).

Isothermal fluctuations

The converse situation is where the timescale for heat exchange between neighbouring material is much shorter than the timescale of the perturbations. Since heat tends to flow from hotter regions to cooler ones, efficient heat exchange will eliminate any temperature fluctuations. Assuming an ideal gas, perturbing equation (2.9) gives

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}. \qquad (2.22)$$

For isothermal fluctuations, $dT = 0$. Hence $dp/p = d\rho/\rho$. In terms of material derivatives,

$$\frac{Dp}{Dt} = \frac{p}{\rho}\frac{D\rho}{Dt}. \qquad (2.23)$$

Linearizing this gives an equation of the same form as equation (2.21) but without the factor Γ_1.

2.3 Lagrangian perturbations

We have previously considered perturbations evaluated at a fixed point in space, so for example $p'(\mathbf{r},t) \equiv p(\mathbf{r},t) - p_0(\mathbf{r},t)$ is the difference between the actual pressure and the value it would take in equilibrium *at that same point in space*. One can also evaluate perturbations as seen by a fluid element (cf. the material derivative).

Such a perturbation will be denoted δp, for example. Now $\delta\mathbf{r}$ is the displacement of a fluid element from the position it would have been at in equilibrium.

$$\begin{aligned}\delta p &\equiv p(\mathbf{r}_0 + \delta\mathbf{r}) - p_0(\mathbf{r}_0) \\ &= p(\mathbf{r}_0) + \delta\mathbf{r}\cdot\nabla p_0 - p_0(\mathbf{r}_0),\end{aligned} \qquad (2.24)$$

where $\mathbf{r}_0$ is the equilibrium position of the fluid element; in the second equation, the first two terms of a Taylor expansion of $p(\mathbf{r}_0 + \delta\mathbf{r})$ have been taken: strictly we should have $\delta\mathbf{r} \cdot \nabla p$, but $\delta\mathbf{r} \cdot \nabla p_0$ is correct up to terms linear in small quantities. Equation (2.24) can be written

$$\delta p(\mathbf{r}_0) = p'(\mathbf{r}_0) + \delta\mathbf{r} \cdot \nabla p_0 \qquad (2.25)$$

where the argument on the left is written $\mathbf{r}_0$ (rather than $\mathbf{r}$) and this is again correct in linear theory. Of course, equation (2.25) holds for any quantity, not just pressure. We note that, in linear theory, $\partial\,\delta f / \partial t = D\,\delta f / Dt$ (where f is any quantity); hence since the velocity of the fluid is just the rate of change of position as seen by a fluid element,

$$\mathbf{u} = \frac{D\,\delta\mathbf{r}}{Dt} = \frac{\partial\,\delta\mathbf{r}}{\partial t}. \qquad (2.26)$$

Perturbations p' at a fixed point in space are called Eulerian; perturbations δp following the fluid are called Lagrangian. See for example [2.1].

2.4 Sound waves

The linearized perturbed Poisson equation (2.19c) has formal solution

$$\psi'(\mathbf{r}) = \int \frac{-G\rho'(\tilde{\mathbf{r}})}{|\mathbf{r} - \tilde{\mathbf{r}}|} d\tilde{V}, \qquad (2.27)$$

the integration being over the whole volume of the fluid. What is under the itegral, is just the perturbation to ψ induced by the mass perturbation in volume $d\tilde{V}$. In the integral on the right-hand side of (2.27) the positive and negative fluctuations in ρ' tend to cancel out, so that it is often a reasonable approximation to say that $\psi' \approx 0$. Thus we will frequently drop ψ' in equation (2.19a). We shall do so in the remainder of this lecture, for example. The term is also absent in equation (2.19a) in problems where self-gravitation is ignored altogether. The term is very important, however, in Lecture 3 when we discuss the Jeans instability.

Suppose now that we have a *homogeneous* medium, so that equilibrium quantities are independent of position (and hence in particular $\nabla p_0 = 0 = \nabla \psi_0$). Equations (2.19a) and (2.19b) can then be rewritten

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla p', \qquad \frac{\partial \rho'}{\partial t} = -\rho_0 \nabla \cdot \mathbf{u}, \qquad (2.28)$$

so taking the divergence of the first of these equations and substituting for $\nabla \cdot \mathbf{u}$ from the second gives

$$\frac{\partial^2 \rho'}{\partial t^2} = \nabla^2 p'. \qquad (2.29)$$

Suppose further that the perturbations are adiabatic. Now equation (2.21) for a homogeneous medium becomes

$$\frac{\partial p'}{\partial t} = c^2 \frac{\partial \rho'}{\partial t},$$

where $c^2 \equiv \Gamma_1 p_0 / \rho_0$ is a constant. Integrating with respect to time gives

$$p' = c^2 \rho',$$

which can be used to eliminate p' from equation (2.29):

$$\frac{\partial^2 \rho'}{\partial t^2} = c^2 \nabla^2 \rho'. \qquad (2.30)$$

This is a wave equation (cf. the 1-D analogue $\partial^2 y / \partial t^2 = c^2 \partial^2 y / \partial x^2$) and describes sound waves propagating with speed c (see [2.2]). In fact, c is called the adiabatic sound speed. (If we had instead assumed isothermal fluctuations, we would have obtained a wave equation with c replaced by a, the isothermal sound speed; cf. section 2.1.1).

One can seek plane wave solutions of eq. (2.30):

$$\rho' = A\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t), \tag{2.31}$$

where the amplitude A, frequency ω and wavenumber $\mathbf{k}$ are constants. (Here and elsewhere, it is understood when writing complex quantities that the real part should be taken to get a physically meaningful solution.)

Substituting equation (2.31) into (2.30), one finds that ρ' is a nontrivial solution ($A \neq 0$) provided

$$\omega^2 = c^2 \, |\mathbf{k}|^2 . \tag{2.32}$$

This is known as the *dispersion relation* for the waves. It specifies the relation that must hold between the freqency and wavenumber for the wave to be a solution of the given wave equation. With a suitable choice of phase, one can deduce from (2.31) that

$$\begin{aligned} \rho' &= A\cos(\mathbf{k}\cdot\mathbf{r} - \omega t), \\ p' &= Ac^2\cos(\mathbf{k}\cdot\mathbf{r} - \omega t), \\ \delta\mathbf{r} &= -\frac{c^2}{\rho_0\omega^2}A\mathbf{k}\sin(\mathbf{k}\cdot\mathbf{r} - \omega t). \end{aligned} \tag{2.33}$$

Note that the adiabatic pressure and density fluctuations are in phase, while the displacement is $\pi/2$ out of phase. A sound wave is called longitudinal, because the fluid displacement is parallel to the wavenumber $\mathbf{k}$.

Exercise 2.3. Derive the expressions (2.33). Explain physically the phase difference between the displacement and the other two quantities in a plane sound wave.

2.5 Surface gravity waves

As a second example of a simple wave solution of the linearized perturbed fluid equations, consider an incompressible fluid ($\nabla \cdot \mathbf{u} = 0$) of constant density ρ_0, occupying the region $z \geq 0$ below the free surface $z = 0$ (so p is constant at the surface). Suppose also that gravity $g\hat{\mathbf{z}}$ is uniform and points downwards, and that self-gravity is negligible. This is a reasonable model for ocean waves on deep water, for example. Equation (2.19b) implies that $\rho' = 0$. Hence equation (2.19a) becomes

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla p', \qquad (2.34)$$

and taking the divergence of this gives

$$\nabla^2 p' = 0. \qquad (2.35)$$

We seek a solution with sinusoidal horizontal variation in the x direction:

$$p'(x,z,t) = f(z)\cos(kx - \omega t), \qquad (2.36)$$

where f is an as yet unknown function; and without loss of generality $k > 0$. Substituting this into (2.35) gives

$$\frac{d^2 f}{dz^2} = k^2 f$$

whence

$$f(z) = A\exp(-kz) + B\exp(kz) \qquad (2.37)$$

(see [2.2]). The fluid is infinitely deep, and the solution should not become infinite as $z \to \infty$; hence $B = 0$.

The boundary condition at the free surface $z = 0$ is that the pressure at the edge of the fluid should be constant: hence $\delta p = 0$ at $z = 0$. Thus at $z = 0$

$$\delta p = p' + \delta \mathbf{r} \cdot \nabla p_0 = p' + \rho_0 g \hat{\mathbf{z}} \cdot \delta \mathbf{r} = 0. \qquad (2.38)$$

Taking the dot product of (2.34) with $\hat{\mathbf{z}}$ and using (2.36) and (2.37) yields

$$\hat{\mathbf{z}} \cdot \delta \mathbf{r} = -\frac{k}{\rho_0 \omega^2} p' \qquad (2.39)$$

everywhere. Hence the boundary condition (2.38) can only be satisfied if ω and k satisfy the dispersion relation

$$\omega^2 = gk. \qquad (2.40)$$

It is clear that these are surface waves; for the perturbed quantities all decrease exponentially with depth. In reality, of course, the fluid cannot be infinitely deep, so B is not identically zero. Instead, A and B will have to be chosen such that some boundary condition is satisfied at the bottom of the fluid layer. However, provided the depth of the layer is much greater than k^{-1}, it will generally be the case that B has to be much less than A.

LITERATURE

[2.1] Batchelor, G. K., 1967. *An Introduction to Fluid Dynamics* (C.U.P.: Cambridge)

[2.2] Landau, L. D. & Lifshitz, E. M., 1959. *Fluid Mechanics* (Pergamon Press: Oxford)

[2.3] Cox, J. P., 1980. *Theory of stellar pulsation* (Princeton University Press: Princeton)

[2.4] Kippenhahn, R. & Weigert, A., 1990. *Stellar structure and evolution* (Springer-Verlag: Berlin)

[2.5] Shu, F. H., 1992. *The Physics of Astrophysics Vol. II. Gas Dynamics* (University Science Books: Mill Valley, CA)

Lecture 3 Jeans instability and star formation. Spherically symmetric accretion and stellar winds

We believe that at least some star formation takes place in interstellar gas clouds. Three pieces of observational evidence are that (1) associations of young, bright massive stars are found in nebulae; (2) nebulosity is seen in young, open clusters; and (3) infrared observations reveal young stellar objects (YSOs) obscured by gas.

