Lecture 8

Energy transport by convection

It is possible to calculate stellar models assuming that energy transport takes place through radiation alone. Such models do not provide a realistic description of real stars, however, because in general they are *unstable*.

Any theoretical model should be tested for possible instabilities, before it can be accepted as realistic. An instability manifests itself through the growth of any small disturbance with time, often exponentially. If the characteristic growth time is less than the evolutionary time scale for the star, the disturbance may in the end dominate the properties of the star. A particular type of instability is often found, namely the instability corresponding to having a layer of higher density on top of a layer with lower density.

An extreme analogy to this instability would be a glass where a layer of mercury had been placed on top of a layer of water. This is evidently an unstable situation.

In a star this type of instability can occur if the temperature decreases too rapidly with distance from the center. The decrease of pressure with r is determined by hydrostatic equilibrium, and is therefore largely given, and the only possibility for compensating for a rapid decrease in temperature is therefore, according to the ideal gas law, that the density decreases slowly or even increases; this leads to the instability. From the equation for radiative transfer (7.18) it follows that that the temperature decreases rapidly with increasing r when the opacity is high or the luminosity is high.

As a result of the instability hotter, relatively light elements of fluid rise and cooler, relatively heavy elements sink. When the motion gets sufficiently strong, the elements are dissolved and the gas is mixed. As a result, the rising elements deposit their excess heat to the surroundings, and this leads to a net transport of energy out through the star. This process is called *convection*, and the instability is called *convective instability*. Convection is well known from everyday life, for example when air raises over a heater. Besides contributing to the energy transport, convection also leads to mixing of the parts of the star where it occurs, which has a substantial effect on the evolution of some stars.

8.1. The instability condition

To determine the condition for instability we consider an element of gas (Figure 8.1) which is moved the distance Δr outwards.

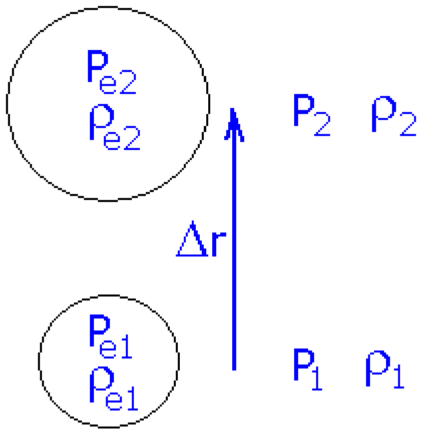

Figure 8.1. The motion of a convective element, from the initial position, indicated by "1", to a later position, indicated by "2".

As indicated we denote the pressure and density outside (inside) the element before and after the motion by P_1 , ρ_1, (P_{e1}, ρ_{e1}), and P_2 , ρ_2, (P_{e2}, ρ_{e2}). The element is initially identical to its surroundings, so that $P_{e1}=P_1$ and $\rho_{e1}=\rho_1$. Its motion is determined by buoyancy, which in turn is given by the difference in density between the element and its surroundings (think of a cork pushed down into water). The force per unit volume at the point 2 is

$$f_{buoy} = -g\left(\rho_{e2} - \rho_2\right) \equiv -g\Delta\rho,$$

$$f_{buoy} = -g\left(\rho_{e2} - \rho_2\right) \equiv -g\Delta\rho, \qquad (8.1)$$

where $g=Gm/r^2$ is the gravitational acceleration. If $f_{buoy}>0$ the force on the element is directed outwards, and the motion is accelerated; hence this corresponds to instability. In the opposite case the force is directed downwards, the element has a tendency to return to its original position, and the situation is stable.

To determine $\Delta\rho$, and hence decide whether between stability and instability, we assume that

(a) the element is always in pressure balance with the surroundings;

(b) the motion is fast enough so that there is no heat exchange between the element and the surroundings.

