PHY101 Electricity and Magnetism I
Topic 4 (Lectures 5 & 6) – Electrostatic Potential

In this topic, we will cover:
1) Potential Energy of a Charge and Electrostatic Potential
2) Potential of Point and Spherical Charge Distributions
3) Deriving Electric Fields from Potentials
4) The Potential Energy of Systems of Charges and Charge Distributions

Introduction

An object which can move vertically in a gravitational field changes its gravitational potential energy. The concept of electrostatic potential energy is equally important in electric fields. This is generalised to electrostatic potential – the potential energy a unit charge would have at a point due to the electric fields around it. We will also see that electric field can be derived from a potential distribution, just as the potential can be evaluated from the field.

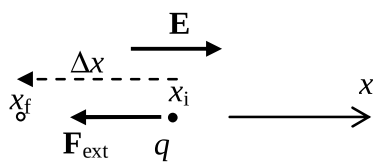
Electrostatic Potential

When a particle moves under the action of a force, it is capable of doing work. The energy to do this must come from somewhere – the potential energy of the object must decrease. Similarly applying a force to move a charge against an electrostatic force does work against the electric field and so builds up the potential energy, U. If the external force is equal and opposite to the electrostatic force (so that the kinetic energy of the object does not change), the work done is just the change in potential energy.

$$W_{\text{ext}} = \Delta U = U_{\text{f}} - U_{\text{i}}$$

where U_{f} is the final value of the potential energy and U_{i} is the initial value.

Consider a test charge q in a *uniform* electric field $\mathbf{E}$. The electrostatic force acting on q is $\mathbf{F} = q\mathbf{E}$. To move the charge through a distance Δx as shown, an external force $\mathbf{F}_{\text{ext}}$, equal and opposite to $\mathbf{F}$, must be applied. In moving the charge, the work done is $W_{\text{ext}} = F\left(x_{\text{f}} - x_{\text{i}}\right) = F\Delta x = -qE\Delta x$.



Note that in this example, the value of Δx is negative.

The change in potential is therefore $\Delta U = -qE\Delta x$.

For a given electric field, the potential energy of a test charge obviously depends on the size of the charge. It is convenient to define the electrostatic potential V as being the potential energy a *unit* charge would possess at that point.

Hence $$\Delta V = \frac{\Delta U}{q} \qquad [1]$$

and $$\Delta V = -E\Delta x. \qquad [2]$$

The S.I unit of potential is the volt (V), which is equal to 1 Joule per Coulomb, 1 V = 1 J C^{-1}.

The absolute potential energy is not something we can observe – we can only measure *changes* in potential energy. This gives us the freedom to choose our zero point. The usual conventions are to define the zero of potential energy (and hence of electrostatic potential) where charges are separated by infinite distances, or in an electric circuit we may choose the ground as being at zero potential.

The potential at a point is the external work required to bring a positive unit charge from a position of zero potential to the given point, with no change in kinetic energy.

We can extend the above arguments to the case when the electric field is not necessarily uniform. For an infinitesimal displacement d**s** of a charge q, the work done by the field is $\mathbf{F}\cdot\mathrm{d}\mathbf{s} = q\mathbf{E}\cdot\mathrm{d}\mathbf{s}$. Note this is the work done *by* the field, so the potential energy of the system (charge plus field) must have changed by $\mathrm{d}U = -q\mathbf{E}\cdot\mathrm{d}\mathbf{s}$. For a finite displacement from point A to point B, the change in potential energy of the system is

$$\Delta U = U_{\mathrm{B}} - U_{\mathrm{A}} = -q\int_{\mathrm{A}}^{\mathrm{B}} \mathbf{E}\cdot\mathrm{d}\mathbf{s}.$$

In terms of electrostatic potential, we can use [1] to deduce that for an infinitesimal displacement

$$\mathrm{d}V = \frac{\mathrm{d}U}{q} = -\mathbf{E}\cdot\mathrm{d}\mathbf{s} \qquad [3]$$

or more generally

$$\Delta V = V_{\mathrm{B}} - V_{\mathrm{A}} = -\int_{\mathrm{A}}^{\mathrm{B}} \mathbf{E}\cdot\mathrm{d}\mathbf{s}. \qquad [4]$$

Note that electric field is conservative, so the value of the integral depends only on the end points A and B, and not on the path taken between them.