3.1 Jeans instability

The linearized equations for small perturbations are equations (2.19). We shall consider the simplest possible system, which is a homogeneous cloud, infinite in all directions, so p_0 and ρ_0 are independent of position, as too is ψ_0 by virtue of the hydrostatic equation. Thus equation (1.19a) becomes

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla p' - \rho_0 \nabla \psi'. \qquad (3.1)$$

Taking the divergence of this equation, and using (2.19c) to eliminate $\nabla^2 \psi'$ gives

$$\rho_0 \frac{\partial (\nabla \cdot \mathbf{u})}{\partial t} = -\nabla^2 p' - 4\pi G \rho_0 \rho'. \qquad (3.2)$$

In a uniform medium, $q' = \delta q$ for any quantity q. Suppose that the gas is ideal and isothermal, so $\delta p / p = \delta \rho / \rho$. Hence

$$p' = a^2 \rho', \qquad (3.3)$$

where $a = \sqrt{p_0 / \rho_0}$ is the isothermal sound speed. Also in a uniform medium (19.b) becomes

$$\frac{\partial \rho'}{\partial t} = -\rho_0 \nabla \cdot \mathbf{u}. \qquad (3.4)$$

Hence, using equations (3.3) and (3.4) to eliminate ρ' and $\nabla \cdot \mathbf{u}$ from (3.2), we obtain

$$\frac{\partial^2 \rho'}{\partial t^2} = a^2 \nabla^2 \rho' + 4\pi G \rho_0 \rho'. \qquad (3.5)$$

This is a linear PDE for ρ', with coefficients that are independent of position and time. Hence we seek solutions $\rho' \propto \exp(i\omega t + i\mathbf{k} \cdot \mathbf{r})$. In this case $\partial \rho' / \partial t = i\omega \rho'$ and $\nabla \rho' = i\mathbf{k} \rho'$. Hence, equation (3.5) implies

$$-\omega^2 \rho' = -k^2 a^2 \rho' + 4\pi G \rho_0 \rho', \qquad (3.6)$$

where $k = |\mathbf{k}|$; and so for a nontrivial solution ($\rho' \neq 0$) we obtain the dispersion relation

$$\omega^2 = k^2 a^2 - 4\pi G \rho_0 . \qquad (3.7)$$

If k and ω are real, this represents an oscillation. However, if the right-hand side of (3.7) is negative, as it will be for sufficiently small k, then ω^2 will be negative and so the cloud will be unstable because there will be a solution ρ' where $|\exp(i\omega t)|$ grows exponentially with time. Thus the cloud is unstable to fluctuations of wavenumber k if

$$k^2 a^2 < 4\pi G \rho_0 ,$$

i.e. if

$$k^{-1} > \left(\frac{a^2}{4\pi G\rho_0}\right)^{1/2} \equiv (2\pi)^{-1}\lambda_J .$$

Now a real cloud is of finite size, so one cannot have arbitrarily large wavelengths λ, *i.e.* arbitrarily small wavenumbers. If the cloud is roughly sperical with radius R, one must have $\lambda < 2R$. So such a cloud is unstable to density perturbations if

$$R > R_J \equiv \frac{1}{2}\lambda_J = \frac{1}{2}\left(\frac{\pi a^2}{G\rho_0}\right)^{1/2} . \qquad (3.8)$$

If the density grows unstably, it is because mass is falling in to some region from surrounding regions. So put another way, the cloud collapses if its mass exceeds the critical Jeans mass

$$M_J = \frac{4}{3}\pi R_J^3 \rho_0 = \frac{\pi}{6}\rho_0 \lambda_J^3 . \qquad (3.9)$$

Note the crucial role of the perturbation to the gravitational potential in this instability. If this had been neglected in equation (3.1), equation (3.7) would have reduced to $\omega^2 = k^2 a^2$, the dispersion relation for isothermal sound waves (cf equation (2.32)), which would give stable oscillatory solutions always.

For a typical HI region, $n_H \approx 10\,\mathrm{cm}^{-3}$ (the number of hydrogen atoms per cubic centimetre), so the density is about $2 \times 10^{-23}\,\mathrm{g\,cm}^{-3}$. The temperature is about 100K. Hence, using $a^2 = \Re T/\mu$ with mean molecular weight $\mu \approx 1$, one gets that $\lambda_J \approx 10^{20}\,\mathrm{cm}$ (about 100 light years) and $M_J \approx 3 \times 10^{37}\,\mathrm{g} \approx 10^4$ solar masses. But this is very much greater than the mass of the most massive known stars. Hence it cannot simply be that a cloud exceeding its Jeans mass collapses under its own self-gravity to form a single star [3.1].

If we suppose that the collapse continues isothermally (*i.e.* a^2 remains constant) then $\lambda_J \propto \rho^{-1/2}$ and similarly $M_J \propto \rho^{-1/2}$. Hence, as the cloud collapses and the density grows, the Jeans mass decreases. One can therefore envisage that subcondensations form in the cloud as smaller and smaller masses start to collapse in upon themselves. This picture is known as fragmentation. However, fragmentation must not continue indefinitely, as we wish eventually to form stars with observed stellar masses. Now in the later stages of collapse, the cloud presumably becomes too opaque for radiation to smooth out temperature fluctuations, so that isothermal collapse is no longer a good assumption. If we assume that we enter the opposite regime, of adiabatic collapse, then

$$\frac{1}{p}\frac{Dp}{Dt} = \frac{\Gamma_1}{\rho}\frac{D\rho}{Dt}$$

so $p \propto \rho^{\Gamma_1}$. Instead of the isothermal sound speed we use the adiabatic sound speed c, with $c^2 = \Gamma_1 p/\rho \propto \rho^{\Gamma_1 - 1}$. From (3.8) and (3.9) we then have $\lambda_J \propto \rho^{(\Gamma_1 - 2)/2}$ and $M_J \propto \rho^{(3\Gamma_1/2 - 2)}$. E.g., for $\Gamma_1 = 5/3, \quad M_J \propto \rho^{1/2}$. Hence the Jeans mass is no longer decreasing with increasing cloud density, and fragmentation halts.

The problem with the fragmentation picture as propounded above is that dispersion relation (3.7) implies that ω^2 is most negative for the smallest values of k. Thus, the cloud collapses fastest at the largest scales. Although the cloud becomes unstable on smaller scales as the density increases, these smaller-scale perturbations would get overwhelmed by the faster overall collapse of the cloud.

Exercise 3.1 You should be at least a little concerned as to whether the infinite homogeneous cloud obeys the equilibrium equations (2.15). Think about this.

Obviously, although the model of a homogenous cloud has led us to a useful criterion (the Jeans mass) for the collapse of a cloud under its own self-gravity, the model is too simplistic to explain in detail the formation of real stars. The Galaxy, and virtually everything within it, rotates. Unless it loses angular momentum by some mechanism, a rotating cloud will rotate faster as it collapses. Indeed, the centrifugal force will eventually balance self-gravity so that the cloud can no longer collapse perpendicularly to the rotation axis. (It can still collapse along the rotation axis.) Thus the cloud tends to flatten into a disk. This could lead ultimately to the formation of planetary systems such as our own solar system.

3.2 Bernoulli's theorem

The problem of how material falls radially onto a central object is sometimes called the Bondi problem, after [3.3] (see also [3.2]).

As a preliminary, we prove a standard result in fluid dynamics, Bernoulli's theorem for steady inviscid flow. A standard identity from vector calculus gives

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\left(\frac{1}{2}u^2\right) - \mathbf{u}\times(\nabla\times\mathbf{u}), \qquad (3.10)$$

where $u = |\mathbf{u}|$. Neglecting viscosity and setting time derivatives to zero, and using equation (3.10), the equation of motion becomes

$$\nabla\left(\frac{1}{2}u^2\right) - \mathbf{u}\times(\nabla\times\mathbf{u}) = -\frac{1}{\rho}\nabla p - \nabla\psi . \qquad (3.11)$$

Suppose that the flow is baratropic - so the pressure is a known function of density, $p = p(\rho)$. (This is a common simplification in astrophysical fluid dynamics - by assuming a given relation between pressure and density, we can often avoid needing to give specific consideration to the energy equation). Define the enthalpy

$$h = \int \frac{dp}{\rho}, \qquad (3.12)$$

so $\nabla h = \rho^{-1}\nabla p$. Then equation (3.11) becomes

$$\mathbf{u} \times (\nabla \times \mathbf{u}) = \nabla\left(\frac{1}{2}u^2 + h + \psi\right), \qquad (3.13)$$

from which it follows by taking the dot product with $\mathbf{u}$ that

$$\mathbf{u} \cdot \nabla\left(\frac{1}{2}u^2 + h + \psi\right) = 0. \qquad (3.14)$$

This result shows that $\frac{1}{2}u^2 + h + \psi$ is constant along a streamline, *i.e.* a line everywhere parallel to $\mathbf{u}$. For a simple physical derivation of the Bernoulli's theorem, press .

An everyday application of Bernoulli's theorem (3.14) is to consider flow from a kitchen tap. In this case ρ is essentially uniform, so $h = p/\rho$. Bernoulli's theorem says that

$$\frac{1}{2}u^2 + \frac{p}{\rho} - gz \qquad (3.15)$$

is constant along streamlines, z being measured downwards. In particular, along a surface streamline the pressure p is everywhere equal to the atmospheric pressure (constant). Therefore as the flow falls from the tap, (3.15) implies that u increases. Now suppose that the stream from the tap has horizontal cross-sectional area $A(z)$. The direction of $\mathbf{u}$ is essentially vertical, and the flow is incompressible. Thus mass conservation implies that $\rho u A$, the mass flow per unit time through a horizontal plane, is independent of z. Thus, as the water falls, u increases and A decreases.

Note, however, that for sufficiently small A, the surface tension cannot be ignored. Also the flow is not stable: a Kelvin-Helmholtz instability sets up an oscillatory disturbance on the surface.

3.3 The de Laval nozzle

We consider now how the picture of incompressible-type flow from a tap will be modified in a situation where the compressibility of the fluid is important. We still take the flow to be steady, baratropic and one-dimensional. As an example, consider the flow from a jet engine. The spatial variation of the cross-section A is given (by the walls of the combustion chamber), and we can neglect gravity. Bernoulli and mass conservation imply

$$\left.\begin{aligned} \tfrac{1}{2}u^2 + h &= \text{constant} \\ \rho u A &= \text{constant} \end{aligned}\right\} \qquad (3.16)$$

The spatial variation of A induces variations in the other quantities. The first of equations (3.16) implies that

$$u\,du + \frac{c^2}{\rho}\,d\rho = 0\,, \qquad (3.17)$$

where $c^2 = dp/d\rho$ (c is thus the sound speed). This equation, relating changes in density and in velocity, can be rewritten

$$\frac{d\rho}{\rho} = -\mathcal{M}^2\,\frac{du}{u}\,, \qquad (3.18)$$

where $\mathcal{M} \equiv u/c$ is the Mach number.

Note that if $\mathcal{M} \ll 1$, fractional changes in density are negligible compared with fractional changes in u. Thus we can generally neglect compressibility if $\mathcal{M} \ll 1$. On the other hand, supersonic flight past obstacles involves substantial compressions and expansions. Also, equation (3.18) and the second of equations (3.16) together give

$$\left(1 - \mathcal{M}^2\right)\frac{du}{u} = -\frac{dA}{A}\,. \qquad (3.19)$$

Exercise 3.2 Derive equation (3.19).

