Fourier Analysis

Fourier series allow you to expand a function on a finite interval as an infinite series of trigonometric functions. What if the interval is infinite? That's the subject of this chapter. Instead of a sum over frequencies, you will have an integral.

15.1 Fourier Transform

For the finite interval you have to specify the boundary conditions in order to determine the particular basis that you're going to use. On the infinite interval you don't have this large set of choices. After all, if the boundary is infinitely far away, how can it affect what you're doing over a finite distance? But see section 15.6.

In section 5.3 you have several boundary condition listed that you can use on the differential equation $u'' = \lambda u$ and that will lead to orthogonal functions on your interval. For the purposes here the easiest approach is to assume periodic boundary conditions on the finite interval and then to take the limit as the length of the interval approaches infinity. On $-L < x < +L$, the conditions on the solutions of $u'' = \lambda u$ are then $u(-L) = u(+L)$ and $u'(-L) = u'(+L)$. The solution to this is most conveniently expressed as a complex exponential, Eq. (5.19)

$$u(x) = e^{ikx}, \qquad \text{where} \qquad u(-L) = e^{-ikL} = u(L) = e^{ikL}$$

This implies $e^{2ikL} = 1$, or $2kL = 2n\pi$, for integer $n = 0, \pm 1, \pm 2, \ldots$. With these solutions, the other condition, $u'(-L) = u'(+L)$ is already satisfied. The basis functions are then

$$u_n(x) = e^{ik_n x} = e^{n\pi i x/L}, \qquad \text{for} \qquad n = 0, \pm 1, \pm 2, \textit{ etc.} \tag{15.1}$$

On this interval you have the Fourier series expansion

$$f(x) = \sum_{-\infty}^{\infty} a_n\, u_n(x), \qquad \text{and} \qquad \big\langle u_m, f\big\rangle = \big\langle u_m, \sum_{-\infty}^{\infty} a_n\, u_n\big\rangle = a_m \big\langle u_m, u_m\big\rangle \tag{15.2}$$

In the basis of Eq. (15.1) this normalization is $\big\langle u_m, u_m\big\rangle = 2L$.

Insert this into the series for f.

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{\big\langle u_n, f\big\rangle}{\big\langle u_n, u_n\big\rangle} u_n(x) = \frac{1}{2L} \sum_{n=-\infty}^{\infty} \big\langle u_n, f\big\rangle u_n(x)$$

Now I have to express this in terms of the explicit basis functions in order to manipulate it. When you use the explicit form you have to be careful not to use the same symbol (x) for two different things in the same expression. Inside the $\big\langle u_n, f\big\rangle$ there is no "x" left over — it's the dummy variable of integration and it is not the same x that is in the $u_n(x)$ at the end. Denote $k_n = \pi n/L$.

$$f(x) = \frac{1}{2L} \sum_{n=-\infty}^{\infty} \int_{-L}^{L} dx' u_n(x')^* f(x')\, u_n(x) = \frac{1}{2L} \sum_{n=-\infty}^{\infty} \int_{-L}^{L} dx' e^{-ik_n x'} f(x')\, e^{ik_n x}$$

Now for some manipulation: As n changes by 1, k_n changes by $\Delta k_n = \pi/L$.

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{\pi}{L} \int_{-L}^{L} dx'\, e^{-ik_n x'} f(x')\, e^{ik_n x} \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{ik_n x}\, \Delta k_n \int_{-L}^{L} dx'\, e^{-ik_n x'} f(x') \end{aligned} \tag{15.3}$$

For a given value of k, define the integral

$$g_L(k) = \int_{-L}^{L} dx' e^{-ikx'} f(x')$$

If the function f vanishes sufficiently fast as $x' \to \infty$, this integral will have a limit as $L \to \infty$. Call that limit $g(k)$. Look back at Eq. (15.3) and you see that for large L the last factor will be approximately $g(k_n)$, where the approximation becomes exact as $L \to \infty$. Rewrite that expression as

$$f(x) \approx \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{ik_n x} \, \Delta k_n \, g(k_n) \tag{15.4}$$

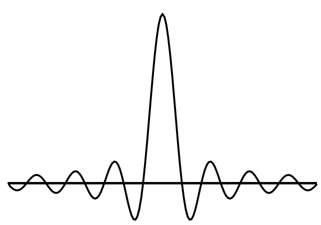
As $L \to \infty$, you have $\Delta k_n \to 0$, and that turns Eq. (15.4) into an integral.