From assumption (a) we have $P_{2e}=P_2$. Assumption (b) expresses that the motion takes place adiabatically; from equation (3.33) it therefore follows that

$$\frac{d\rho_e}{\rho_e} = \frac{1}{\gamma}\frac{dP_e}{P_e} = \frac{1}{\gamma}\frac{dP}{P},$$

$$\frac{d\rho_e}{\rho_e} = \frac{1}{\gamma}\frac{dP_e}{P_e} = \frac{1}{\gamma}\frac{dP}{P},$$

(8.2)

where $d\rho_e$ and dP_e are the changes in ρ and P inside the element. From Taylor expansion we therefore obtain

$$\begin{aligned}\Delta\rho &= \rho_{e2} - \rho_2 = \rho_{e2} - \rho_{e1} - (\rho_2 - \rho_1) \\ &\approx \rho_1 \frac{1}{\gamma}\frac{1}{P_1}\frac{dP}{dr}\Delta r - \frac{d\rho}{dr}\Delta r \\ &\approx \left(\frac{\rho_1}{P_1}\frac{1}{\gamma}\frac{dP}{dr} - \frac{d\rho}{dr}\right)\Delta r = \left[\left(\frac{d\rho}{dr}\right)_{ad} - \frac{d\rho}{dr}\right]\Delta r,\end{aligned}$$

$$\begin{aligned}\Delta\rho &= \rho_{e2} - \rho_2 = \rho_{e2} - \rho_{e1} - (\rho_2 - \rho_1) \\ &\approx \rho_1 \frac{1}{\gamma}\frac{1}{P_1}\frac{dP}{dr}\Delta r - \frac{d\rho}{dr}\Delta r \\ &\approx \left(\frac{\rho_1}{P_1}\frac{1}{\gamma}\frac{dP}{dr} - \frac{d\rho}{dr}\right)\Delta r = \left[\left(\frac{d\rho}{dr}\right)_{ad} - \frac{d\rho}{dr}\right]\Delta r,\end{aligned}$$

(8.3)

where we introduced

$$\left(\frac{d\rho}{dr}\right)_{ad} \equiv \frac{1}{\gamma}\frac{\rho}{P}\frac{dP}{dr},$$

$$\left(\frac{d\rho}{dr}\right)_{ad} \equiv \frac{1}{\gamma}\frac{\rho}{P}\frac{dP}{dr},$$

(8.4)

the density gradient resulted from adiabatic motion in the given pressure gradient.

The condition for *instability* is that $\Delta\rho < 0$ (see equation (8.1)), i.e.

$$\frac{d\ln\rho}{d\ln P} < \frac{1}{\gamma}.$$

$$\frac{d\ln\rho}{d\ln P} < \frac{1}{\gamma}. \quad (8.5)$$

Note that for a completely ionized ideal gas $1/\gamma = 3/5$.

The instability condition is normally expressed in terms of the gradient in temperature, rather than the gradient in density. We use the ideal gas law, written in the form

$$\rho = \frac{\mu m_H P}{kT}.$$

$$\rho = \frac{\mu m_H P}{kT}. \quad (8.6)$$

It is normally assumed that the chemical composition is independent of position. If we further assume that the gas is fully ionized, μ is constant, and we obtain by differentiation

$$\frac{1}{\rho}\frac{d\rho}{dr} = \frac{1}{P}\frac{dP}{dr} - \frac{1}{T}\frac{dT}{dr}.$$

$$\frac{1}{\rho}\frac{d\rho}{dr} = \frac{1}{P}\frac{dP}{dr} - \frac{1}{T}\frac{dT}{dr}. \quad (8.7)$$

