Chapter 10

Rigid Bodies

10.1 Moments of Inertia

A *rigid body* is a collection of particles, as in Chapter 9, for which the relative distance $|\mathbf{x}_i - \mathbf{x}_j|$ between *every* pair (i, j) of particles is fixed. Such a body can be described completely by the position of its centre of mass, $\mathbf{X}$, and the rotation of the body about $\mathbf{X}$; no further information is necessary in order to know the position of every particle in the body.

We shall use all the results of Chapter 9, except that we shall use a continuum limit: we will regard the body V as being of density ρ and made up of infinitesimal "particles" with volume $\mathrm{d}V$, mass $\rho\,\mathrm{d}V$ and position vector $\mathbf{x}$. So, for example, the centre of mass is at

$$\mathbf{X} = \frac{1}{M} \iiint_V \rho\mathbf{x}\,\mathrm{d}V$$

where

$$M = \iiint_V \rho\,\mathrm{d}V.$$

The angular momentum about $\mathbf{a}$ is given by

$$\mathbf{H} = \iiint_V \rho(\mathbf{x} - \mathbf{a}) \times (\dot{\mathbf{x}} - \dot{\mathbf{a}})\,\mathrm{d}V$$

(compare with the definition in §9.2).

Suppose that $\mathbf{a}$ is a point which is part of the body, and that the body is rotating with angular velocity $\boldsymbol{\omega}$ about $\mathbf{a}$. For any other point $\mathbf{x}$ in the body, $\mathbf{x} - \mathbf{a}$ is fixed *relative to the body* (any two parts of the body move and/or rotate together); so from §7.1,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x}-\mathbf{a}) = \boldsymbol{\omega} \times (\mathbf{x}-\mathbf{a}). \tag{10.1}$$

Therefore

$$\begin{aligned}\mathbf{H} &= \iiint_V \rho(\mathbf{x}-\mathbf{a}) \times [\boldsymbol{\omega} \times (\mathbf{x}-\mathbf{a})]\,\mathrm{d}V \\ &= \iiint_V \rho\big\{|\mathbf{x}-\mathbf{a}|^2\boldsymbol{\omega} - [(\mathbf{x}-\mathbf{a})\,\boldsymbol{.}\,\boldsymbol{\omega}](\mathbf{x}-\mathbf{a})\big\}\,\mathrm{d}V. \end{aligned} \tag{10.2}$$

Using summation convention,

$$\begin{aligned}H_i &= \iiint_V \rho\big\{|\mathbf{x}-\mathbf{a}|^2\delta_{ij}\omega_j - [\mathbf{x}-\mathbf{a}]_j\omega_j[\mathbf{x}-\mathbf{a}]_i\big\}\,\mathrm{d}V \\ &= I_{ij}\omega_j\end{aligned}$$

where

$$I_{ij} = \iiint_V \rho\big\{|\mathbf{x}-\mathbf{a}|^2\delta_{ij} - (x_i-a_i)(x_j-a_j)\big\}\,\mathrm{d}V$$

is the *inertia tensor* at $\mathbf{a}$. This matrix relationship ($\mathbf{H} = \mathsf{I}\boldsymbol{\omega}$ where I is the matrix (I_{ij})) means that $\mathbf{H}$ and $\boldsymbol{\omega}$ are not necessarily parallel; but if $\boldsymbol{\omega}$ is along an eigenvector of I — known as a *principal axis* of I — then they will be.

Many rigid bodies have symmetries of various kinds: for example, a cylinder is rotationally symmetric about its axis. Since the matrix I is an intrinsic property of the body and nothing else, it must share these symmetries; in particular, its principal axes must coincide with the axes of symmetry. So if the angular velocity vector is aligned with one of these axes then $\mathbf{H} \parallel \boldsymbol{\omega}$. We shall consider only this kind of rotation in this course: more complicated motions in which $\mathbf{H} \nparallel \boldsymbol{\omega}$, such as the spinning of a gyroscope, are studied in the Part II course *Classical Dynamics*.