If we consider a path that is parallel to the field line, $\mathbf{E}\cdot\mathrm{d}\mathbf{s}$ must be positive, so ΔV is negative. *Electric field lines therefore always point in the direction of decreasing potential.*

Equation [1] shows us that $\Delta U = q\Delta V$. A positive charge therefore loses potential energy when it moves in the direction of a field line. (This is the direction it would naturally tend to move under the influence of the field, and it may gain kinetic energy as a result.) On the other hand a negative charge gains energy when it moves in the direction of the field (and this must happen as a result of an external force).

An *equipotential surface* is a surface which joins a continuous distribution of points all having the same potential. Electric field lines run perpendicular to such surfaces. No work is required to move a particle along an equipotential surface.

<u>Electrostatic Potential of a Point Charge</u>

We will now consider the potential in the vicinity of various charge distributions. For a point charge, we saw in Topic 2 that the electric field is given by

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$$

Equation [4] enables us to find the change in potential moving from point A to point B. If we choose point A to be at zero potential – by being an infinite distance from the charge – we can find the absolute potential at a point B, a distance r_{B} from the charge. We can also simplify the integral by choosing a path **s** which is radial, and so parallel to the field lines. Hence

$$V(r_B) = -\int_{\infty}^{r_{\mathrm{B}}} \mathbf{E}\cdot\mathrm{d}\mathbf{s} = -\int_{\infty}^{r_{\mathrm{B}}} \frac{Q}{4\pi\varepsilon_0 r^2}\mathrm{d}r = \left[\frac{Q}{4\pi\varepsilon_0 r}\right]_{\infty}^{r_{\mathrm{B}}} = \frac{Q}{4\pi\varepsilon_0 r_B}. \qquad [5]$$

(Be careful not to confuse the expression for electric field about a point charge, $\mathrm{E} = \dfrac{Q}{4\pi\varepsilon_0 r^2}$ *with that for potential* $V = \dfrac{Q}{4\pi\varepsilon_0 r}$ *!)*

Electrostatic Potential about a Charged Conducting Sphere of Radius R

We have seen that electric charge distributes itself over the surface of a conductor. Gauss's law also told us that the flux through a surface only depends on the total charge contained within it. The electric field outside a charged sphere is therefore identical to that around a point charge of the same value as the charge contained on the sphere. The potential in this region must also be identical to that evaluated above, $V = \dfrac{Q}{4\pi\varepsilon_0 r}$. [6]

There is no electric field *inside* the conductor. There can therefore be no change in the potential from point to point – throughout the sphere it must have the same value as at its surface:

For $r < R$
$$V = \frac{Q}{4\pi\varepsilon_0 R}.$$

In general, all points within and on the surface of a conductor in electrostatic equilibrium are at the same potential.

Calculating Electric Field from Potential

We have calculated the potential from a known electric field distribution, using

$$\Delta V = V_\mathrm{B} - V_\mathrm{A} = -\int_\mathrm{A}^\mathrm{B} \mathbf{E}\cdot \mathrm{d}\mathbf{s}. \qquad [4]$$

This relationship can be inverted, to allow us to determine the electric field strength in situations where we know the potential. For an example where we know the direction of the field, for instance in the case of a point charge where we know E is radial, we can write

$$\Delta V = -\int_\mathrm{A}^\mathrm{B} E\,\mathrm{d}r$$

so
$$E = -\frac{\mathrm{d}V}{\mathrm{d}r}. \qquad [7]$$

(Check this, by deriving E from the expression for V in [6] above.)

More generally, we must separate the dot product into its components. If we write d**s** as $(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$, equation [4] becomes

$$\Delta V = -\int_\mathrm{A}^\mathrm{B} E_x \mathrm{d}x + E_y \mathrm{d}y + E_z \mathrm{d}z .$$

Hence
$$E_x = -\frac{\partial V}{\partial x}; \quad E_y = -\frac{\partial V}{\partial y}; \quad E_z = -\frac{\partial V}{\partial z}, \qquad [8]$$

or, using the *del* operator
$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\boxed{\mathbf{E} = -\nabla V}. \qquad [9]$$

[Note: $\dfrac{\partial V}{\partial x}$ is a *partial derivative*, the differential of V with respect to x while keeping y and z constant.]