This leads to

$$\left(\frac{d\rho}{dr}\right)_{ad} - \frac{d\rho}{dr} = \frac{1}{\gamma}\frac{\rho}{P}\frac{dP}{dr} - \frac{\rho}{P}\frac{dP}{dr} + \frac{\rho}{T}\frac{dT}{dr}$$
$$= -\frac{\gamma - 1}{\gamma}\frac{\rho}{P}\frac{dP}{dr} + \frac{\rho}{T}\frac{dT}{dr}.$$

$$\left(\frac{d\rho}{dr}\right)_{ad} - \frac{d\rho}{dr} = \frac{1}{\gamma}\frac{\rho}{P}\frac{dP}{dr} - \frac{\rho}{P}\frac{dP}{dr} + \frac{\rho}{T}\frac{dT}{dr}$$
$$= -\frac{\gamma - 1}{\gamma}\frac{\rho}{P}\frac{dP}{dr} + \frac{\rho}{T}\frac{dT}{dr}. \quad (8.8)$$

Hence the instability condition becomes

$$\left(\frac{dT}{dr}\right)_{ad} > \frac{dT}{dr},$$

$$\left(\frac{dT}{dr}\right)_{ad} > \frac{dT}{dr},$$

(8.9)

where

$$\left(\frac{dT}{dr}\right)_{ad} \equiv \frac{\gamma - 1}{\gamma}\frac{T}{P}\frac{dP}{dr}$$

$$\left(\frac{dT}{dr}\right)_{ad} \equiv \frac{\gamma - 1}{\gamma}\frac{T}{P}\frac{dP}{dr}$$

(8.10)

is the adiabatic temperature gradient. In analogy with equation (8.5), equation (8.9) can also be written as

$$\frac{d\ln T}{d\ln P} > \frac{\gamma - 1}{\gamma}.$$

$$\frac{d\ln T}{d\ln P} > \frac{\gamma - 1}{\gamma}.$$

(8.11)

This equation shows that there is instability if the temperature decreases too rapidly out through the star, in perfect agreement with our simple discussion. For a fully ionized ideal gas, $(\gamma-1)/\gamma=2/5$.

8.2. Where does convection occur?

To determine the circumstances under which one may expect convection, we consider a model where energy transport takes place through radiation and investigate its stability. Here (equation 7.18)

$$\frac{dT}{dr} = -\frac{3\kappa\rho L(r)}{16\pi acr^2T^3}.$$

$$\frac{dT}{dr} = -\frac{3\kappa\rho L(r)}{16\pi acr^2T^3}.$$

(8.12)

By using the equation for hydrostatic equilibrium (4.5) and the ideal gas law (8.6), we have

$$\frac{d\ln T}{d\ln P} = \frac{3k}{16\pi acGm_H} \frac{\kappa}{\mu} \frac{L(r)}{m(r)} \frac{\rho}{T^3}.$$

$$\frac{d\ln T}{d\ln P} = \frac{3k}{16\pi acGm_H} \frac{\kappa}{\mu} \frac{L(r)}{m(r)} \frac{\rho}{T^3}.$$

(8.13)

From equations (8.11) and (8.13) it is evident that, roughly speaking, one may expect convection if

(a) $L(r)/m(r)$ is large. This condition expresses that the average rate of energy generation per unit mass within the radius r is large. This is typically the case in the interiors of massive stars. The energy generation in such stars is a rapidly increasing function of temperature (cf. equation (6.11)) and hence is strongly concentrated towards the center of the star. Therefore L/m is large, and the star has a convective core.

(b) κ is large. This is satisfied in the outer parts of relatively light stars on the main sequence, or more generally in stars with low surface temperatures, where the temperature in the outer parts of the star is low, and the opacity consequently high (cf. equation (7.19)). A further contribution to the high opacities in these regions comes from the ionization of hydrogen.

(c) ρ/T^3 is large. This is also typically satisfied in the outer parts of relatively cool stars.

(d) $(\gamma-1)/\gamma$, i.e. the adiabatic temperature gradient, is small. This is satisfied in the ionization zone of hydrogen, i.e. again in the outer parts of cool stars.

Thus condition (a) predicts convection in the core of massive stars, whereas the remaining conditions indicate a tendency for convection in the outer parts of cool stars, i.e. in relatively light stars on the main sequence, and in the so-called red giants. These locations of convection zones are summarized in Figure 8.2.