Definition: the *moment of inertia* of a body *about a particular axis* is

$$\boxed{I = \iiint_V \rho r_\perp^2\,\mathrm{d}V}$$

where $r_\perp$ is the perpendicular distance from $\mathbf{x}$ to the axis.

This definition is only really useful when the axis coincides with one of the principal axes of the inertia tensor at $\mathbf{a}$.

Suppose that the body is spinning with angular velocity $\boldsymbol{\omega}$ about $\mathbf{a}$. Let $\omega = |\boldsymbol{\omega}|$ and $\hat{\mathbf{n}} = \boldsymbol{\omega}/\omega$, so that $|\hat{\mathbf{n}}| = 1$. By Pythagoras,

$$r_\perp^2 + [(\mathbf{x}-\mathbf{a})\,\boldsymbol{.}\,\hat{\mathbf{n}}]^2 = |\mathbf{x}-\mathbf{a}|^2.$$

Now assume that $\mathbf{H} \parallel \boldsymbol{\omega}$, and let $\mathbf{H} = H\hat{\mathbf{n}}$. From (10.2) we have

$$\mathbf{H} = \omega \iiint\limits_V \rho\big\{|\mathbf{x}-\mathbf{a}|^2\hat{\mathbf{n}} - [(\mathbf{x}-\mathbf{a}) \,.\, \hat{\mathbf{n}}](\mathbf{x}-\mathbf{a})\big\}\,\mathrm{d}V$$

$$\implies \qquad \hat{\mathbf{n}} \,.\, \mathbf{H} = \omega \iiint\limits_V \rho\big\{|\mathbf{x}-\mathbf{a}|^2 - [(\mathbf{x}-\mathbf{a}) \,.\, \hat{\mathbf{n}}]^2\big\}\,\mathrm{d}V$$

$$\implies \qquad H = \omega \iiint\limits_V \rho r_\perp^2 \,\mathrm{d}V$$

$$= I\omega.$$

Hence $\mathbf{H} = I\omega\hat{\mathbf{n}}$, i.e.,

$$\boxed{\mathbf{H} = I\boldsymbol{\omega}.} \qquad (10.3)$$

Examples:

- A uniform cylinder of mass M, length l and radius a, rotating about its axis through the centre of mass $\mathbf{X}$. Use cylindrical polar coordinates (r, ϕ, z) with origin at $\mathbf{X}$ and z-axis along the axis of rotation: it is obvious that $r_\perp$ is simply r in this coordinate system. The density is $\rho = M/(\pi a^2 l)$, so

$$\begin{aligned} I &= \iiint\limits_V \rho r^2 \,\mathrm{d}V \\ &= \frac{M}{\pi a^2 l}\int_{z=-\frac{1}{2}l}^{\frac{1}{2}l}\int_{\phi=0}^{2\pi}\int_{r=0}^{a} r^2\, r\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}z \qquad \text{(using the Jacobian } J = r\text{)} \\ &= \frac{M}{\pi a^2 l}(2\pi l)(\tfrac{1}{4}a^4) \\ &= \tfrac{1}{2}Ma^2. \end{aligned}$$

- A uniform rod of mass M and length l, rotating about an axis perpendicular to the rod through an end-point. Such a mathematically ideal rod is one-dimensional, so we replace $\iiint \mathrm{d}V$ by $\int \mathrm{d}x$. The density is M/l per unit length, so

$$I = \int_0^l \frac{M}{l}\,x^2\,\mathrm{d}x = \tfrac{1}{3}Ml^2.$$

(Note that this is *not* the same as if the whole mass M were concentrated at $x = \frac{1}{2}l$.)