If we consider [7], we can see an alternative set of units for electric field would be Volts per metre or V m^{-1}. In fact, these units are used more commonly than N C^{-1}.

The Potential and Potential Energy of a System of Charges

We saw above that the electrostatic potential at a distance r from a point charge Q was

$$V_Q = \frac{Q}{4\pi\varepsilon_0 r}. \qquad [6]$$

The potential energy of a charge q at this point is therefore (from [1])

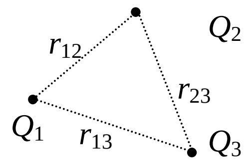
$$U = qV_Q = \frac{qQ}{4\pi\varepsilon_0 r}. \qquad [10]$$

You might think that the charge Q, seeing the electric field and potential due to q, would also have a potential energy $U = QV_q = \frac{Qq}{4\pi\varepsilon_0 r}$

so that the total potential energy would be $2\frac{Qq}{4\pi\varepsilon_0 r}$. **This would be wrong!!** The energy would have been double-counted.

The potential energy expressed in [10] is that contained in the *system* of charge plus field or charge plus other charge.

Since potential can be derived from electric field, we can use the principle of superposition to determine the potential at a point due to a system of charges. In the figure, Q_3 is at a point where the potential due to the other two charges is $V = \frac{Q_1}{4\pi\varepsilon_0 r_{13}} + \frac{Q_2}{4\pi\varepsilon_0 r_{23}}$. [11]



The total potential energy of the three charges, due to their interactions with each other, is just

$$U = \frac{Q_1 Q_3}{4\pi\varepsilon_0 r_{13}} + \frac{Q_2 Q_3}{4\pi\varepsilon_0 r_{23}} + \frac{Q_1 Q_2}{4\pi\varepsilon_0 r_{12}}$$

The Potential Energy of a Continuous Charge Distribution

We have seen that bringing charges closer together requires work to be done, which is stored as potential energy of the system. The same must be true of building up a continuous charge distribution, for example charging up a sphere with charge.

Consider a conducting sphere of radius R already carrying a charge Q. The work done in adding another element of charge dq (e.g. bringing it from infinity) is equal to the product of the charge element and the potential at the surface of the sphere:

$$\mathrm{d}U = V\,\mathrm{d}q = \frac{Q}{4\pi\varepsilon_0 R}\mathrm{d}q\,.$$

We can calculate the total potential energy of the charged sphere by considering building its charge up from 0 to a final value, say $Q_{\mathrm{tot.}}$

$$U = \int_0^{Q_{\mathrm{tot}}} \frac{q}{4\pi\varepsilon_0 R}\mathrm{d}q = \frac{1}{2}\frac{Q_{\mathrm{tot}}^2}{4\pi\varepsilon_0 R}\,.$$

[Note the integral produces a factor of ½ compared with considering the simple product of the total charge with the potential due to the total charge. This automatically avoids the "double counting" error discussed above when considering the interaction of discrete charges.]

Motion of Charges and the Electron-Volt

A free charge in an electric field will move in such a way that its total energy is conserved. In terms of kinetic energy K and potential energy U, this means

$$\Delta K + \Delta U = 0 .$$

We have already seen that for a charge q $\Delta U = q\Delta V$, so the kinetic energy obeys

$$\Delta K = -q\Delta V .$$

We might consider an electron moving between cathode and anode of a cathode ray tube. Here ΔV is positive, q is negative, so the change in K is positive, and kinetic energy is gained.