We now consider equation (3.19) for three cases. If $\mathcal{M} < 1$ (subsonic flow), an increase in u corresponds to a decrease in A. This was the situation for the running tap.

If $\mathcal{M} > 1$ (supersonic flow), an increase in u requires an increase in the area of the nozzle! The explanation for this is that the density decreases faster than the velocity increases (equation 3.18); thus mass conservation requires an increase in A.

For $\mathcal{M} = 1$, the sonic transition between subsonic and supersonic flow, for a smooth transition (du finite) equation (3.19) implies that dA must be zero at the transition point. This is important for jet design. The nozzle needs to converge (A decreasing) to provide the necessary acceleration from subsonic speeds, but should smoothly stop converging and start to diverge where the flow gets to supersonic speeds. In astrophysical situations, the same acceleration can be achieved by external body forces, such as gravity.

3.4 The Bondi problem

We consider the steady, spherically symmetric accretion of gas onto a gravitating point mass M. We assume a baratropic flow, so $p = p(\rho)$. Also we neglect the self-gravity of the infalling gas, which is a good approximation if its total mass is much less than that of the central point mass.

The velocity is wholly in the inward radial direction. Since the flow is steady, integrating the continuity equation over the region between concentric spherical surfaces and using the divergence theorem gives the mass conservation equation

$$4\pi r^2 \rho u = \text{constant} = -\dot{M} \qquad (3.20)$$

where $\dot{M}$ is a positive constant. Bernoulli's theorem yields

$$\frac{1}{2}u^2 + h - \frac{GM}{r} = 0 \qquad (3.21)$$

where

$$h = \int_{\rho_\infty}^{\rho} \frac{dp}{\rho}, \qquad (3.22)$$

ρ_∞ being the density at infinity. Note that Bernoulli's theorem applies to a given radial streamline, and following a streamline out to infinity shows that the constant on the right-hand side of (3.21) is zero: for at infinity $u = 0$, and $h = 0$ there by equation (3.22). Since every point in space is on some radial streamline, and the constant is zero on each one of them, equation (3.21) holds not just on a single streamline but everywhere in space.

In the particular case of isothermal flow, $p = c_\infty^2 \rho$ where c_∞ is a constant. Evaluating equation (3.22) gives

$$h = c_\infty^2 \ln(\rho / \rho_\infty). \qquad (3.23)$$

A characteristic length is

$$r_B = \frac{GM}{c_\infty^2}. \qquad (3.24)$$

This is called the Bondi radius.

We may define a dimensionless radial variable, speed and density by

$$x = \frac{r}{r_B}, \qquad v = \frac{u}{c_\infty}, \qquad a = \frac{\rho}{\rho_\infty} \qquad (3.25)$$

and a dimensionless accretion rate λ by measuring $\dot{M}$ in units of a mass flux $\rho_\infty c_\infty$ across an area $4\pi r_B^2$:

$$\lambda = \frac{\dot{M}}{4\pi\rho_\infty (GM)^2 / c_\infty^3}. \qquad (3.26)$$

The governing equations (3.20), (3.21) can then be written in dimensionless form as

$$x^2 a v = \lambda \qquad (3.27)$$

and

$$\frac{1}{2}v^2 + H(a) - \frac{1}{x} = 0 \qquad (3.28)$$

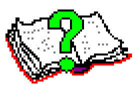
where $H(a) = \ln a$ for isothermal flow. Equations (3.27) and (3.28) imply that changes in the different dimensionless variables are related by

$$2\frac{dx}{x} + \frac{da}{a} + \frac{dv}{v} = 0, \qquad (3.29)$$

$$v\,dv + \frac{da}{a} + \frac{dx}{x^2} = 0; \qquad (3.30)$$

and eliminating da between these gives a relation between dx and dv:

$$\left(v - \frac{1}{v}\right)dv = \left(\frac{2}{x} - \frac{1}{x^2}\right)dx. \qquad (3.31)$$

Exercise 3.3 Derive equations (3.27) - (3.31) for yourself. 

The sonic transition ($v = 1$) occurs when $x = 1/2$. At this point, equation (3.28) implies that $a = \exp(3/2)$ and (3.27) gives $\lambda = (1/4)\exp(3/2) \equiv \lambda_c$. This implies that

$$\dot{M} = \lambda_c 4\pi\rho_\infty \frac{(GM)^2}{c_\infty^3} \qquad (3.32)$$

and so this is the rate at which mass will be accreted (steadily) onto a point mass, assuming spherical symmetry and isothermal flow.

3.5 The Parker solar-wind solution

Parker's model for a thermally-driven solar wind is closely related mathematically to the Bondi problem. It is discussed in detail in [3.4]. There are some differences since the material is now being accelerated from rest at the central object to a large velocity far away, and the conditions at infinity are no longer specified *a priori*. You are encouraged to look at Q. 3f of Problem Set 2 of [3.2]. For a review on the subject of the solar wind, see [3.5].

LITERATURE

[3.1] Hoyle, F., 1953. *Astrophys. J.*, **118**, 513.

[3.2] Shu, F. H., 1992. *The Physics of Astrophysics Vol. II. Gas Dynamics* (University Science Books: Mill Valley, CA).

[3.3] Bondi, H., 1952. *Mon. Not. R. astr. Soc.* **112**, 195.

[3.4] Parker, E. N., 1963. *Interplanetary Dynamical Processes* (Wiley).

[3.5] Holtzer, T. E. & Axford, W. I., 1970. *Ann. Reviews Astron. Astrophys.* **8**, 31.

Lecture 4 Theory of rotating bodies

4.1 Equilibrium equations for a slowly rotating body

In this lecture we shall be considering principally how to calculate the shape of a fluid body that is rotating slowly with a uniform rotation rate. We shall consider in particular the case of a slowly rotating star; but the equations apply equally well to, for example, a slowly rotation gaseous planet. It will be assumed that in the absence of rotation the body would be spherically symmetric, and that rotation induces a weak distortion of the shape from spherical symmetry. For slow rotation, the distorted body is axisymmetric about the rotation axis, as one would expect. Although we shall not consider it here, faster rotation can give rise to some surprises, notably the Jacobi ellipsoids which are triaxial figures of equilibrium. For a fuller exposition of the subject, see the classic texts [4.1] and [4.2].

The momentum equation (1.9) is

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p - \nabla\psi, \qquad (4.1)$$

assuming that the only body forces are due to gravity. Since the rotation is axially symmetric, it is convenient to introduce cylindrical polar coordinates (r_c, φ, z), z being the rotation axis; r_c is the distance from the rotation axis, and φ is the angular coordinate, so that the fluid velocity $\mathbf{u}$ is directed along φ.

For a steady rotation, we have $\partial\mathbf{u}/\partial t = 0$, but $D\mathbf{u}/Dt$ differs from zero: the velocity of a given fluid element does not change with time in its absolute value, but changes in its direction. Simple geometrical considerations show that for an observer moving together with the fluid element,

$$d\mathbf{u} = -\hat{\mathbf{r}}_c u\, d\varphi, \qquad (4.2)$$

where $\hat{\mathbf{r}}_c$ is unit vector along r_c and $u = |\mathbf{u}|$. Introducing the angular velocity of rotation $\Omega = D\varphi/Dt$, we have

$$\frac{D\mathbf{u}}{Dt} = -\hat{\mathbf{r}}_c u\,\Omega = -\hat{\mathbf{r}}_c r_c \Omega^2, \qquad (4.3)$$

since $u = r_c\Omega$. Indeed, the equation (4.2) gives nothing else but the centripetal acceleration of the circular motion.

For a uniform rotation (Ω is constant in space), the right-hand side of the equation (4.3) can be written as a gradient of scalar function:

$$-\hat{\mathbf{r}}_c r_c \Omega^2 = -\nabla\left(\frac{1}{2}\Omega^2 r_c^2\right), \qquad (4.4)$$

and the momentum equation (4.1) reduces to

$$\nabla p = -\rho \nabla \Phi \,, \tag{4.5}$$

where

$$\Phi = \psi - \frac{1}{2}\Omega^2 r_c^2 \,. \tag{4.6}$$

Exercise 4.1 Now, instead if the uniform rotation, consider a general rotaton law around the z- axis. Show that the centripetal acceleration $-\hat{\mathbf{r}}_c r_c \Omega^2$ can be written as a gradient of a scalar function if and only if the rotation rate is independent of φ and z, *i.e.* $\Omega = \Omega(r_c)$ only.

We can argue qualitatively from equation (4.4) what the effect of rotation on the equilibrium shape of the body will be. For an observer moving together with the fluid - in the corotating coordinate system, the fluid is at rest; instead of the centripetal force of the circular motion, in the corotating frame we have the fictitious centrifugal force (a body force), which has to be balanced by pressure and gravity. On the stellar surface, the centrifugal acceleration is zero, and it is radially outwards at the equator. It thus reduces the "effective gravitational acceleration'' at the equator, *i.e*. the centreward pull is not so strong there as at the poles and so instead of being spherical the body "bulges'' at the equator.

The gradient vector ∇f of any scalar is perpendicular to surfaces of constant f. Thus it follows from equation (4.5) that surfaces of constant p are also surfaces of constant Φ, and vice versa. Thus we can write $p = p(\Phi)$, and so

$$\nabla p = \frac{dp}{d\Phi}\nabla \Phi \,.$$

Substituting this into equation (4.5) yields

$$\rho = -\frac{dp}{d\Phi},$$

so ρ is also a function of Φ: $\rho = \rho(\Phi)$.

Henceforward, for definiteness, we shall speak of the body as being a star, but it could equally be a gaseous planet, for example. The outer surface of the star is a surface of constant pressure (because the pressure outside is constant, say zero) and so on the surface, Φ is constant.

We consider the case where if the star were not rotating it would be spherically symmetric, and rotation induces a *weak* distortion from sphericity. We suppose that the star has mass M and (in the nonrotating case) radius R.

Let us approximate the gravitational potential ψ by what it would be in the nonrotating case:

$$\psi = -\frac{GM}{r}$$

at the surface and outside the star.