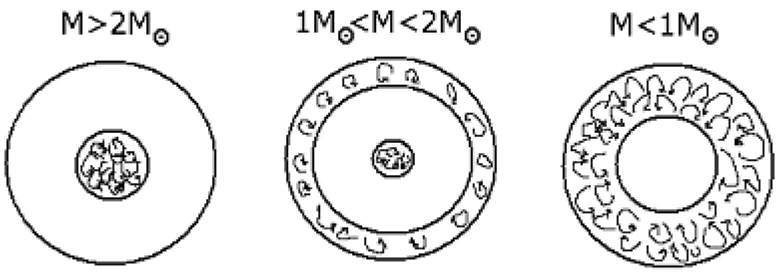

Figure 8.2. The typical occurrence of convection zones in main-sequence stars. In relatively massive stars there is a convective core, whereas in relatively light stars

on the main sequence, and in general in stars with low effective temperature, there is an outer convection zone.

8.3. Energy transport by convection.

The motion of a convective element after the onset of instability is extremely difficult to describe. As a result, there is no definitive method for calculating the motion or the convective energy transport. Presumably the velocity of the element increases up to the point where new hydrodynamical instabilities set in, making the motion turbulent and dissolving the element. In this way the excess heat in the element is deposited in the surroundings, hence leading to energy transport. The description of such turbulent process is, and has for a long time been, the subject of intensive investigations. A satisfactory understanding, or sufficiently efficient methods for numerical computation, has not been achieved so far. We are still very far from incorporating a complete numerical description of convection into computations of stellar models.

Fortunately a less complete description is adequate for such computations, at least as far as the overall properties of the stars are concerned. This only requires a relation for the temperature gradient required to transport the luminosity by convection, to replace equation (8.12) for radiative transport. It is possible to make a very rough estimate of the relationship between the temperature gradient and the luminosity. This is carried out in the remainder of this section. The result is that in most of the star the temperature gradient is only slightly steeper than the adiabatic gradient determined by equation (8.10).

As usual in these estimates, we neglect factors of order unity, and make rough approximations of the physics involved. We assume that a given convective element moves a distance Δr, before being destroyed. In the destruction, the surroundings receive the energy $\Delta u \approx \rho c_P \Delta T$ per unit volume, where

$$\Delta T = \left[\left(\frac{dT}{dr} \right)_{ad} - \frac{dT}{dr} \right] \Delta r$$

$$\Delta T = \left[\left(\frac{dT}{dr} \right)_{ad} - \frac{dT}{dr} \right] \Delta r$$

(8.14)

is the temperature difference between the element and the surroundings. If the mean speed of the element is v, the convective energy flux can thus be estimated as

$$F_{con} \approx v \rho c_P \Delta T.$$

(8.15)

$$F_{con} \approx v \rho c_P \Delta T.$$

To obtain an estimate of v we equate the kinetic energy $\frac{1}{2}\rho v^2$ of the element per unit volume to the work of buoyancy over the distance Δr. From equations (8.1), (8.3) and (8.8) we obtain (neglecting the factor $\frac{1}{2}$)

$$\rho v^2 \approx f_{buoy}\Delta r \approx -\left[\left(\frac{d\rho}{dr}\right)_{ad} - \frac{d\rho}{dr}\right] g\Delta r^2$$
$$= \frac{\rho}{T}\left[\left(\frac{dT}{dr}\right)_{ad} - \frac{dT}{dr}\right] g\Delta r^2.$$

$$\rho v^2 \approx f_{buoy}\Delta r \approx -\left[\left(\frac{d\rho}{dr}\right)_{ad} - \frac{d\rho}{dr}\right] g\Delta r^2$$
$$= \frac{\rho}{T}\left[\left(\frac{dT}{dr}\right)_{ad} - \frac{dT}{dr}\right] g\Delta r^2.$$

(8.16)