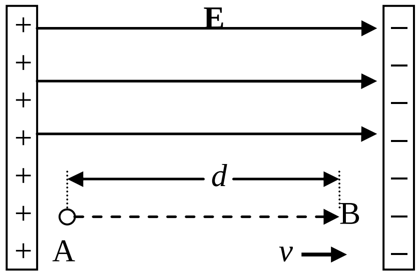
For elementary particles like electrons, it is often convenient to measure kinetic energy in a unit called the electron-volt (eV). When a particle with charge of magnitude e moves through a potential difference of 1 V, its kinetic energy changes by 1 eV

$$\boxed{1\ \text{eV} = 1.6\times10^{-19}\ \text{J}} \qquad [12]$$

Putting What You Have Learnt Into Practice

Question 1

A proton is released from rest in a uniform electric field of magnitude 8.0×10^4 V m^{-1} (see figure). The proton travels a distance of 0.5 m in the direction of **E**.



(a) Find the change in electric potential between points A and B.

(b) Find the change in potential energy of the proton-field system as a result of this displacement.

(c) Find the speed of the proton when it arrives at point B.

Solution

(a) $\Delta V = -Ed = -8.0\times10^4 \times 0.5 = -4.0\times10^4$ V .

(b) $\Delta U = q\Delta V = e\Delta V = 1.6\times10^{-19}\times\left(-4.0\times10^4\right) = -6.4\times10^{-15}$ J .

Note the negative sign means potential energy has been lost from the system as the proton moves in the direction of the field. This energy is gained as kinetic energy by the proton.

(c) The charge-field system is isolated, so its total energy is conserved.

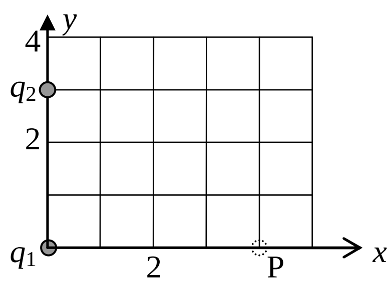
$$\Delta K + \Delta U = 0$$

$$\tfrac{1}{2}mv^2 = -\Delta U$$

$$v = \sqrt{-\frac{2\Delta U}{m}} = \sqrt{-\frac{2\times\left(-6.4\times10^{-15}\right)}{1.67\times10^{-27}}} = 2.8\times10^6 \text{ m s}^{-1}.$$

Question 2

A charge q_1 = 2.0 μC is located at the origin, and a charge q_2 = –6.0 μC is located at (0,3.0) m, as shown in the figure.



(a) Find the total electric potential due to these charges at the point P, with co-ordinates (4.0,0) m.

(b) Find the change in potential energy of the system of two charges plus a third charge q_3 = 3.0 μC as the latter is moved from infinity to the point P.

Solution

(a) Using [11], and Pythagoras for the distance from q_2 to P, we have

$$V_P = \frac{q_1}{4\pi\varepsilon_0 r_{1P}} + \frac{q_2}{4\pi\varepsilon_0 r_{2P}} = \frac{1}{4\pi\times8.85\times10^{-12}}\left(\frac{2.0\times10^{-6}}{4} + \frac{-6.0\times10^{-6}}{5}\right)$$
$$= -6.3\times10^3 \text{ V}$$

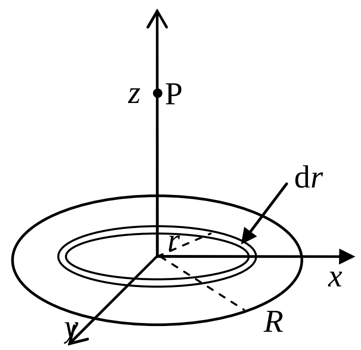
(b) By definition, when q_3 is at infinity it has zero potential energy. On moving to P, its potential energy changes to $q_3 V_P$. The change in potential energy of the system is therefore

$$\Delta U = q_3 V_P = 3.0\times10^{-6}\times\left(-6.3\times10^3\right) = -1.89\times10^{-2} \text{ J}.$$

The fact that this is negative means that (positive) work would have to be done to return q_3 to its position at infinity.

Question 3

What is the electric potential due to a disc of radius R carrying a uniform charge density σ C m^{-2}, at a point a distance z from the disc and on the axis of the disc?