We suppose that the surface of the rotating star is described by

$$r = R\left(1 + \varepsilon(\theta)\right), \qquad (4.7)$$

where $\varepsilon(\theta)$ is a small function of the colatitude θ. Then, using $r_c = r\sin\theta$, we have on the surface

$$\Phi_{surf} = -\frac{GM}{R(1+\varepsilon)} - \frac{1}{2}\Omega^2 R^2 (1+\varepsilon)^2 \sin^2\theta \qquad (4.8)$$

constant (*i.e.* independent of θ). The rotation is slow and the distortion weak, so Ω^2 and ε are small and we neglect products of small quantities. Then (4.8) implies that

$$-\frac{GM}{R}(1-\varepsilon)-\frac{1}{2}\Omega^2R^2\sin^2\theta$$

is independent of θ, *i.e.*

$$\varepsilon(\theta)=\frac{1}{2}\frac{\Omega^2R^3}{GM}\sin^2\theta \qquad (4.9)$$

(possibly plus a constant). Note that Ω^2R is the equatorial acceleration due to centrifugal forces; and GM/R^2 is the gravitational acceleration. So the dimensionless quantity Ω^2R^3/GM is the ratio of centrifugal acceleration to gravitational acceleration.

The radius at the pole and at the equator are obtained from equation (4.7) by putting $\theta=0$ and $\theta=\pi/2$ respectively. Thus the relative difference between equatorial and polar radii is

$$\frac{R(1+\varepsilon(\pi/2))-R(1+\varepsilon(0))}{R}=\frac{1}{2}\frac{\Omega^2R^3}{GM}. \qquad (4.10)$$

The only thing wrong with this argument is the use of $\psi=-GM/r$. In general, we should properly use the gravitational potential appropriate to the distorted star. The relevant generalization of our analysis is known as the Chandrasekhar-Milne expansion. A description of the procedure is given in [4.3].

The distorsion of the gravitational field can indeed modify our result (equation 4.10) significantly. The magnitude of this effect depends on the mass distribution inside the star: the distorsion of the gravitational field is bigger when more mass is localized closer to the surface, where centrifugal forces are bigger. For a star with uniform density, the coefficient 1/2 in the right-hand side of (4.10) appears to be replaced with 5/4. The rotational distortion of the stellar configuration becomes bigger; it is indeed quite a general property of gravity to have a destabilizing effect. In the oposite limit, when almost all the mass is concentrated very near the stellar center, the equation (4.10) is adequate. For a real star, we can well expect the result to be in between the two limits.

4.2 Binary stars

Consider a binary system in which the two components are in circular orbits about their common centre of mass O, and in which the two stars corotate so as to always show the same side to the other star. In this system there is a rotating frame in which the stars are completely stationary. If Ω is the angular velocity of each star about O, in an inertial frame, then of course Ω is also the angular velocity of the rotating frame.

Suppose that the separation distance between the two stars is a, that their masses are M_1, M_2, and that their respective distances from O are μa and $(1-\mu)a$. Since O is the centre of mass,

$$\mu = \frac{M_2}{M_1 + M_2} \,. \qquad (4.11)$$

Also the gravitational force on star 1 towards star 2 (and hence towards O) must be equal to $M_1\Omega^2(\mu a)$, since μa is the radius of its circular orbit; hence it is straightforward to show that

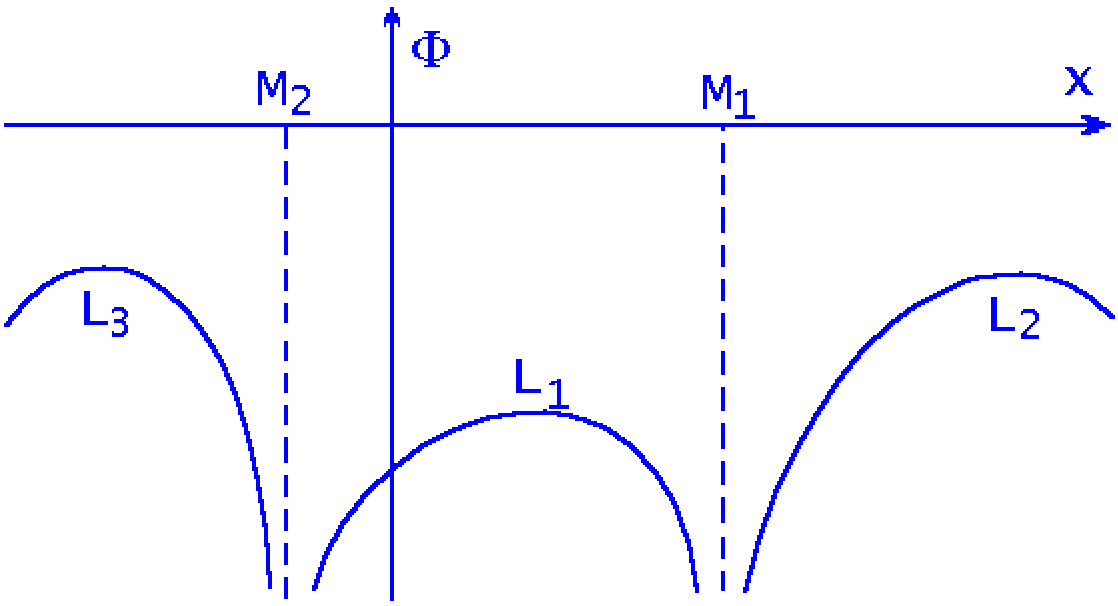
$$\Omega^2 = \frac{G\left(M_1 + M_2\right)}{a^3} \,. \qquad (4.12)$$

Now equations (4.5) and (4.6) hold for this system (in the rotating frame), where the gravitational potential Ψ is given by the sum of the potentials due to the two stars. Choosing Cartesian coordinates such that the angular velocity of the frame is in the z- direction, with the stars at $(\mu a, 0, 0)$ and $(-(1-\mu)a, 0, 0)$, Φ can be written from equation (4.6) as

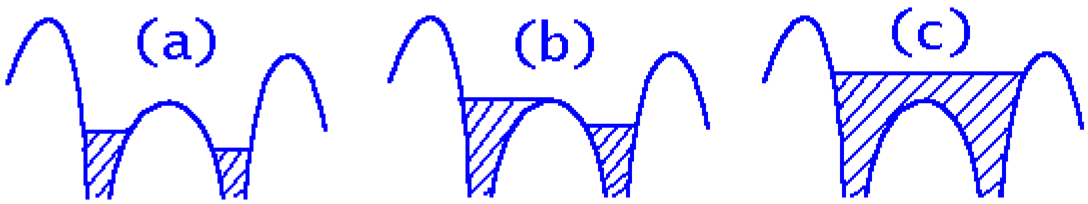
$$\Phi = -\frac{GM_1}{(x-\mu a)^2+y^2+z^2} - \frac{GM_2}{(x+(1-\mu)a)^2+y^2+z^2}$$
$$-\frac{1}{2}\Omega^2\left(x^2+y^2\right) \qquad (4.13)$$

(the Roche potential). Here we have made the same approximation as in section 4.1, namely that we can use the undistorted gravitational potential of each star: this is reasonable for centrally condensed stars.

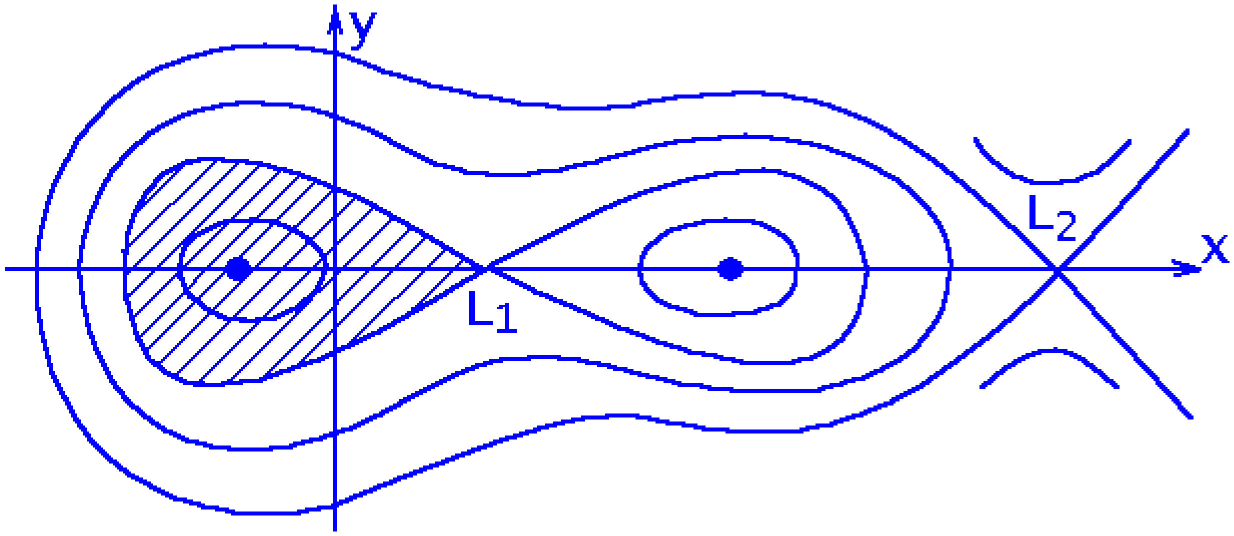
The potential (4.13) is plotted schematically as a function of x along the line $y = z = 0$ below.The Lagrangian points L_1 , L_2 and L_3, where $\nabla\Phi = 0$, are indicated.



As in the case of a single star, the surface of each star in the binary system is a surface of constant Φ. Now provided the surface potential of each star is less than the potential Φ_{L1} at the L_1 Lagrangian point, each star occupies a well in the Roche potential and the stars form a detached binary system (Fig. a):



Suppose though that M_2 expands (perhaps attempting to become a red giant) until its surface potential is equal to Φ_{L1} (Fig. b). Any further expansion will cause matter to fall from star 2 to star 1, since it will fall to the lower potential. Algol is an example of such a binary. Finally, if the surface potentials of both stars are greater than Φ_{L1}, then we have a "common envelope binary'' or "contact binary'' (Fig. c). It is also instructive to plot contours of constant Φ in the $x - y$ plane:



The shaded region, called a Roche lobe, is the maximum region the star can occupy before it starts to lose mass to its companion.

4.3 Dynamics of rotating stellar models

We have not so far considered how energy is transported in the rotating star. A well-known result, which is discussed at length in [4.3] is that one cannot have a uniformly rotating star in strict radiative equilibrium. Assuming the contrary leads to what is known as "von Zeipel's paradox''. The same is true if the rotation rate is a function only of distance from the rotation axis. We conclude therefore that the rotation rate must have a more general form, depending on cylindrical polar coordinate z as well as distance from the axis, or that strict radiative equilibrium does not hold.

The von Zeipel paradox in effect says that the radiative flux cannot be balanced everywhere by the energy generation. Some regions have a net influx of heat: these will heat up and tend to rise under buoyancy. Others will cool and sink. This tends to set up motions in meridional planes: this is called meridional circulation.