To simplify the notation, we introduce the dimensionless measure

$$\delta \equiv \frac{R}{T}\left[\left(\frac{dT}{dr}\right)_{ad} - \frac{dT}{dr}\right]$$

$$\delta \equiv \frac{R}{T}\left[\left(\frac{dT}{dr}\right)_{ad} - \frac{dT}{dr}\right]$$

(8.17)

of the departure of the temperature gradient from its adiabatic value. Then we finally obtain

$$F_{con} \approx \rho c_p T \delta^{3/2}\left(\frac{\Delta r}{R}\right)^2 (gR)^{1/2},$$

$$F_{con} \approx \rho c_p T \delta^{3/2}\left(\frac{\Delta r}{R}\right)^2 (gR)^{1/2},$$

(8.18)

and hence the convective luminosity

$$L_{con} \approx R^2 F_{con}$$

$$\approx R^3 \rho c_P T \delta^{3/2} \left(\frac{\Delta r}{R}\right)^2 \left(\frac{g}{R}\right)^{1/2} .$$

$$L_{con} \approx R^2 F_{con}$$

$$\approx R^3 \rho c_P T \delta^{3/2} \left(\frac{\Delta r}{R}\right)^2 \left(\frac{g}{R}\right)^{1/2} . \quad (8.19)$$

In the interior of the star we can estimate L_{con} as

$$L_{con} \approx U \delta^{3/2} \left(\frac{\Delta r}{R}\right)^2 t_{dyn}^{-1} ,$$

$$L_{con} \approx U \delta^{3/2} \left(\frac{\Delta r}{R}\right)^2 t_{dyn}^{-1} , \quad (8.20)$$

where we used that $t_{dyn} \approx (R/g)^{1/2}$ (equation 1.3), and $U \approx \rho c_P T R^3$ is the total internal energy of the star. This equation has a simple physical interpretation. If we neglect the factor $(\Delta r/R)^2$ we have that $L_{con} \approx (U\delta)(\delta^{1/2}/t_{dyn})$. Here $U\delta$ is a measure of the internal energy that is transported; the factor δ reduces the energy transport, since it is only the *excess* internal energy which contributes to the energy transport. Correspondingly $t_{dyn} / \delta^{1/2}$ is the *convective timescale* t_{con} , which can be defined as

$$t_{con} = \frac{\Delta r}{v} \approx \delta^{-1/2} \left(\frac{R}{g}\right)^{1/2} \approx \delta^{-1/2} t_{dyn} ,$$

$$t_{con} = \frac{\Delta r}{v} \approx \delta^{-1/2} \left(\frac{R}{g}\right)^{1/2} \approx \delta^{-1/2} t_{dyn} , \quad (8.21)$$

which determines the time taken to transport the energy; t_{con} is a dynamical timescale, but the effective gravitational acceleration is reduced, since it is only the *difference* in density which provides the force. Therefore the timescale is increased by the factor $\delta^{-1/2}$.

In the case of radiative transport the temperature gradient was determined as being sufficiently large to transport the energy by radiation. Correspondingly, in the case of convection, δ must be sufficiently large that the energy can be transported by convection. If we assume that $L=L_{con}$ we obtain

$$\delta \approx \left[\frac{L}{U}\left(\frac{\Delta r}{R}\right)^{-2} t_{dyn}\right]^{2/3}$$

$$\approx \left(\frac{t_{dyn}}{t_{KH}}\right)^{2/3} \left(\frac{\Delta r}{R}\right)^{-4/3},$$

$$\delta \approx \left[\frac{L}{U}\left(\frac{\Delta r}{R}\right)^{-2} t_{dyn}\right]^{2/3}$$

$$\approx \left(\frac{t_{dyn}}{t_{KH}}\right)^{2/3} \left(\frac{\Delta r}{R}\right)^{-4/3},$$