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} g(k), \qquad \text{where} \qquad g(k) = \int_{-\infty}^{\infty} dx \, e^{-ikx} f(x) \tag{15.5}$$

The function g is called* the Fourier transform of f, and f is the inverse Fourier transform of g.

Examples

For an example, take the step function



$$f(x) = \begin{cases} 1 & (-a < x < a) \\ 0 & (\text{elsewhere}) \end{cases} \qquad \text{then}$$

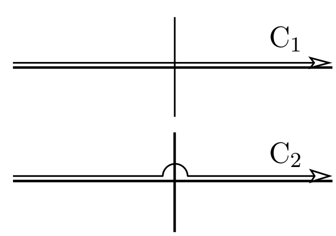
$$\begin{aligned} g(k) &= \int_{-a}^{a} dx \, e^{-ikx} \, 1 \\ &= \frac{1}{-ik} \left[e^{-ika} - e^{+ika} \right] = \frac{2 \sin ka}{k} \end{aligned} \tag{15.6}$$

The first observation is of course that the dimensions check: If dx is a length then so is $1/k$. After that, there is only one parameter that you can vary, and that's a. As a increases, obviously the width of the function f increases, but now look at g. The first place where $g(k) = 0$ is at $ka = \pi$. This value, π/a *decreases* as a increases. As f gets broader, g gets narrower (and taller). This is a general property of these Fourier transform pairs.

Can you invert this Fourier transform, evaluating the integral of g to get back to f? Yes, using the method of contour integration this is very easy. Without contour integration it would be extremely difficult, and that is typically the case with these transforms; complex variable methods are essential to get anywhere with them. The same statement holds with many other transforms (Laplace, Radon, Mellin, Hilbert, *etc.*)

The inverse transform is

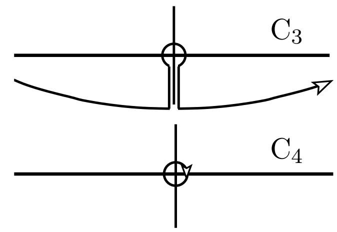
$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \frac{2 \sin ka}{k} &= \int_{C_1} \frac{dk}{2\pi} e^{ikx} \frac{e^{ika} - e^{-ika}}{ik} \\ &= -i \int_{C_2} \frac{dk}{2\pi} \frac{1}{k} \left[e^{ik(x+a)} - e^{ik(x-a)} \right] \end{aligned}$$



* Another common notation is to define g with an integral $dx/\sqrt{2\pi}$. That will require a corresponding $dk/\sqrt{2\pi}$ in the inverse relation. It's more symmetric that way, but I prefer the other convention.

1. If $x > +a$ then both $x+a$ and $x-a$ are positive, which implies that both exponentials vanish rapidly as $k \to +i\infty$. Push the contour C_2 toward this direction and the integrand vanishes exponentially, making the integral zero.
2. If $-a < x < +a$, then only $x + a$ is positive. The integral of the first term is then zero by exactly the preceding reasoning, but the other term has an exponential that vanishes as $k \to -i\infty$ instead, implying that you must push the contour down toward $-i\infty$.

$$= i \int_{\mathrm{C}_3} \frac{dk}{2\pi} \frac{1}{k} e^{ik(x-a)} = \int_{\mathrm{C}_4}$$
$$= +i\frac{1}{2\pi}(-1)2\pi i \operatorname*{Res}_{k=0} \frac{e^{ik(x-a)}}{k} = -i\frac{1}{2\pi}\cdot 2\pi i = 1$$



The extra (-1) factor comes because the contour is clockwise.
3. In the third domain, $x < -a$, both exponentials have the form e^{-ik}, requiring you to push the contour toward $-i\infty$. The integrand now has both exponentials, so it is analytic at zero and there is zero residue. The integral vanishes and the whole analysis takes you back to the original function, Eq. (15.6).