LITERATURE

[4.1] Chandrasekhar, S., 1969. *Ellipsoidal Figures of Equilibrium* (Yale University Press: New Haven)

[4.2] Lyttleton, R. A., 1953. *The stability of rotating liquid masses* (C.U.P.: Cambridge)

[4.3] Tassoul, J.-L., 1978. *Theory of rotating stars* (Princeton University Press: Princeton).

Lecture 5 Radial oscillations of stars

In this lecture we consider oscillations of a star about its spherically symmetric equilibrium state, with the oscillatory motion being purely radial.This is relevant to a number of classical variable stars, *e.g.* Cepheids. A readable account of the theory of radial stellar pulsations may be found in [5.1], while [5.2] provides a more mathematical treatment.

5.1 Linear adiabatic wave equations for radial oscillations

We shall assume that the amplitude of the oscillations is small, so that linear perturbation theory will suffice, and in the present section we shall suppose that the period of the oscillations is sufficiently short that no heat is exchanged between neighbouring fluid elements, *i.e.* the oscillations are adiabatic. Then the linearized equations for small perturbations are equations (2.19,
2.21):

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla p' - \rho' \nabla \psi_0 - \rho_0 \nabla \psi', \qquad (5.1a)$$

$$\frac{\partial \rho'}{\partial t} = -\nabla \cdot (\rho_0 \mathbf{u}), \qquad (5.1b)$$

$$\nabla^2 \psi' = 4\pi G \rho', \qquad (5.1c)$$

$$\frac{\partial p'}{\partial t} + \mathbf{u} \cdot \nabla p_0 = \frac{\Gamma_1 p_0}{\rho_0}\left(\frac{\partial \rho'}{\partial t} + \mathbf{u} \cdot \nabla \rho_0\right). \qquad (5.1d)$$

We now seek solutions with sinusoidal time dependence. Writing

$$\begin{aligned}
\delta \mathbf{r} &= \hat{\mathbf{r}} U(r) \exp(i\omega t), \\
p' &= p_1(r) \exp(i\omega t), \\
\rho' &= \rho_1(r) \exp(i\omega t), \qquad (5.2) \\
\psi' &= \psi_1(r) \exp(i\omega t),
\end{aligned}$$

and $\mathbf{u} = \partial \delta \mathbf{r} / \partial t = i\omega \delta \mathbf{r}$, equations (5.1) become

$$-\rho_0\omega^2 U = -\frac{d}{dr}p_1 - \rho_1 g_0 - \rho_0 \frac{d}{dr}\psi_1, \qquad (5.3a)$$

$$\rho_1 = -\frac{1}{r^2}\frac{d}{dr}\left(r^2\rho_0 U\right), \qquad (5.3b)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\psi_1\right) = 4\pi G\rho_1, \qquad (5.3c)$$

$$p_1 - \rho_0 g_0 U = c^2\left(\rho_1 + \frac{d\rho_0}{dr}U\right), \qquad (5.3d)$$

by using $d\psi_0/dr = g_0$, $dp_0/dr = -\rho_0 g_0$,
and $\Gamma_1 p_0/\rho_0 = c^2$, where c is the adiabatic sound speed.
Eliminating ρ_1 and ψ_1 in equations (5.3), we arrive to

$$\frac{dU}{dr} = \left(\frac{g_0}{c^2} - \frac{2}{r}\right)U - \frac{1}{\rho_0 c^2}p_1, \qquad (5.4a)$$

$$\frac{dp_1}{dr} = \left(\omega^2 - N^2 + 4\pi G\rho_0\right)\rho_0 U - \frac{g_0}{c^2}p_1, \qquad (5.4b)$$

where N is the so-called Brunt-V�is�l� frequency,

$$N^2 = -g_0\left(\frac{d\ln\rho_0}{dr} - \frac{1}{\Gamma_1}\frac{d\ln p_0}{dr}\right). \qquad (5.5)$$

Exercise 5.1. Convince yourself, using the divergence theorem, that

$$\nabla\cdot\mathbf{u} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 u\right)$$

for any spherically-symmetric vector field $\mathbf{u} = u(r)\hat{\mathbf{r}}$. Further, fill in the missing steps to derive equations (5.4). [Hint: from equations (5.3b,c) we have $d\psi_1 / dr = -4\pi G\rho_0 U$].

5.2 Boundary conditions

Equations (5.4) represent a system of two ordinary differential equations for $U(r)$ and $p_1(r)$. In order to solve this system, we require boundary conditions at the center of the star ($r = 0$) and at its surface ($r = R$).

The boundary condition at the center is that the solution be regular, not divergent, at the origin. We are looking for the solutions $U(r)$ in the vicinity of $r = 0$ as a power series expansion

$$U(r) = a_0 r^{\alpha} + a_1 r^{\alpha+1} + \ldots . \qquad (5.6)$$

The first term in the series specifies the leading-order behaviour of $U(r)$ near the origin; we require $a_0 \neq 0$ and allow α to be an unknown constant which has to be determined. With this expansion for $U(r)$, equation (5.4a) provides a corresponding expansion for $p_1(r)$:

$$p_1(r) = -(\alpha+2)\rho_0(0)c^2(0)a_0 r^{\alpha-1} + \ldots , \qquad (5.7)$$

where $\rho_0(0)$ and $c(0)$ are the central values of the equilibrium density and the sound speed, and we limit our analysis by the leading-order term only.

We now substitute the expansions (5.6, 5.7) into the second equation (5.4b):

$$-(\alpha-1)(\alpha+2)\rho_0(0)c^2(0)a_0 r^{\alpha-2} + \ldots = 0. \qquad (5.8)$$

As before, dots designate terms of higher order, proportional to $r^{\alpha-1},\ r^{\alpha}$ etc. From equation (5.8), we get $\alpha = 1$ or $\alpha = -2$. The requirement that the solution be regular rules out $\alpha = -2$; hence we must have $\alpha = 1$. The regularity condition thus selects one of the two linearly-independent solutions of the oscillation equations, which behaves as

$$U(r) \simeq a_0 r\,, \qquad p_1(r) \simeq -3\rho_0(0)c^2(0)a_0 \qquad (5.9)$$

in the vicinity of $r = 0$. Constant a_0 is arbitrary, since the equations are homogeneous (solutions are only determined with an arbitrary scaling factor).

At the surface, we require the Lagrangian pressure perturbation δp to be zero. With $\delta p = p' + \delta\mathbf{r} \cdot \nabla p_0$ (equation 2.25 of Lecture 2), the outer boundary condition is

$$p_1 - \rho_0 g_0 U = 0\,, \qquad r = R\,. \qquad (5.10)$$

In a realistic stellar model, we do not have any well-defined "surface", but rather a smooth transition to the low-density atmosphere. the boundary condition $\delta p = 0$ is still applicable, however, when the "stellar radius" is taken to be sufficiently high in the atmosphere. The physical reason is that the atmospheric layers above $r = R$ have essentially no dynamical influence on the global oscillations due to their very small mass. Also when $\rho_0(R)$ is small, the boundary condition (5.10) can be replaced with $p_1(R) = 0$.

5.3 Eigenvalue nature of the problem

The second-order system of differential equations (5.4) has two linearly-independent solutions. Regularity condition at $r = 0$ selects one of them; this solution does not, in general, satisfy the surface boundary condition (5.10). The second boundary condition can be satisfied for certain values of ω^2 only: these values are called eigenvalues, and corresponding solutions for $U(r)$ and $p_1(r)$ are called eigenfunctions. The eigenvalues give the resonant frequencies (eigenfrequencies) at which the star can oscillate radially.

5.4 Local dispersion relation and mode classification

Let us suppose for a moment that we have some solutions $U(r)$ and $p_1(r)$ to the differential equations (5.4) with a sinusoidal character, which oscillate rapidly with r. We will see later in this section that this is indeed the case in the high-frequency limit (when ω is high). We write these solutions as

$$U(r)=\bar{U}(r)\exp(ikr), \quad p_1(r)=\bar{p}_1(r)\exp(ikr) \quad (5.11)$$

with slowly-varying amplitude functions $\bar{U}(r)$ and $\bar{p}_1(r)$, and rapidly-varying exponent (k is high). Differentiation with respect to r gives

$$\frac{dU}{dr}=ikU, \qquad \frac{dp_1}{dr}=ikp_1, \qquad (5.12)$$

when k is large (contribution of terms with derivatives of the amplitude functions can be neglected). We substitute (5.12) into the oscillation equations (5.4), getting

$$ikU+\frac{1}{\rho_0 c^2}p_1=0, \qquad (5.13a)$$

$$\left(\omega^2-N^2+4\pi G\rho_0\right)\rho_0 U-ikp_1=0. \qquad (5.13b)$$

Note that we were using the asymptotic limit of large k once more, neglecting g_0/c^2 and $2/r$ compared with k.

Equations (5.13) are algebraic equations. This homogeneous system of two linear equations for U and p_1 has a non-trivial solution if and only if the determinant of matrix of its coefficients iz zero, *i.e.* when

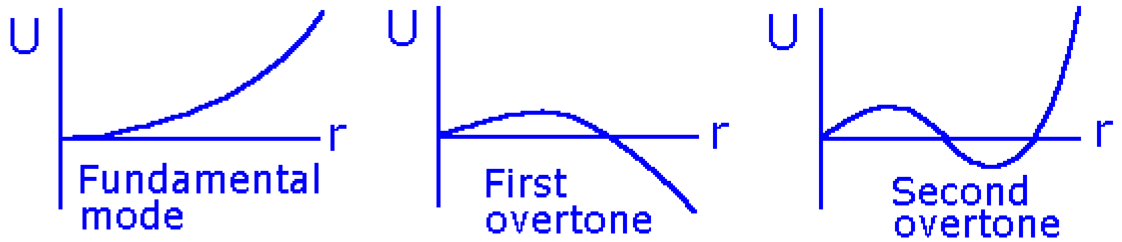
$$k^2=\frac{\omega^2}{c^2}\left(1-\frac{N^2}{\omega^2}+\frac{4\pi G\rho_0}{\omega^2}\right). \qquad (5.14)$$

We observe from this relation that k^2 is large when ω^2 is large, so that our local analysis is indeed applicable in the asymptotic limit of high frequencies.

What we also observe is that when ω is high, k^2 is approximately ω^2 / c^2, which is nothing else but the dispersion relation for sound waves developed earlier in Lecture 2 (equation 2.32). We thus arrive to the simple physical interpretation of the radial oscillations: at least in the high-frequency limit, they are formed by the acoustic waves propagating along stellar radius. A wave travelling upwards is reflected back at the stellar surface, and the downward wave is reflected back at the stellar center. When added together, these waves can form a standing wave (a particular mode of stellar oscillations), which can only happen at frequencies of acoustic resonances (radial oscillation frequencies).

