homologous stellar models

We have already seen that it is often more convenient to write the equations of stellar structure in terms of M instead of r. Dividing each of the equations of stellar structure by the equation of mass conservation, and assuming that no energy is carried by convection, we can write:

I. $dP / dM = - GM / 4\pi r^4$,

II. $dr / dM = 1 / 4\pi r^2\rho$,

III. $dL / dM = \varepsilon$,

IV. $dT / dM = - 3\kappa L / 64\pi^2 acr^4 T^3$,

where we have also inverted the equation of mass conservation. The corresponding boundary conditions become:

$r = L = 0$ at $M = 0$ and

$\rho = T = 0$ at $M = M_s$.

These differential equations must be supplemented by the three additional relations for P, κ and ε. We derived approximate forms for these parameters in the physics of stellar interiors part of the course; if we assume that radiation pressure is negligible, that stellar material behaves as an ideal gas, and that the laws of opacity and energy generation can be approximated by power laws, we can write:

$P = R\rho T / \mu$,

$\kappa = \kappa_0 \rho^\alpha T^\beta$, and

$\varepsilon = \varepsilon_0 \rho T^\eta$,

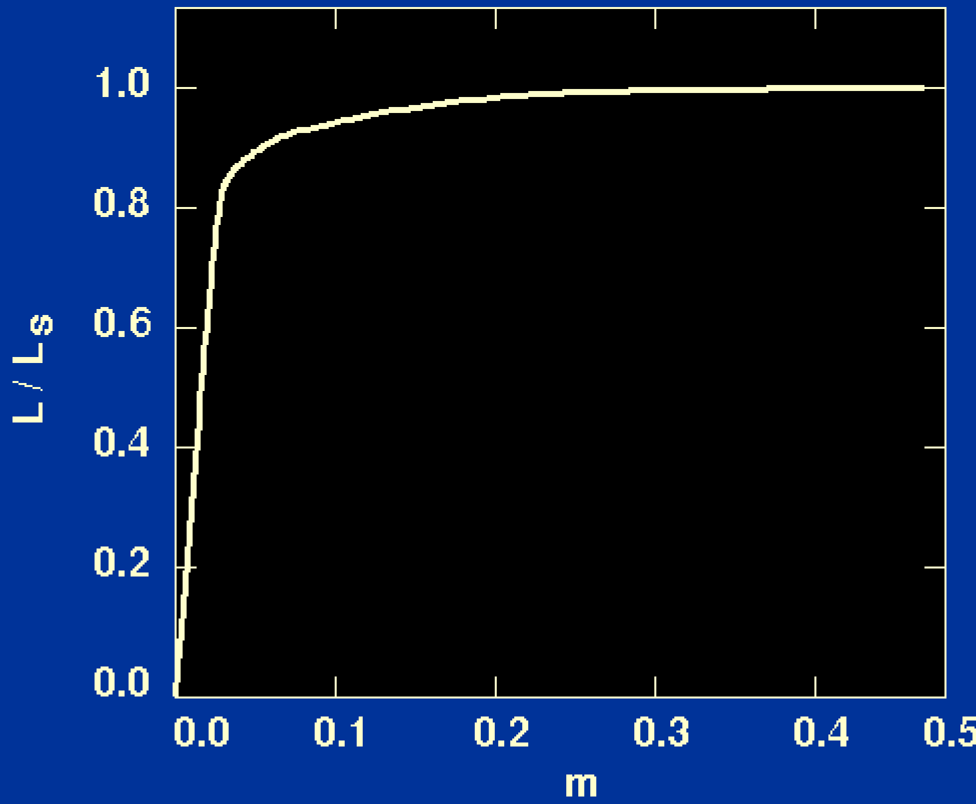
where α, β and η are constants and κ_0 and ε_0 are constants for a given chemical composition.

Using the above approximations in conjunction with the differential equations and associated boundary conditions, it is possible to solve for the structure of a star of any given total mass, M_s. If we then wish to consider the structure of a star of a different mass, the equations must be solved again for the new M_s. If we then wish to consider the structures of a large number of stars covering a wide range of M_s, such as when attempting to explain the observed mass-luminosity relation of main-sequence stars, we must solve the equations of stellar structure many times. This is clearly very tedious and time-consuming. Fortunately, there is a much simpler way of formulating the equations of stellar structure which ensures that, once the structure of a star of any one mass has been determined, the structure of a star of any mass (**but with the same homogeneous chemical composition**) can be obtained by a simple scaling of variables. Such a sequence of stellar models is known as a *homologous series* and we shall now describe how such a sequence can be obtained.

Our aim is to formulate the equations of stellar structure in such a way that they are independent of the total mass, M_s. In so doing, we are assuming that the way in which any physical quantity such as luminosity varies from the centre of the star to the surface is the same for stars of all masses and only the absolute value of the luminosity varies from star to star. This is illustrated schematically in figure 21, where the ratio of luminosity to surface luminosity (L/L_s) is plotted against the *fractional mass*, m, defined as the ratio of mass to total mass:

$$m = M / M_s.$$

Figure 21: Fractional luminosity as a function of fractional mass.



The curve shown in figure 21 is the same for all stars with the same laws of opacity and energy generation, but the value of L_s depends on M_s and it is proportional to some power of M_s, which depends on the values of α, β and η. The same is also true of the other physical quantities such as radius (r_s) and temperature (T).

