kepler's second law

Kepler also investigated the speeds of the planets and found that the closer in its orbit a planet is to the Sun, the faster it moves. Drawing a straight line connecting the Sun and the planet (the *radius vector*), he discovered that he could express this fact in his second law - the law of areas:

For any planet the radius vector sweeps out equal areas in equal times.

figure 32: Kepler's second law.

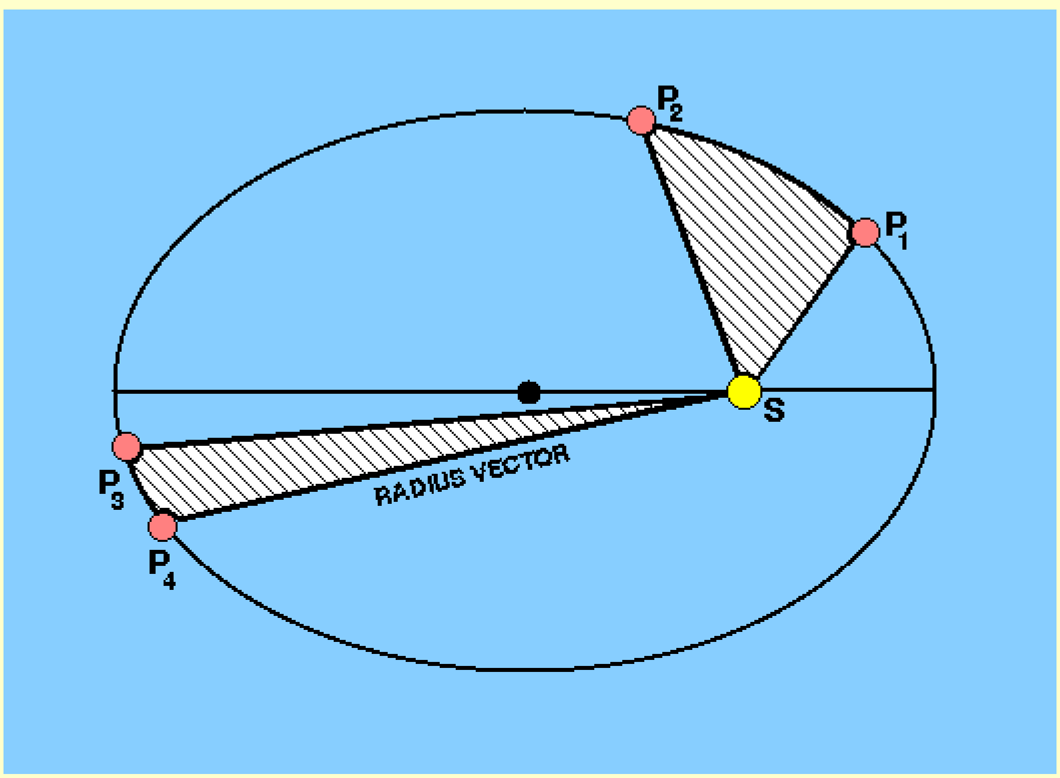

Figure 32 shows the positions of a planet, P_1, P_2, P_3, P_4, at times t_1, t_2, t_3, t_4. Between times t_1 and t_2 the planet's radius vector sweeps out the area bounded by the radius vectors SP_1, SP_2 and the arc P_1P_2. Similarly, the area swept out by the radius vector in the time interval (t_4-t_3) is the area SP_3P_4. Then, Kepler's second law states that:

area SP_1P_2 / $(t_2 - t_1)$ = area SP_3P_4 / $(t_4 - t_3)$ = constant.

If $t_2 - t_1 = t_4 - t_3$, then area SP_1P_2 = area SP_3P_4.

If the area is the area of the ellipse itself (which is given by πab), the radius vector will be back to its original position and Kepler's second law therefore implies that the planet's orbital period is constant.

It is possible to derive a mathematical expression for Kepler's second law by supposing that the time interval $(t_2 - t_1)$ is very small and equal to interval $(t_4 - t_3)$. Position P_2 will be very close to P_1, just as P_4 will be close to P_3. The area SP_1P_2 is then approximately the area of triangle SP_1P_2, or

$½SP_1$ x SP_2 x $\sin P_1SP_2$.

If angle P_1SP_2 is expressed in radians we may write

$\sin P_1SP_2$ = angle $P_1SP_2 = \theta_1$,

since angle P_1SP_2 is very small. Also,

$SP_1 = SP_2 = r_1$, say,

so that the area SP_1P_2 is given by

$½r^2{}_1\theta_1$.

Similarly, area SP_3P_4 is given by

$½r^2{}_2\theta_2$,

where $SP_3 = r_2$ and angle $P_3SP_4 = \theta_2$. Let $t_4 - t_3 = t_2 - t_1 = t$. Then, from Kepler's second law,

$½r^2{}_1(\theta_1 / t) = ½r^2{}_2(\theta_2 / t)$ = constant.

But θ/t is the angular velocity, ω, in the limit when t tends to zero. Hence

^{2} ω ^{2} ω

$\frac{1}{2}r_1 \quad _1 = \frac{1}{2}r_2 \quad _2$ = constant,

is the mathematical expression of Kepler's second law. In order for this law to be obeyed, the planet has to move fastest when its radius vector is shortest, at perihelion, and slowest when it is at aphelion, as shown in this java applet.