When going to smaller frequencies, simple local analysis can loose its accuracy. Note also that it is always in trouble in the vicinity of $r = 0$: indeed, we were neglecting $2 / r$ compared with k. By doing that we have the effects of spherical geometry discarded: we are using a plane-wave approximation to describe spherical waves.

When ω gets higher, the acoustic wavelength becomes smaller, and radial displacement function $U(r)$ acquires more and more modes in its variation with radius. Different modes of radial oscillations are classified according to the number of nodes in $U(r)$. The oscillation with no nodes, which has the lowest frequency, is called the fundamental radial mode. Higher in frequency is the first overtone, with one node in $U(r)$, and so on.

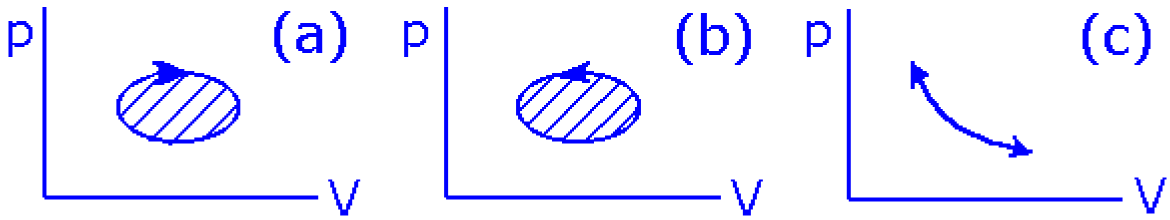


5.5 Non-adiabatic oscillations: physical discussion of driving and damping

Consider a small fluid element of volume V. Let it be small enough so that at any time, pressure p can be considered as uniform everywhere inside V. If V increases by an ammount dV, the mechanical work done by the fluid element against the external pressure is $p\,dV$. Integrating over one cycle of the oscillation, this work will be

$$\oint p\,dV.$$

If this work is positive, it goes to the increase of the total mechanical energy of the surrounding fluid, *i.e.* to the increase of of the pulsational energy of the star; the fluid element acts as a driving source (Fig.a):



In the driving case, at point of maximum compression (V minimum, $DV/Dt = 0$), p is still increasing, $Dp/Dt > 0$ (Fig.a), which means that some heat is being added. What we have is assentially a small heat engine, which transfers thermal energy into mechanical energy.

One way in which oscillations can be driven is by the so-called opacity mechanism, or κ- mechanism. It operates in the near-surface layers of partial ionization with anomalous behaviour of the opacity κ. If the opacity increases when the star is compressed, the heat is gained due to the blocking of the radiative flux coming from the stellar interior. Another driving mechanism is the so-called ε- mechanism. When the star is compressed, density and temperature increase, hence the nuclear energy generation rate ε increases and so more heat is generated.

If the work integral is negative (Fig.b), the fluid element acts as an energy sink for the oscillation, absorbing the mechanical energy from the surrounding fluid and transfering it into heat. At maximum compression, the heat is being lost. This scenario is realized in the so-called radiative damping, when the compressed (and hence hotter) fluid element looses its heat to the surrounding. Note that in the adiabatic approximation, we neglect any heat exchange, and the work integral is zero (Fig.c). Further discussion of the various excitation mechanisms may be found in [5.2].

LITERATURE

[5.1] Cox, J. P., 1980. *Theory of stellar pulsation* (Princeton University Press: Princeton)

[5.2] Unno, W., Osaki, Y., Ando, H., Saio, H. & Shibahashi, H., 1989. *Nonradial oscillations of stars* (2nd edition) (University of Tokyo Press)

Lecture 6 Linear adiabatic nonradial oscillations. Helioseismology

6.1 Nonradial modes of oscillations of a star

In the last lecture we considered stellar oscillations where the motion was wholly in the radial direction. In this lecture we shall consider more general motions. The equilibrium structure about which the oscillations take place is still presumed to be spherically symmetric, but the velocity will now have a horizontal as well as a radial component, and the velocity and other perturbations will depend not only on r but also on θ and φ, where (r, θ, φ) are spherical polar coordinates.

A well-studied star in which nonradial oscillations are observed is the Sun. For this reason, we shall draw upon the example of the Sun frequently in this lecture. However, it should not be forgotten that other pulsating stars are known to exhibit nonradial oscillations (*e.g.* ZZ Cetis, Delta Scutis and Ap stars).

As a preliminary, we list some few mathematical expressions which will be used later in this Lecture. In spherical coordinates (r, θ, φ), the gradient, divergence and Laplacian operators can be written as

$$\nabla f = \hat{\mathbf{r}} \frac{\partial f}{\partial r} + \frac{1}{r} \nabla_1 f , \tag{6.1}$$

$$\nabla_1 f = \hat{\theta} \frac{\partial f}{\partial \theta} + \frac{1}{\sin\theta} \hat{\varphi} \frac{\partial f}{\partial \varphi} ,$$

$$\nabla \cdot \mathbf{u} = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r} \nabla_1 \cdot \mathbf{u} , \tag{6.2}$$

$$\nabla_1 \cdot \mathbf{u} = \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}(\sin\theta\, u_\theta) + \frac{1}{\sin\theta} \frac{\partial}{\partial \varphi} u_\varphi ,$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2} \nabla_1^2 f , \tag{6.3}$$

$$\nabla_1^2 f = \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2 f}{\partial \varphi^2} .$$

As we will see below, it will be possible to separate out the angular dependences and to reduce the oscillation equations to ordinary differential equations with one independent variable r. For variable separation, we will use normalized spherical harmonics

$$Y_{\ell m}(\theta,\varphi) = (-1)^m \left[\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}\right]^{\frac{1}{2}} P_\ell^m(\cos\theta)\, e^{im\varphi} , \tag{6.4}$$

where $P_\ell^m(\cos\theta)$ are the associated Legendge polynomials

$$P_\ell^m(z) = \frac{(1-z^2)^{m/2}}{\ell!\, 2^\ell} \frac{d^{(\ell+m)}}{dz^{(\ell+m)}} (z^2-1)^\ell , \qquad z = \cos\theta . \tag{6.5}$$

The degree ℓ of the spherical harmonic takes any integer value $\ell = 0, 1, 2, \ldots$; at each ℓ, the azimuthal order m can take $2\ell + 1$ value, $m = -\ell, \ldots, \ell$. Spherical harmonics satisfy the second-order partial differential equation

$$\nabla_1^2 Y_{\ell m} + \ell(\ell + 1) Y_{\ell m} = 0. \qquad (6.6)$$

6.2 Linear adiabatic wave equations in Cowling approximation

As with radial oscillations, we assume that the oscillations are adiabatic, and start with the same equations for linear perturbations (2.19, 2.21). To simplify the analysis, we will neglect in this lecture the effects of gravity perturbations; in stellar pulsation theory this approximation is known as Cowling approximation. This approximation is not very restrictive and can in general be easily relaxed; we adopt it here with the only reason to make mathematical derivations more transparent. In Cowling approximation, our starting equations are

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla p' - \rho' \nabla \psi_0 \,, \qquad (6.7a)$$

$$\frac{\partial \rho'}{\partial t} = -\nabla \cdot (\rho_0 \mathbf{u}), \qquad (6.7b)$$

$$\frac{\partial p'}{\partial t} + \mathbf{u} \cdot \nabla p_0 = \frac{\Gamma_1 p_0}{\rho_0} \left(\frac{\partial \rho'}{\partial t} + \mathbf{u} \cdot \nabla \rho_0 \right). \qquad (6.7c)$$

We now seek the solutions of these equations with sinusoidal time dependence and with angular dependence specified by a particular spherical harmonic as

$$\delta \mathbf{r} = \left[\hat{\mathbf{r}} U(r) Y_{\ell m}(\theta,\varphi) + V(r) \nabla_1 Y_{\ell m}(\theta,\varphi) \right] e^{i\omega t},$$

$$p' = p_1(r) Y_{\ell m}(\theta,\varphi) e^{i\omega t},$$

$$\rho' = \rho_1(r) Y_{\ell m}(\theta,\varphi) e^{i\omega t}, \qquad (6.8)$$

with $\mathbf{u} = \partial \delta \mathbf{r} / \partial t = i\omega \delta \mathbf{r}$. The expressions (6.8) allow to separate the angular dependences in equations (6.7), reducing them to the system of ordinary differential equations

$$-\rho_0\omega^2 U = -\frac{d}{dr}p_1 - \rho_1 g_0, \qquad (6.9a)$$

$$-\rho_0\omega^2 V = -\frac{1}{r}p_1, \qquad (6.9b)$$

$$\rho_1 = -\frac{1}{r^2}\frac{d}{dr}\left(r^2\rho_0 U\right) + \frac{\ell(\ell+1)}{r}\rho_0 V, \qquad (6.9c)$$

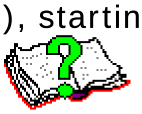
$$p_1 - \rho_0 g_0 U = c^2\left(\rho_1 + \frac{d\rho_0}{dr}U\right), \qquad (6.9d)$$

using $d\psi_0/dr = g_0,\ \ dp_0/dr = -\rho_0 g_0$ and $\Gamma_1 p_0/\rho_0 = c^2$. The first of the equations (6.9) comes from the radial component of the momentum equation (6.7a), the second - from its horizontal component. Eliminating ρ_1 and V in equations (6.9), we arrive to

$$\frac{dU}{dr} = \left(\frac{g_0}{c^2} - \frac{2}{r}\right)U - \frac{1}{\rho_0 c^2}\left(1 - \frac{\ell(\ell+1)c^2}{r^2\omega^2}\right)p_1, \qquad (6.10a)$$

$$\frac{dp_1}{dr} = \left(\omega^2 - N^2\right)\rho_0 U - \frac{g_0}{c^2}p_1, \qquad (6.10b)$$

where N stands for the Brunt-V�is�l� frequency defined earlier by equation (5.5).

Exercise 6.1 Fill in the missing steps to derive the oscillation equations (6.10), starting with variable separation in (6.7) by using equations(6.1-6.6). 