Expressing this mathematically, we can write:

V. $r = M_s^{a_1} r^*(m)$

VI. $\rho = M_s^{a_2} \rho^*(m)$

VII. $L = M_s^{a_3} L^*(m)$

VIII. $T = M_s^{a_4} T^*(m)$

IX. $P = M_s^{a_5} P^*(m)$

where a_1 , a_2 , a_3 , a_4 and a_5 are constants and (as indicated) r^*, ρ^*, L^*, T^* and P^* depend only on the fractional mass, m.

If we now substitute the expressions V, VI, VII, VIII and IX for r, ρ, L, T and P into equations I, II, III and IV, we can eliminate M_s provided that the values of a_1 , a_2 , a_3 , a_4 and a_5 are chosen correctly. So, for equation I,

$$dP / dM = - GM / 4\pi r^4,$$

we can substitute equation V and IX and write:

$$M_s^{a_5}\, dP^* / dM = - GM / 4\pi M_s^{4a_1}\, r^{*4}.$$

The definition of fractional mass tells us that $M = mM_s$. Substituting this expression into the above equation gives:

$$M_s^{(a_5-1)}\, dP^* / dm = - GmM_s^{(1-4a_1)} / 4\pi r^{*4}.$$

In order to eliminate M_s, the powers of M_s on both sides of the equation must be equal, i.e.:

$$a_5 - 1 = 1 - 4a_1,$$

and hence,

$$4a_1 + a_5 = 2.$$

The equation then becomes:

$$dP^* / dm = - Gm / 4\pi r^{*4}.$$

Similarly, equation II,

$$dr / dM = 1 / 4\pi r^2\rho,$$

can be written in the form:

$$M_s^{(a_1-1)}\, dr^* / dm = 1 / 4\pi M_s^{(2a_1+a_2)} r^{*2}\rho^*,$$

which reduces to

$$dr^* / dm = 1 / 4\pi r^{*2}\rho^*,$$

in the case that

$$3a_1 + a_2 = 1.$$

In the same way, equation III,

$$dL / dM = \varepsilon = \varepsilon_0 \rho T^{\eta},$$

can be written as follows:

$$M_s^{(a_3-1)} dL^* / dm = \varepsilon_0 M_s^{(a_2+\eta a_4)} \rho^* T^{*\eta}.$$

This is independent of M_s if

$$a_3 = 1 + a_2 + \eta a_4,$$

in which case:

$$dL^* / dm = \varepsilon_0 \rho^* T^{*\eta}.$$

Similarly, equation IV,

$$dT / dM = - 3\kappa L / 64\pi^2 acr^4 T^3 = - 3\kappa_0 \rho^{\alpha} T^{(\beta-3)} L / 64\pi^2 acr^4,$$

can be written in the form:

$$M_s^{(a_4-1)} dT^* / dm = - 3\kappa_0 M_s^{[\alpha a_2 + a_4(\beta-3) + a_3 - 4a_1]} \rho^{*\alpha} T^{*(\beta-3)} L^* / 64\pi^2 acr^{*4}.$$

This equation is independent of M_s if

$$a_4 - 1 = \alpha a_2 + (\beta-3)a_4 + a_3 - 4a_1,$$

which reduces to

$$4a_1 + (4-\beta)a_4 = \alpha a_2 + a_3 + 1,$$

in which case:

$$dT^* / dm = - 3\kappa_0 \rho^{*\alpha} T^{*(\beta-3)} L^* / 64\pi^2 acr^{*4}.$$

Finally, treating the equation of state,

$P = R\rho T / \mu$,

in a similar manner gives:

$M_s^{a_5} P^* = R\, M_s^{(a_2+a_4)}\, \rho^*\, T^* / \mu$.

This equation is independent of M_s if

$a_5 = a_2 + a_4$,

in which case:

$P^* = R\, \rho^*\, T^* / \mu$.

We have now obtained 5 equations for the five constants a_1, a_2, a_3, a_4 and a_5. We have also obtained 5 new equations of stellar structure which are independent of M_s. They are only independent of M_s, however, if the 5 equations for a_1, a_2, a_3, a_4 and a_5 have consistent solutions. It can be shown that this is indeed the case for all reasonable values of α, β and η (see page 114 of Tayler). Note that the general solution for the a's is too complicated to be given here, but we will be considering solutions for special values of α, β and η later.

Collecting things together, the 5 new equations which govern the structure of a star of any mass are:

$dP^* / dm = - Gm / 4\pi r^{*4}$.

$dr^* / dm = 1 / 4\pi r^{*2}\rho^*$,

$dL^* / dm = \varepsilon_0\, \rho^*\, T^{*\eta}$.

$dT^* / dm = - 3\kappa_0\, \rho^{*\alpha}\, T^{*(\beta-3)} L^* / 64\pi^2 ac r^{*4}$.

$P^* = R\, \rho^*\, T^* / \mu$.

These equations can now be solved to find r^*, ρ^*, L^*, T^* and P^* in terms of m, using the boundary conditions:

$r^* = L^* = 0$ at $m = 0$ and

$\rho^* = T^* = 0$ at $m = 1$,

where the centre and surface of the star are at $m = 0$ and $m = 1$, respectively. The above equations must be solved on a computer and, after the solution has been obtained, the quantities r^*, ρ^*, L^*, T^* and P^* can be converted into r, ρ, L, T and P for a star of any given mass M_s by using the relations V, VI, VII, VIII and IX and the values of the constants a_1, a_2, a_3, a_4 and a_5 previously found. Hence the equations of stellar structure have only to be solved once in this manner and the properties of stars of all masses can then be obtained.