- A sphere of mass M and radius a rotating about its centre of mass. The direction of the axis does not matter, by symmetry; choose it along the z-axis without loss of generality. Then

$$I = \iiint_V \rho(x^2 + y^2)\,\mathrm{d}V.$$

We could simply evaluate this; or instead we can use the following trick which makes the calculation a little easier. If we had chosen the x-axis rather than the z-axis then we would have obtained

$$I = \iiint_V \rho(y^2 + z^2)\,\mathrm{d}V$$

instead, and for the y-axis

$$I = \iiint_V \rho(x^2 + z^2)\,\mathrm{d}V.$$

All three values must be equal by symmetry, so by adding and then converting to spherical polar coordinates (r, θ, ϕ) we obtain

$$\begin{aligned} 3I &= 2 \iiint_V \rho(x^2 + y^2 + z^2)\,\mathrm{d}V \\ \implies \qquad I &= \tfrac{2}{3} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} \frac{M}{\frac{4}{3}\pi a^3}\, r^2\, r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= \frac{M}{2\pi a^3}\, 2\pi \big[-\cos\theta\big]_0^\pi \big[\tfrac{1}{5} r^5\big]_0^a \\ &= \tfrac{2}{5} M a^2. \end{aligned}$$

Somewhat contrary to intuition, a body can be said to be rotating about *any* point within the body with the *same* angular velocity $\boldsymbol{\omega}$. To see this, suppose that the body is rotating with angular velocity $\boldsymbol{\omega}$ about $\mathbf{a}$. Then for any point $\mathbf{x}$ within the body we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x} - \mathbf{a}) = \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{a})$$

from (10.1). Let $\mathbf{b}$ be another point in the body. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{b} - \mathbf{a}) = \boldsymbol{\omega} \times (\mathbf{b} - \mathbf{a}),$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x} - \mathbf{b}) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x} - \mathbf{a} + \mathbf{a} - \mathbf{b}) = \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{a}) - \boldsymbol{\omega} \times (\mathbf{b} - \mathbf{a}) = \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{b}).$$

So the body can equally well be said to be rotating with angular velocity $\boldsymbol{\omega}$ about $\mathbf{b}$.

For example, a uniform rod rotating with angular velocity $\boldsymbol{\omega}$ about a fixed end-point $\mathbf{w}$ can be described either as rotating about $\mathbf{a} = \mathbf{w}$, with $\dot{\mathbf{a}} = \mathbf{0}$, or as rotating about its centre of mass $\mathbf{a} = \mathbf{X}$, in which case $\mathbf{a}$ is moving. Since the moment of inertia I depends on both the direction of the axis *and* the point about which the rotation is taking place, we need to calculate different values of I for these two cases.

10.2 Perpendicular and Parallel Axes

The Perpendicular Axes Theorem

The *perpendicular axes theorem* applies to a lamina S, i.e., a flat, perfectly thin body lying without loss of generality in the (x, y)-plane. If I_x and I_y are the moments of inertia about the x- and y-axes respectively then the moment of inertia about the z-axis is

$$\boxed{I_z = I_x + I_y.}$$

Proof: $I_x = \iint_S \rho y^2 \, \mathrm{d}S$ and $I_y = \iint_S \rho x^2 \, \mathrm{d}S$, but

$$I_z = \iint\limits_S \rho(x^2 + y^2) \, \mathrm{d}S$$

by definition. The result follows. □

Example: what is the moment of inertia of a flat circular wire loop of radius a and total mass M about a diameter?

Taking axes about the centre of the circle, we have $I_z = Ma^2$ because every part of the loop is a distance a from the origin. Clearly $I_x = I_y$ by symmetry; so the moment of inertia about a diameter is

$$I_x = \tfrac{1}{2} I_z = \tfrac{1}{2} Ma^2.$$

The Parallel Axis Theorem

The *parallel axis theorem* applies to a general rigid body (not necessarily a lamina). Let I_{CoM} be its moment of inertia about an axis through the centre of mass $\mathbf{X}$, and let I be its moment of inertia about a parallel axis. Then

$$\boxed{I = I_{\text{CoM}} + Md^2}$$

where M is the total mass and d is the perpendicular distance between the axes.