6.3 Boundary conditions

The boundary conditions are derived in a manner similar to radial equations. At $r = 0$, the second-order system of linear differential equations (6.10) has a regular singular point; one of the two linearly-independent solutions is regular, another is singular. Near the origin, the physically relevant regular solution behaves as

$$U(r) \simeq \ell\, a_0 r^{\ell-1}\,, \qquad p_1(r) \simeq \rho_0(0)\omega^2 a_0 r^{\ell}. \quad (6.11)$$

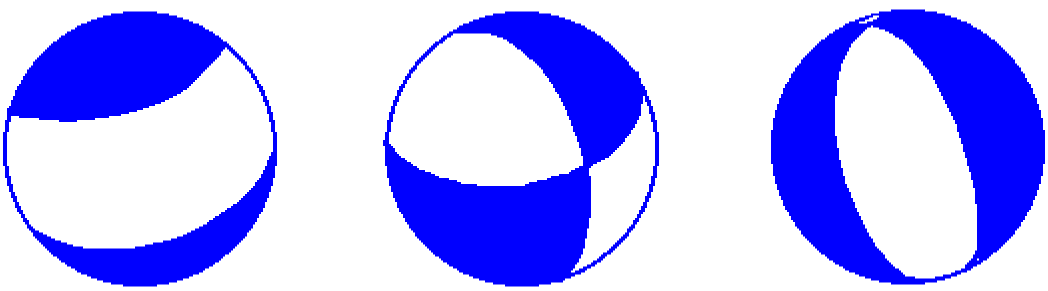
At the stellar surface, with Lagrangian pressure perturbation set to zero, the outer boundary condition is

$$p_1 - \rho_0 g_0 U = 0\,, \qquad r = R\,, \qquad (6.12)$$

the same as for radial modes.

6.4 Mode classification in degree ℓ.

As with radial oscillations, the boundary-value problem specified by the differential equations (6.10) and boundary conditions (6.11, 6.12) has non-trivial solutions only for certain values of ω^2, called eigenvalues. The major difference is that we now have a separate set of eigenfrequencies for each particular value of degree ℓ. The spherical harmonic $Y_{\ell m}(\theta,\varphi)$ determines the angular dependence of the eigenfunctions, and hence the surface distribution of the oscillation amplitudes as seen by an observer. Radial oscillations is just a particular case with $\ell = 0$. Oscillations with $\ell = 1$ are called dipole oscillations, $\ell = 2$ - quadrupole, etc. The surface amplitudes of $\ell = 2$ modes are shown schematically below. Dark and light areas correspond to $Y_{\ell m}(\theta,\varphi)$ positive and negative (*e.g.* when dark areas move upwards, light areas move downwards):



$\ell = 2, m = 0$ $\qquad$ $\ell = 2, m = 1$ $\qquad$ $\ell = 2, m = 2$

The degree ℓ is the total number of the node lines $Y_{\ell m}(\theta,\varphi) = 0$ on the solar surface. The azimuthal order m is the number of the node lines going along a meridian; there are $\ell - m$ node lines parallel to the equator. Note that m does not enter the oscillation equations (6.10), which means that modes of the same ℓ but with different m have the same frequencies. This degeneracy comes from the spherical symmetry of the equilibrium solar configuration.

6.5 Local dispersion relation. Mode classification in radial order n.

Assume that we have solutions of the wave equations (6.10) which oscillate rapidly in radial direction,

$$U(r)=\bar{U}(r)\exp(ik_r r), \quad p_1(r)=\bar{p}_1(r)\exp(ik_r r) \quad (6.13)$$

with amplitudes $\bar{U}(r)$ and $\bar{p}_1(r)$ varying much slower than the exponents (k_r is large). The analysis similar to that of radial modes now leads to the dispersion relation

$$k_r^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\ell(\ell+1)}{r^2}\frac{c^2}{\omega^2}\right)\left(1 - \frac{N^2}{\omega^2}\right). \qquad (6.14)$$

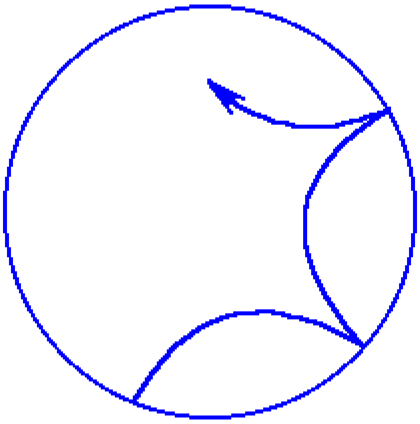
We observe that when $\ell \neq 0$, there are now two possibilities of having k_r^2 large and positive: ω^2 shall be either very large, when we have approximately

$$k_r^2 = \frac{\omega^2}{c^2} - \frac{\ell(\ell+1)}{r^2}, \qquad (6.15)$$

or very small, when we have approximately

$$k_r^2 = \frac{\ell(\ell+1)}{r^2}\left(\frac{N^2}{\omega^2} - 1\right). \qquad (6.16)$$

The physics behind this result is that we now have two different types of waves which can propagate in the solar interior. The last relation (6.16) refers to internal gravity waves, with restoring forces provided by buoyancy. These waves are known in the oceans, and in the Earth's atmosphere. The gravity waves can be "trapped" to form the low-



frequency standing waves, called gravity modes, or g-modes.

At high frequencies, we have (6.15) as a proper dispersion relation, which can be rewritten as

$$k_r^2 + k_h^2 = \frac{\omega^2}{c^2}, \quad k_h = \frac{\sqrt{\ell(\ell+1)}}{r}$$

Here k_h can be interpreted as a horizontal wavenumber, *i.e.* the horizontal component of the total wavevector $\mathbf{k}$. We have nothing else but the dispersion relation of the sound waves $k^2 = \omega^2 / c^2$. When $\ell = 0$, we have radial waves. A non-radial wave propagation and trapping of the acoustic wave in solar interior are shown schematically below. When travelling downwards,the acoustic wave suffers the refraction because of the larger temperature, and hence larger sound speed deeper in the Sun, reflects back after approaching the surface, and so on. If after a closed path the wave returns back in a proper phase, a standing wave is formed. This standing wave represents a particular mode of solar oscillations. These high-frequency modes are called acoustic modes, or p-modes. At given degree ℓ, different p modes are classified according to the number of nodes in their radial displacement function $U(r)$; the number of nodes is called the radial order n. Mode p_1 has one node in $U(r)$, mode p_2 has two nodes, etc.

The frequencies of p modes increase when the radial order n increases; frequencies of g modes decrease when their radial order increases. Between the two frequency domains of the high-frequency p modes and the low-frequency g modes there is usually an extra mode, which physical nature is that of a surface gravity wave: this is the so-called fundamental, or f-mode. A more comprehensive discussion of the classification of nonradial oscillations can be found in [6.1].

6.6 Inversion of the sound speed in solar interior

In solar seismology we have precise measurements of a large number of p-mode frequencies of different degree ℓ (from zero to few thousands) and of different radial order n. These observational data allow to address the inverse problem of solar seismology - the reconstruction of the internal structure of the Sun from its oscillation frequencies.

Using the local dispersion analysis of the previous section, the resonant frequencies must satisfy

$$\int_{r_1}^{R} k_r(r)\,dr = \pi(n+\alpha)\,, \tag{6.18}$$

where r_1 is an inner turning point with $k_r(r_1) = 0$. Equation (6.18) states that at frequencies of acoustic resonances, integer number of half-waves in $U(r)$ shall fit within the acoustic cavity $r_1 < r < R$; this number is radial order n. An additional phase shift α, which we will not specify explicitly here, accounts for a proper boundary conditions at the top and at the bottom of the acoustic cavity.

With radial wavenumber k_r specified by equation (6.15), we have

$$F(\tilde{w}) = \pi\frac{n+\alpha}{\omega}\,, \tag{6.19}$$

$$F(\tilde{w}) = \int_{r_1}^{R}\left(\frac{r^2}{c^2} - \tilde{w}^2\right)^{1/2}\frac{dr}{r}\,, \tag{6.20}$$

where

$$\tilde{w}^2 = \frac{\ell(\ell+1)}{\omega^2}. \tag{6.21}$$

The oscillation frequencies provide the right-hand side of the equation (6.19), and thus function $F(\tilde{w})$ is available from the observations. From its definition (6.20), $F(\tilde{w})$ is determined by the sound-speed profile $c(r)$ only. The integral relation (6.20) between the two functions $F(\tilde{w})$ and $c(r)$ can be inverted analytically. The result is

$$\ln\frac{r_1}{R} = \frac{2}{\pi}\int_{r_1/c_1}^{R/c_s}\left(\tilde{w}^2 - \frac{r_1^2}{c_1^2}\right)^{-1/2}\frac{dF}{d\tilde{w}}\,d\tilde{w}, \qquad (6.22)$$

where $c_1 = c(r_1)$ and $c_s = c(R)$. With expression (6.22), we obtain r_1 as a function of r_1/c_1, and hence c as a function of r.

Exercise 6.2 Prove the equation (6.22). [Hint: differentiate $F(\tilde{w})$ as given by (6.20), substitute the resulted integral into (6.22), and change the order of integration]. This exercise is an optional one: try it only if you feel yourself confident with double integrals. 

LITERATURE

[6.1] Unno, W., Osaki, Y., Ando, H., Saio, H. & Shibahashi, H., 1989. *Nonradial oscillations of stars* (2nd edition) (University of Tokyo Press).

Web
http://www.maths.qmul.ac.uk/~svv/

EXERCISE 6.1

With $\partial\mathbf{u}/\partial t = \partial^2\delta\mathbf{r}/\partial t^2 = -\omega^2\delta\mathbf{r}$, and

$$\nabla p' = \hat{\mathbf{r}}\frac{\partial}{\partial r}p' + \frac{1}{r}\nabla_1 p',$$

and $\nabla\psi_0 = \hat{\mathbf{r}}g_0$, the radial component of the momentum equation (6.7a) reduces to (6.9a), and its horizontal component - to (6.9b).

With

$$\nabla\cdot(\rho_0\mathbf{u}) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\rho_0 u_r\right) + \frac{1}{r}\nabla_1\cdot(\rho_0\mathbf{u})$$

and

$$\nabla_1\cdot(\rho_0\mathbf{u}) = \rho_0\nabla_1\cdot\mathbf{u} = i\omega\rho_0\nabla_1\cdot\delta\mathbf{r}$$

$$= i\omega\rho_0 V\,\nabla_1^2 Y_{\ell m} = -i\omega\ell\left(\ell+1\right)\rho_0 V\,Y_{\ell m},$$

the continuity equation (6.7b) gives (6.9c). With $\nabla p_0 = \hat{\mathbf{r}} \dfrac{dp_0}{dr}$ and $\nabla \rho_0 = \hat{\mathbf{r}} \dfrac{d\rho_0}{dr}$, the adiabatic energy equation (6.7c) gives (6.9d).

Now express ρ_1 in terms of p_1 and U from the equation (6.9d):

$$\rho_1 = \frac{1}{c^2} p_1 - \left(\frac{d\rho_0}{dr} - \frac{1}{c^2} \frac{dp_0}{dr} \right) U = \frac{1}{c^2} p_1 + \frac{\rho_0}{g_0} N^2 U$$

and $V = p_1 / (\rho_0 \omega^2)$ from the equation (6.9b), and substitute into (6.9a,c) to eliminate ρ_1 and V and to get (6.10b,a).

