## Tensors for Real

It is not intended that the full detail of this handout be examinable.

Proper time $\tau$ is an example of a scalar: it has no space-time indices and its value is independent of coordinate frame.
Four velocity $\frac{d x^{\mu}}{d \tau}$ is a [four-]vector: it has one index and its components are sensitive to our choice of coordinates.
The metric tensor $g_{\mu \nu}$ is a tensor of rank 2: it has two indices. We can also refer to vectors as tensors of rank 1 , and to scalars as tensors of rank 0 .

It is common practice to refer to the whole of a tensor (more correctly, say, $\underline{\underline{g}}$ ) in terms of its arbitrary component (say $g_{\mu \nu}$ ), and we have done so above.

We have to distinguish contravariant indices, written as superscripts as in $\frac{d x^{\mu}}{d \tau}$, from covariant indices written as subscripts, as in $g_{\mu \nu}$. Be careful not to confuse contravariant indices with powers!

The key defining properties of tensors are their transformation laws under a change of coordinates, from old coordinates $x^{\mu}$ to new coordinates $x^{\prime \mu}$ where each of the new coordinates is in general a function of all of the old ones, for example $x^{\prime 2}=f^{(2)}\left(x^{0}, x^{1}, x^{2}, x^{3}\right)$.

For a contravariant vector we have new components

$$
A^{\prime \mu}=\frac{\partial x^{\prime \mu}}{\partial x^{v}} A^{v} \quad \text { (summation } \sum_{v=0}^{3} \text { implied over repeated index } v \text { ) }
$$

and you can verify that this is obeyed by the four-velocity, where it corresponds to the chain rule of partial differentiation.

For a covariant vector we have $\quad B_{\mu}^{\prime}=\frac{\partial x^{v}}{\partial x^{\prime \mu}} B_{v} \quad$ (summation over $v$ implied).
For higher rank tensors the rules apply separately on each index, so for example we have

$$
g_{\mu \nu}^{\prime}=\frac{\partial x^{\alpha}}{\partial x^{\prime \mu}} \frac{\partial x^{\beta}}{\partial x^{\prime \nu}} g_{\alpha \beta} \text { (which you can also confirm from preservation of the metric). }
$$

The metric tensor plays a special role.
$A_{\mu}=g_{\mu \nu} A^{\nu}$ interrelates covariant and contravariant index forms of the same physical tensor, and
$A^{\mu}=g^{\mu v} A_{v}$ where (evidently) $g^{\mu v}$ is the matrix inverse of $g_{\mu \nu}$.

Differentiation is delicate because the metric tensor varies through space-time.
The covariant differential compares a tensor between two nearby events in space-time:

$$
D A^{\mu}=d A^{\mu}+\Gamma^{\mu}{ }_{\alpha \beta} A^{\alpha} d x^{\beta}
$$

components of physical change $=$ change in component values + correction due to distortion and rotation of coordinate frame.
The covariant derivative $\frac{D A^{\mu}}{D \tau}$ is a valid tensor, whilst the derivative of components $\frac{d A^{\mu}}{d \tau}$ is not.
The affine connections or Christoffel symbols are given by

$$
\Gamma^{\mu}{ }_{\alpha \beta}=\frac{1}{2} g^{\mu v}\left(\frac{\partial g_{v \alpha}}{\partial x^{\beta}}+\frac{\partial g_{v \beta}}{\partial x^{\alpha}}-\frac{\partial g_{\alpha \beta}}{\partial x^{v}}\right),
$$

and the covariant derivative of higher rank tensors involves one affine connection term for each index (and hence none for scalars, thankfully).

There is a subtlety about the four-velocity $\frac{d x^{\mu}}{d \tau}$ being a valid tensor: it is, without any affine connection terms, whilst the bare coordinates $\chi^{\mu}$ are not themselves a valid tensor.
Thus a little confusingly the co[ntra]variant four-acceleration is given by:

$$
\frac{D}{D \tau} \frac{d x^{\mu}}{d \tau}=\frac{d^{2} x^{\mu}}{d \tau^{2}}+\Gamma^{\mu}{ }_{\alpha \beta} \frac{d x^{\alpha}}{d \tau} \frac{d x^{\beta}}{d \tau}
$$

and geodesic paths are defined by this being zero.

The affine connections, which are first derivatives of the metric tensor, are also not tensors (otherwise everything else would have to be, in the equation $\frac{D A^{\mu}}{D \tau}=\frac{d A^{\mu}}{d \tau}+\Gamma^{\mu}{ }_{\alpha \beta} A^{\alpha} \frac{d x^{\beta}}{d \tau}$ ). There are no tensors you can make from first derivatives of the metric tensor, because $D g_{\mu \nu}=0$.

Importantly, at a given event in space-time there always exist choices of coordinates such that at that event the metric tensor is simply that of Minkowski and the affine connections are all zero.

The only way you can tell locally that spacetime might not be flat is to look at the curvature, most generally the Riemann Tensor

$$
R^{\alpha}{ }_{\beta \gamma \delta}=\frac{\partial \Gamma^{\alpha}{ }_{\beta \delta}}{\partial x^{\gamma}}-\frac{\partial \Gamma^{\alpha}{ }_{\beta \gamma}}{\partial x^{\delta}}+\Gamma^{\alpha}{ }_{\rho \gamma} \Gamma^{\rho}{ }_{\beta \delta}-\Gamma^{\alpha}{ }_{\rho \delta} \Gamma^{\rho}{ }_{\beta \gamma}
$$

which clearly involves second derivatives of the metric tensor. This is a true tensor, and if written with all covariant (i.e. 'down') indices it is: anti-symmetric with respect to interchange of its first or of its second pair of indices, and symmetric with respect to interchange of the first pair with the second pair. That leaves 20 distinct non-zero components in 4 dimensional space-time.

The simpler rank 2 curvature tensor is given by the Ricci tensor:

$$
R_{\beta \gamma}=R^{\alpha}{ }_{\beta \gamma \alpha} \quad \text { (note summation on the repeated index) }
$$

and from this the Ricci Scalar: $\quad R=g^{\mu \nu} R_{\mu \nu}$.
For gravity Einstein needed a combination which is rank 2 and has zero divergence:

$$
G_{\mu \nu}=R_{\mu \nu}-\frac{1}{2} R g_{\mu \nu}
$$