(8.22)

using $t_{KH} \approx U/L$ (cf equation (1.6) and discussion which follows it). In the interior of a star we may assume, roughly, that $\Delta r \approx R$. Using the values t_{dyn} and t_{KH} for the Sun, we obtain

$$\delta \approx 5 \times 10^{-8}.$$

$$\delta \approx 5 \times 10^{-8}.$$

(8.23)

Although these estimates are very uncertain, it is obvious that even an extremely small superadiabatic gradient is sufficient to transport the entire energy by convection. This simplifies the treatment of convection tremendously: at a given point in the star one determines, by means of equation (8.11), whether the layer is unstable; if this is the case, energy transport occurs through convection, and $\delta \approx 0$, and hence

$$\frac{dT}{dr} = \left(\frac{dT}{dr}\right)_{ad} \equiv \frac{\gamma - 1}{\gamma} \frac{T}{P} \frac{dP}{dr}$$
$$= -\frac{\gamma - 1}{\gamma} \frac{T}{P} \frac{Gm\rho}{r^2}.$$

$$\frac{dT}{dr} = \left(\frac{dT}{dr}\right)_{ad} \equiv \frac{\gamma - 1}{\gamma} \frac{T}{P} \frac{dP}{dr}$$
$$= -\frac{\gamma - 1}{\gamma} \frac{T}{P} \frac{Gm\rho}{r^2}.$$

(8.24)

At such a point equation (8.24) replaces the usual equation (8.12) for the temperature gradient.

From equations (8.21, 8.22) we can estimate t_{con} as

$$t_{con} \approx \delta^{-1/2} t_{dyn} \approx \left(\frac{t_{KH}}{t_{dyn}}\right)^{1/3} \left(\frac{\Delta r}{R}\right)^{2/3} t_{dyn}$$
$$= t_{KH}^{1/3} t_{dyn}^{2/3} \left(\frac{\Delta r}{R}\right)^{2/3}.$$

$$t_{con} \approx \delta^{-1/2} t_{dyn} \approx \left(\frac{t_{KH}}{t_{dyn}}\right)^{1/3} \left(\frac{\Delta r}{R}\right)^{2/3} t_{dyn}$$
$$= t_{KH}^{1/3} t_{dyn}^{2/3} \left(\frac{\Delta r}{R}\right)^{2/3}.$$

(8.25)

Assuming again that $\Delta r \approx R$, we find in the case of the Sun that $t_{con} \approx 1 \text{year}$. This is much shorter than the characteristic evolutionary time scale. Over a time scale not much longer than t_{con} matter in a convection zone must be completely mixed. Hence, we can assume that convection zones are chemically homogeneous with the same chemical composition everywhere.

8.4. Numerical calculation of stellar structure.

One almost always considers the evolution of a star of a given mass. Hence it is convenient to rewrite the equations of stellar structure with the mass $m = m(r)$ as the independent variable. This may be done by noting that for any quantity φ

$$\frac{d\varphi}{dm} = \frac{d\varphi}{dr}\frac{dr}{dm} = \frac{1}{4\pi\rho r^2}\frac{d\varphi}{dr},$$

$$\frac{d\varphi}{dm} = \frac{d\varphi}{dr}\frac{dr}{dm} = \frac{1}{4\pi\rho r^2}\frac{d\varphi}{dr},$$

(8.26)

by using equation (4.7).

By transforming equations (4.7), (4.5), (6.1), (8.12) and (8.24) in this manner, we obtain the following set of equations:

$$\frac{dr}{dm} = \frac{1}{4\pi\rho r^2},$$

$$\frac{dr}{dm} = \frac{1}{4\pi\rho r^2},$$

(8.27a)

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4},$$

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4},$$

(8.27b)

$$\frac{dL}{dm} = \varepsilon,$$

$$\frac{dL}{dm} = \varepsilon,$$

(8.27c)

$$\frac{dT}{dm} = \begin{cases} -\frac{3\kappa}{4acT^3}\frac{L}{16\pi^2 r^4} & \text{radiative energy transport} \\ \frac{\gamma-1}{\gamma}\frac{T}{P}\frac{dP}{dm} & \text{convective energy transport} \end{cases}$$

$$\frac{dT}{dm} = \begin{cases} -\frac{3\kappa}{4acT^3}\frac{L}{16\pi^2 r^4} & \text{radiative energy transport} \\ \frac{\gamma-1}{\gamma}\frac{T}{P}\frac{dP}{dm} & \text{convective energy transport} \end{cases}$$