Solution

This calculation proceeds in a similar way to that of the electric field, in question 4 of Topic 2. Consider the disc to be made up of concentric rings, of thickness dr, one of which is shown. The charge dQ on this ring is equal to σ × its area, or $2\pi r\sigma\,\mathrm{d}r$. The distance from point P of all points on the ring is $\sqrt{r^2+z^2}$, so the potential at P due to one ring of radius r is

$$\mathrm{d}V = \frac{\mathrm{d}Q}{4\pi\varepsilon_0\left(z^2+r^2\right)^{1/2}} = \frac{r\sigma\,\mathrm{d}r}{2\varepsilon_0\left(z^2+r^2\right)^{1/2}}\,.$$

Let $u = r^2 \qquad \Rightarrow \qquad \mathrm{d}u = 2r\,\mathrm{d}r$

$$V = \int_0^R \frac{r\sigma\,\mathrm{d}r}{2\varepsilon_0\left(z^2+r^2\right)^{1/2}} = \int_0^{R^2} \frac{\sigma\,\mathrm{d}u}{4\varepsilon_0\left(z^2+u\right)^{1/2}}$$

$$= \frac{\sigma}{4\varepsilon_0}\left[2\left(z^2+u\right)^{1/2}\right]_0^{R^2} = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{z^2+R^2}-z\right]$$

Note that this result gives us an alternative way of calculating the electric field due to the disc, as was discussed in Topic 2 Question 4. Since by symmetry the electric field on the axis must be parallel to **z**,

$$E = E_z = -\frac{\partial V}{\partial z}$$

$$= -\frac{\sigma}{2\varepsilon_0}\left(\frac{1}{2}\frac{2z}{\sqrt{z^2+R^2}}-1\right) = \frac{\sigma}{2\varepsilon_0}\left(1-\frac{1}{\sqrt{1+R^2/z^2}}\right)$$

Question 4
A charge Q is distributed uniformly throughout a sphere of radius R. Calculate the electric potential (a) at a point outside the sphere; (b) at a point within the sphere.

Solution
In the previous topic, we used Gauss's law to calculate the electric field due to this charge distribution. We can now use that result to find the potential. From equation [4]

$$\Delta V = V_{\mathrm{B}} - V_{\mathrm{A}} = -\int_{\mathrm{A}}^{\mathrm{B}} \mathbf{E} \cdot \mathrm{d}\mathbf{s}$$

and we know by symmetry that the field is radial. We may also take the potential at infinity as being zero, so

$$V(r) = -\int_{\infty}^{\mathrm{r}} \mathrm{E}\,\mathrm{d}r\,.$$

(a) For $r > R$, then

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}\,.$$

$$V(r) = -\int_{\infty}^{r} \frac{Q}{4\pi\varepsilon_0 r^2}\,\mathrm{d}r = -\left[\frac{-Q}{4\pi\varepsilon_0 r}\right]_{\infty}^{r} = \frac{Q}{4\pi\varepsilon_0 r} \qquad [13]$$

(which is the same as when Q is concentrated at a point at the centre of the sphere).

(b) When we consider a point inside the sphere, $r < R$, we cannot take a single expression for E all the way from infinity to the point. We can use [13] to find the potential at the surface of the sphere

$$V(R) = \frac{Q}{4\pi\varepsilon_0 R}$$

and then integrate from the surface in to the required point, using the previously calculated expression for E within the sphere

$$E = \frac{Qr}{4\pi\varepsilon_0 R^3}\,.$$

Since (from [4])

$$\Delta V = V(r) - V(R) = -\int_{R}^{r} E\,\mathrm{d}r$$

$$V(r) = -\int_{R}^{r} E\,\mathrm{d}r + V(R) = -\int_{R}^{r} \frac{Qr}{4\pi\varepsilon_0 R^3}\,\mathrm{d}r + \frac{Q}{4\pi\varepsilon_0 R}$$

$$= -\left[\frac{Qr^2}{2\times 4\pi\varepsilon_0 R^3}\right]_{R}^{r} + \frac{Q}{4\pi\varepsilon_0 R} = -\frac{Qr^2}{8\pi\varepsilon_0 R^3} + \frac{Q}{8\pi\varepsilon_0 R} + \frac{Q}{4\pi\varepsilon_0 R}$$

$$= \frac{Q}{8\pi\varepsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)$$