Proof: let $\hat{\mathbf{n}}$ be a unit vector parallel to both axes. Place the origin of coordinates at $\mathbf{X}$ (i.e., set $\mathbf{X} = \mathbf{0}$ by translating the origin), and let Π be the plane through the origin perpendicular to $\hat{\mathbf{n}}$. Let $\mathbf{d}$ be the position vector of the point where the second axis meets Π. Choose Cartesian coordinate axes through the origin as follows: let the z-axis lie parallel to $\hat{\mathbf{n}}$, and let the x-axis lie parallel to $\mathbf{d}$, so that $\mathbf{d} = (d, 0, 0)$.

Using this coordinate system, the perpendicular distance from a general point $\mathbf{x}$ within the body to the first axis is $\sqrt{x^2+y^2}$, while the distance from $\mathbf{x}$ to the second axis is $r_\perp = \sqrt{(x-d)^2+y^2}$. Hence

$$\begin{aligned} I &= \iiint\limits_V \rho r_\perp^2 \,\mathrm{d}V \\ &= \iiint\limits_V \rho(x^2 - 2dx + d^2 + y^2)\,\mathrm{d}V \\ &= \iiint\limits_V \rho(x^2+y^2)\,\mathrm{d}V - 2d\iiint\limits_V \rho x\,\mathrm{d}V + d^2 \iiint\limits_V \rho\,\mathrm{d}V. \end{aligned}$$

Now, because we have set $\mathbf{X} = \mathbf{0}$, we must have in this coordinate system that

$$\iiint\limits_V \rho\mathbf{x}\,\mathrm{d}V = \mathbf{0}.$$

The x-component of this vector equation gives us that $\iiint \rho x\,\mathrm{d}V = 0$. We also have that $I_{\text{CoM}} = \iiint \rho(x^2+y^2)\,\mathrm{d}V$ and $M = \iiint \rho\,\mathrm{d}V$ by definition. Hence

$$I = I_{\text{CoM}} + Md^2. \qquad \square$$

Corollary: the moment of inertia about an axis which passes through the centre of mass is lower than about any parallel axis.

Examples (using the results already obtained in §10.1):

- The moment of inertia of a uniform sphere of mass M and radius a about an axis tangential to the surface is given by $I = \frac{2}{5}Ma^2 + Ma^2 = \frac{7}{5}Ma^2$.
- The moment of inertia of a rod of mass M and length l about an axis through its midpoint (i.e., through the centre of mass) is $I_{\text{CoM}} = \frac{1}{3}Ml^2 - M(\frac{1}{2}l)^2 = \frac{1}{12}Ml^2$.

10.3 Energy

From §9.3, the kinetic energy of the body is $T = \frac{1}{2}M\dot{\mathbf{a}}^2 + T_{\text{rel}}$ where T_{rel} is the kinetic energy relative to $\mathbf{a}$. Let $\mathbf{y} = \mathbf{x} - \mathbf{a}$; then

$$T_{\text{rel}} = \iiint\limits_V \tfrac{1}{2}\rho\dot{\mathbf{y}}^2\,\mathrm{d}V.$$

Suppose that the body has angular velocity $\boldsymbol{\omega}$ about $\mathbf{a}$ with moment of inertia I; then from (10.1), $\dot{\mathbf{y}} = \boldsymbol{\omega} \times \mathbf{y}$, so

$$\begin{aligned} T_{\text{rel}} &= \tfrac{1}{2} \iiint\limits_V \rho(\boldsymbol{\omega} \times \mathbf{y}) \,.\, \dot{\mathbf{y}} \,\mathrm{d}V \\ &= \tfrac{1}{2} \iiint\limits_V \rho\boldsymbol{\omega} \,.\, (\mathbf{y} \times \dot{\mathbf{y}}) \,\mathrm{d}V \\ &= \tfrac{1}{2}\boldsymbol{\omega} \,.\, \iiint\limits_V \rho\mathbf{y} \times \dot{\mathbf{y}} \,\mathrm{d}V \\ &= \tfrac{1}{2}\boldsymbol{\omega} \,.\, \mathbf{H} \end{aligned}$$

since $\mathbf{H} = \iiint_V \rho(\mathbf{x} - \mathbf{a}) \times (\dot{\mathbf{x}} - \dot{\mathbf{a}}) \,\mathrm{d}V$ is the angular momentum about $\mathbf{a}$. But from (10.3), $\mathbf{H} = I\boldsymbol{\omega}$, so we find that the "kinetic energy relative to $\mathbf{a}$" is $\frac{1}{2}I\omega^2$ and

$$\boxed{T = \tfrac{1}{2}M\dot{\mathbf{a}}^2 + \tfrac{1}{2}I\omega^2.} \tag{10.4}$$