(8.27d)

These equations must be supplemented by expressions for ρ , γ , κ and ε , as functions of P , T and the chemical composition. As we have seen, these expressions are obtained from thermodynamics, atomic physics and nuclear physics.

The differential equations (8.27) must be supplemented by suitable boundary conditions. At the stellar center, we have obviously

$$\left.\begin{aligned} r &= 0 \\ L &= 0 \end{aligned}\right\} \text{ for } \quad m = 0.$$

$$\left.\begin{aligned} r &= 0 \\ L &= 0 \end{aligned}\right\} \text{ for } \quad m = 0.$$

(8.28)

For the surface of the model, we can choose the point where $T=T_{eff}$. What we need as the outer boundary condition, is the value of pressure at this temperature.

In the stellar atmosphere the photons are radiated directly to space, without being substantially absorbed, and hence the transport of energy does not require a big temperature gradient, and the energy can be transported by radiation. The optical properties of the atmosphere are usually described in terms of the so-called *optical depth*, defined as

$$\tau = \int_r^\infty \kappa\rho\, dr.$$

$$\tau = \int_r^\infty \kappa\rho\, dr.$$

(8.29)

Since $\kappa\rho = 1/\lambda_{ph}$ $\kappa\rho = 1/\lambda_{ph}$,where λ_{ph} is the mean free path of the photon (equation 7.2 of Lecture 7), a photon can be radiated directly to space from layers which have an optical depth of about 1 (or smaller).

Rewriting equation (7.16) for the diffusive flux of radiative energy F as

$$\frac{d}{dr}T^4 = -\frac{3F}{ac}\kappa\rho = \frac{3F}{ac}\frac{d\tau}{dr},$$

$$\frac{d}{dr}T^4 = -\frac{3F}{ac}\kappa\rho = \frac{3F}{ac}\frac{d\tau}{dr},$$

(8.30)

we have

$$T^4 = \frac{3F}{ac}(\tau + \text{const}),$$

$$T^4 = \frac{3F}{ac}(\tau + \text{const}),$$

(8.31)

since the thin atmosphere can be considered as a plane-parallel layer with $dF/dr = 0$.

Of coarse, the equation (7.16), and hence equation (8.31), become invalid when one approaches optically thin layers high in the atmosphere. Because of the increasing density, the mean free path of the photons will there become comparable with (and finally larger than) the distance which is left for the photons to reach the free space; hence the whole diffusion approximation breaks down, and one has to solve the far more complicated full set of transport equations for radiation in the stellar atmosphere.

We can employ equation (8.31) in lower atmosphere, where the diffusion approximation becomes relevant, but we need the value of the constant of integration which appears in this equation. The theory of stellar atmospheres suggests the value of 2/3 as a simple approximation for this constant, and we obtain

$$T^4 = \frac{3F}{ac}\left(\tau + \frac{2}{3}\right).$$

$$T^4 = \frac{3F}{ac}\left(\tau + \frac{2}{3}\right).$$

(8.32)

Using the definition of the effective temperature T_{eff} (equation 2.13 of Lecture 2), we also have

$$F = \sigma T_{eff}^4 = \frac{ac}{4} T_{eff}^4,$$

$$F = \sigma T_{eff}^4 = \frac{ac}{4} T_{eff}^4,$$

(8.33)

and hence equation (8.32) can be rewritten as

$$T^4 = \frac{3}{4} T_{eff}^4 \left(\tau + \frac{2}{3} \right),$$

$$T^4 = \frac{3}{4} T_{eff}^4 \left(\tau + \frac{2}{3} \right),$$

(8.34)

which shows that $T = T_{eff}$ when $\tau = 2/3$ $\tau = 2/3$. This level in the atmosphere is known as *photosphere*; it is the level from where the bulk of the radiation is emitted to space.