EXERCISE 6.2

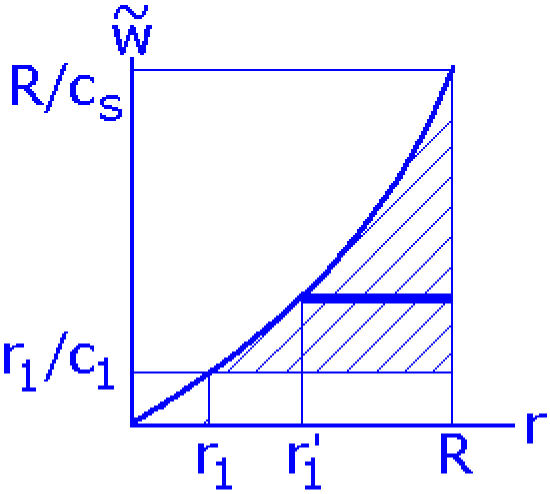
We need to prove that the equation

$$F(\tilde{w}) = \int_{r_1}^{R} \left(\frac{r^2}{c^2} - \tilde{w}^2 \right)^{1/2} \frac{dr}{r}, \qquad (6.20)$$

when considered as an integral equation with function $F(\tilde{w})$ specified, has a solution

$$\ln \frac{r_1}{R} = \frac{2}{\pi} \int_{r_1/c_1}^{R/c_s} \left(\tilde{w}^2 - \frac{r_1^2}{c_1^2} \right)^{-1/2} \frac{dF}{d\tilde{w}} d\tilde{w}, \qquad (6.22)$$

which allows to infer r_1 as function of r_1 / c_1, and hence c as function of r.



Take the derivative of the both sides of (6.20) with respect to $\tilde{w}$, considered as a parameter in the integral:

$$\frac{dF(\tilde{w})}{d\tilde{w}} = -\tilde{w}\int_{r_1}^{R}\left(\frac{r^2}{c^2} - \tilde{w}^2\right)^{-1/2}\frac{dr}{r},$$

and substitute the result into (6.22):

$$\ln\frac{r_1}{R} = -\frac{2}{\pi}\int_{\frac{r_1}{c_1}}^{\frac{R}{c_s}}\left(\tilde{w}^2 - \frac{r_1^2}{c_1^2}\right)^{-\frac{1}{2}} d\tilde{w}\,\tilde{w}\int_{r_1'}^{R}\left(\frac{r^2}{c^2} - \tilde{w}^2\right)^{-\frac{1}{2}}\frac{dr}{r},$$

where r_1' is such that

$$r_1'/c(r_1') = \tilde{w}.$$

Now change the order of integration, using $2\tilde{w}\,d\tilde{w} = d(\tilde{w}^2)$:

$$\ln\frac{r_1}{R} = -\frac{1}{\pi}\int_{r_1}^{R}\frac{dr}{r}\int_{\frac{r_1^2}{c_1^2}}^{\frac{r^2}{c^2}}\left(\tilde{w}^2 - \frac{r_1^2}{c_1^2}\right)^{-\frac{1}{2}}\left(\frac{r^2}{c^2} - \tilde{w}^2\right)^{-\frac{1}{2}} d(\tilde{w}^2).$$

To evaluate the inner integral, simplify it by using new variable t (just linear rescaling of $\tilde{w}^2$) as

$$t = 2\,\frac{\tilde{w}^2 - r_1^2/c_1^2}{r^2/c^2 - r_1^2/c_1^2} - 1,$$

so that

$$\tilde{w}^2 - \frac{r_1^2}{c_1^2} = \frac{r^2/c^2 - r_1^2/c_1^2}{2}(1+t),$$

$$\frac{r^2}{c^2} - \tilde{w}^2 = \frac{r^2/c^2 - r_1^2/c_1^2}{2}(1-t),$$

$$d(\tilde{w}^2) = \frac{r^2/c^2 - r_1^2/c_1^2}{2}\,dt.$$

We thus have

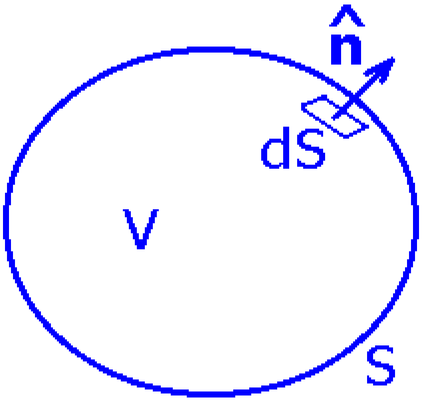
$$\ln\frac{r_1}{R} = -\frac{1}{\pi}\int_{r_1}^{R}\frac{dr}{r}\int_{-1}^{1}\frac{dt}{(1-t^2)^{1/2}}.$$

With substitution $t = \sin(\theta)$, the inner integral is

$$\int_{-1}^{1}\frac{dt}{(1-t^2)^{1/2}} = \int_{-\pi/2}^{\pi/2}\theta\,d\theta = \pi,$$

and we arrive to the identity

$$\ln\frac{r_1}{R} = -\int_{r_1}^{R}\frac{dr}{r}.$$



THE DIVERGENCE THEOREM

EXERCISE 1.1

Let temperature $T(\mathbf{r}, t)$ varies in both space ($\nabla T \neq 0$) and time ($\partial T / \partial t \neq 0$), but appears to be constant in time for an observer moving together with a fluid element ($DT / Dt = 0$). In this situation, the contribution to the temperature variation in the element due to the variation of its position $\mathbf{r}$, this contribution being $\mathbf{u} \cdot \nabla T$, is compensated by the contribution due to the time dependence of $T(\mathbf{r}, t)$, this second contribution being $\partial T / \partial t$. We have $\partial T / \partial t = -\mathbf{u} \cdot \nabla T$.

When $\partial T / \partial t = 0$, the temperature field $T(\mathbf{r}, t)$ is stationary: at any point is space, T remains constant in time. When the fluid is in motion, the temperature of a fluid element can change in time due to its displacement to hotter or cooler regions. The rate of change will be $DT / Dt = \mathbf{u} \cdot \nabla T$.

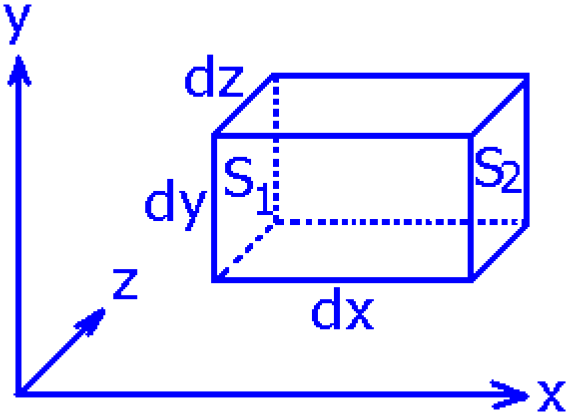
THE DIVERGENCE THEOREM

The divergence theorem (or Gauss theorem) states that for an arbitrary vector field $\mathbf{F}(\mathbf{r})$

$$\int_V \nabla \cdot \mathbf{F} \, dV = \int_S \mathbf{F} \cdot \hat{\mathbf{n}} \, dS,$$

for an arbitrary volume V bounded by a closed surface S. Here $\hat{\mathbf{n}}$ is unit vector normal to S and directed to the outside of volume V.

Proof. In cartesian coordinates x, y, z, consider a small rectangular element with dimensions dx, dy, dz. Let the element is small enough so that the divergence of the vector field $\mathbf{F} = (F_x, F_y, F_z)$, which is



$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z},$$

can be considered uniform inside the element. The vector flux across the surface surrounding the element is the sum of six fluxes. Consider the contribution of the vector fluxes crossing the two surfaces S_1 and S_2, both orthogonal to x. This contribution is provided by a small increment in F_x due to the increment in x,

$$dF_x = \frac{\partial F_x}{\partial x} dx,$$

times the area of S_1 and S_2, which is $dy\,dz$. Adding similar contributions of the vector fluxes crossing the surfaces orthogonal to z and y, the total vector flux to the outside of the rectangular element is

$$\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dx\,dy\,dz = \nabla \cdot \mathbf{F}\, dV.$$

Now any finite volume V can be considered as a sum of the small rectangular elements; summing $\nabla \cdot \mathbf{F}\, dV$, we get the volume integral in the divergence theorem. Summing the vector fluxes, the contributions coming from the rectangular areas separating any two neighbouring elements cancel out, and we are left with vector flux across the outer boundary S of V.

Application of the divergence theorem here might look unusual, because p is scalar and hence

$$\int_S p\hat{\mathbf{n}}\, dS$$

is vector, not a scalar. The procedure is nevertheless straightforward, if we consider the unit vector $\hat{\mathbf{n}}$ separately in tree components

$$\hat{\mathbf{n}} = (\hat{\mathbf{x}} \cdot \hat{\mathbf{n}})\hat{\mathbf{x}} + (\hat{\mathbf{y}} \cdot \hat{\mathbf{n}})\hat{\mathbf{y}} + (\hat{\mathbf{z}} \cdot \hat{\mathbf{n}})\hat{\mathbf{z}}$$

were hats denote unit vectors in cartesian coordinates (x, y, z). We then have

$$\int_S p\hat{\mathbf{n}}\, dS = \hat{\mathbf{x}}\int_S p\hat{\mathbf{x}}\cdot\hat{\mathbf{n}}\, dS + \hat{\mathbf{y}}\int_S p\hat{\mathbf{y}}\cdot\hat{\mathbf{n}}\, dS + \hat{\mathbf{z}}\int_S p\hat{\mathbf{z}}\cdot\hat{\mathbf{n}}\, dS\,.$$

We now apply the divergence theorem to the three integrals separately:

$$\int_S p\hat{\mathbf{n}}\, dS = \hat{\mathbf{x}}\int_V \nabla\cdot(p\hat{\mathbf{x}})\, dV + \hat{\mathbf{y}}\int_V \nabla\cdot(p\hat{\mathbf{y}})\, dV$$
$$+\hat{\mathbf{z}}\int_V \nabla\cdot(p\hat{\mathbf{z}})\, dV\,,$$

and with

$$\nabla\cdot(p\hat{\mathbf{x}}) = \partial p/\partial x,\ \nabla\cdot(p\hat{\mathbf{y}}) = \partial p/\partial y,\ \nabla\cdot(p\hat{\mathbf{z}}) = \partial p/\partial z\,,$$

the result is

$$\int_S p\hat{\mathbf{n}}\, dS = \int_V \left(\frac{\partial p}{\partial x}\hat{\mathbf{x}} + \frac{\partial p}{\partial y}\hat{\mathbf{y}} + \frac{\partial p}{\partial z}\hat{\mathbf{z}}\right) dV = \int_V \nabla p\, dV\,.$$