This result applies in either of the "safe cases", namely $\dot{\mathbf{a}} = \mathbf{0}$ or $\mathbf{a} = \mathbf{X}$. In the latter case we obtain

$$T = \tfrac{1}{2}M\dot{\mathbf{X}}^2 + \tfrac{1}{2}I_{\text{CoM}}\omega^2.$$

10.4 Motion with Rotation and Translation

Problems involving rotation and/or translation of a rigid body may be approached by making extensive use of the results

$$\mathbf{H} = I\boldsymbol{\omega}, \tag{10.3}$$

$$\dot{\mathbf{H}} = \mathbf{G}, \tag{9.5}$$

$$M\ddot{\mathbf{X}} = \mathbf{F}, \tag{9.2}$$

$$T = \tfrac{1}{2}M\dot{\mathbf{a}}^2 + \tfrac{1}{2}I\omega^2. \tag{10.4}$$

Example: a rod of mass M and length l is pivoted at an end and oscillates in a vertical plane. Let θ be the angle to the vertical; then $\boldsymbol{\omega} = \dot{\theta}\hat{\mathbf{n}}$. The moment of inertia of the rod about the pivot is $I = \frac{1}{3}Ml^2$ from an example in §10.1; and the total couple is

$$\mathbf{G} = (\mathbf{X} - \mathbf{a}) \times M\mathbf{g} = -Mg(\tfrac{1}{2}l\sin\theta)\hat{\mathbf{n}}$$

where $\mathbf{a}$ is the pivot point. Note that we have the "safe" case $\dot{\mathbf{a}} = \mathbf{0}$. Using $\mathbf{H} = I\boldsymbol{\omega}$ and $\dot{\mathbf{H}} = \mathbf{G}$, we obtain

$$I\ddot{\theta}\hat{\mathbf{n}} = -\tfrac{1}{2}Mgl\sin\theta\,\hat{\mathbf{n}}$$

$$\implies \qquad \ddot{\theta} = -\frac{3g}{2l}\sin\theta = -\frac{g}{\frac{2}{3}l}\sin\theta.$$

This is equivalent to a simple pendulum with length $\frac{2}{3}l$.

The Rolling Point of Contact

Consider a curved body, specifically, a cylinder or sphere of radius a, rolling along a flat surface. Let X be the distance moved by the centre of mass and let θ be the angle through which the body has turned.

What is the instantaneous speed of the contact point? The centre of mass is moving forward with speed $\dot{X}$, while the whole body is rotating about its centre of mass with angular velocity $\dot{\theta}$. The speed of the contact point relative to the centre of mass is therefore $a\dot{\theta}$ (by considering $\boldsymbol{\omega} \times (\mathbf{x} - \mathbf{X})$); so the speed of the contact point relative to the flat surface is $\dot{X} - a\dot{\theta}$. This applies whether the body is rolling or slipping.

But if the body is *rolling* (and therefore, by implication, *not slipping*), we must have $X = a\theta$ by considering arc length along the circumference of the body. Therefore

$$\boxed{\dot{X} = a\dot{\theta}}$$

and we deduce that *the rolling point of contact is instantaneously at rest.* When the body is slipping, the speed of the point of contact is non-zero.

One consequence of this result is that any frictional force $\mathbf{R}$ which acts on a rolling body does no work, because it acts at the rolling point of contact which (instantaneously) always has zero velocity. Hence friction does not slow down a rolling body. Needless to say, this is a mathematical idealisation: a real body deforms slightly so that a whole section of it is in contact with the surface rather than just a single point. Friction acts on that section, not all of which is at rest, resulting in non-zero retardation.