To evaluate the photospheric value of pressure, we define a mean opacity $\langle \kappa \rangle$ $\langle \kappa \rangle$, averaged over the stellar atmosphere above the photospheric radius R_{ph}, by the relation

$$\tau_{ph} = \frac{2}{3} = \langle \kappa \rangle \int_{R_{ph}}^{\infty} \rho \, dr.$$

$$\tau_{ph} = \frac{2}{3} = \langle \kappa \rangle \int_{R_{ph}}^{\infty} \rho \, dr.$$

(8.35)

We now approximate the gravitational acceleration in the atmosphere by a constant value $GM/R_{ph}{}^2$, and obtain

$$P_{ph} = \int_{R_{ph}}^{\infty} \frac{GM}{R_{ph}^2} \rho \, dr = \frac{2}{3 \langle \kappa \rangle} \frac{GM}{R_{ph}^2}.$$

$$P_{ph} = \int_{R_{ph}}^{\infty} \frac{GM}{R_{ph}^2} \rho \, dr = \frac{2}{3 \langle \kappa \rangle} \frac{GM}{R_{ph}^2}.$$

(8.36)

Together with $L = 4 \Pi R^2 \sigma T^4$, this is the required surface boundary condition, applied at $T = T_{eff}$.

More accurate surface boundary conditions, which are implemented in modern numerical computations, can be formulated as follows: the inner solution shall fit smoothly to a solution of the stellar-atmosphere problem.

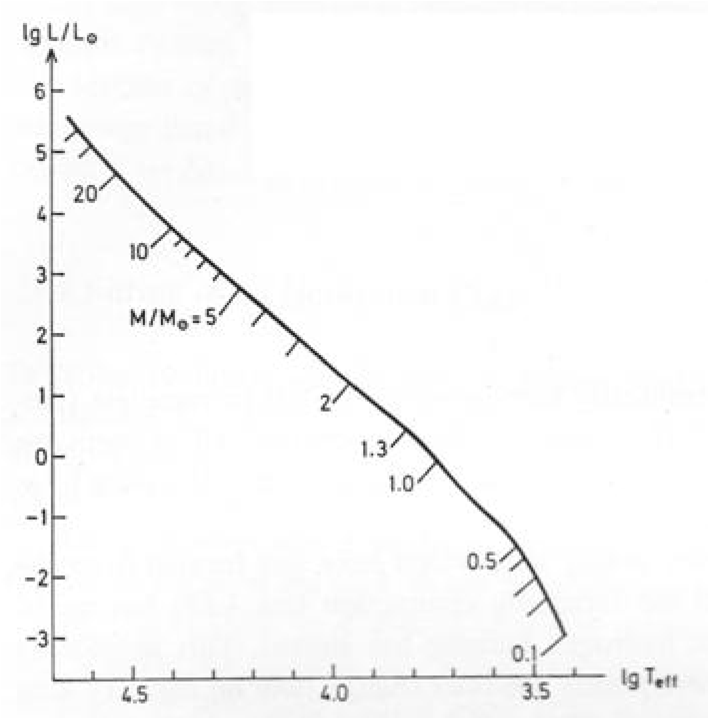

Figure 8.3. Hertzsprung-Russel diagram of the zero-age main sequence computed for the composition X=0.685, Y=0.294. The location of several models with masses between 0.1 and 22 solar masses are indicated below the sequence (from Kippenhahn and Weigert 1990, Stellar Structure and Evolution, Springer-Verlag).

As an example of the numerical results, Figure 8.3 shows the location of the main-sequence stars of uniform chemical composition (the so-called *zero-age main sequence*) on the theoretical H-R diagram (cf Figure 2.4).