Another example of a Fourier transform, one that shows up often in quantum mechanics

$$f(x) = e^{-x^2/\sigma^2}, \qquad \text{so} \qquad g(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} e^{-x^2/\sigma^2} = \int_{-\infty}^{\infty} dx\, e^{-ikx-x^2/\sigma^2}$$

The trick to doing this integral is to complete the square inside the exponent.

$$-ikx - x^2/\sigma^2 = \frac{-1}{\sigma^2}\left[x^2 + \sigma^2 ikx - \sigma^4k^2/4 + \sigma^4k^2/4\right] = \frac{-1}{\sigma^2}\left[(x + ik\sigma^2/2)^2 + \sigma^4k^2/4\right]$$

The integral of f is now

$$g(k) = e^{-\sigma^2k^2/4} \int_{-\infty}^{\infty} dx' e^{-x'^2/\sigma^2} \qquad \text{where} \qquad x' = x + ik\sigma/2$$

The change of variables makes this a standard integral, Eq. (1.10), and the other factor, with the exponential of k^2, comes outside the integral. The result is

$$g(k) = \sigma\sqrt{\pi}\, e^{-\sigma^2k^2/4} \tag{15.7}$$

This has the curious result that the Fourier transform of a Gaussian is* a Gaussian.

15.2 Convolution Theorem

What is the Fourier transform of the product of two functions? It is a convolution of the individual

* Another function has this property: the hyperbolic secant. Look up the quantum mechanical harmonic oscillator solution for an infinite number of others.

transforms. What that means will come out of the computation. Take two functions f_1 and f_2 with Fourier transforms g_1 and g_2.

$$\begin{aligned}\int_{-\infty}^{\infty} dx\, f_1(x) f_2(x) e^{-ikx} &= \int dx \int \frac{dk'}{2\pi} g_1(k')e^{ik'x} f_2(x) e^{-ikx} \\ &= \int \frac{dk'}{2\pi} g_1(k') \int dx\, e^{ik'x} f_2(x) e^{-ikx} \\ &= \int \frac{dk'}{2\pi} g_1(k') \int dx\, f_2(x) e^{-i(k-k')x} \\ &= \int_{-\infty}^{\infty} \frac{dk'}{2\pi} g_1(k') g_2(k-k') \end{aligned} \tag{15.8}$$

The last expression (except for the 2π) is called the convolution of g_1 and g_2.

$$\int_{-\infty}^{\infty} dx\, f_1(x) f_2(x) e^{-ikx} = \frac{1}{2\pi}(g_1 * g_2)(k) \tag{15.9}$$

The last line shows a common notation for the convolution of g_1 and g_2.

What is the integral of $|f|^2$ over the whole line?

$$\begin{aligned}\int_{-\infty}^{\infty} dx\, f^*(x) f(x) &= \int dx\, f^*(x) \int \frac{dk}{2\pi} g(k) e^{ikx} \\ &= \int \frac{dk}{2\pi} g(k) \int dx\, f^*(x) e^{ikx} \\ &= \int \frac{dk}{2\pi} g(k) \left[\int dx\, f(x) e^{-ikx}\right]^* \\ &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} g(k) g^*(k) \end{aligned} \tag{15.10}$$

This is Parseval's identity for Fourier transforms. There is an extension to it in problem 15.10.

15.3 Time-Series Analysis

Fourier analysis isn't restricted to functions of x, sort of implying position. They're probably more often used in analyzing functions of time. If you're presented with a complicated function of time, how do you analyze it? What information is present in it? If that function of time is a sound wave you may choose to analyze it with your ears, and if it is music, the frequency content is just what you will be listening for. That's Fourier analysis. The Fourier transform of the signal tells you its frequency content, and sometimes subtle periodicities will show up in the transformed function even though they aren't apparent in the original signal.

A function of time is $f(t)$ and its Fourier transform is

$$g(\omega) = \int_{-\infty}^{\infty} dt\, f(t)\, e^{i\omega t} \qquad \text{with} \qquad f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, g(\omega)\, e^{-i\omega t}$$

The sign convention in these equations appear backwards from the one in Eq. (15.5), and it is. One convention is as good as the other, but in the physics literature you'll find this pairing more common because of the importance of waves. A function $e^{i(kx-\omega t)}$ represents a wave with (phase) velocity ω/k, and so moving to the right. You form a general wave by taking linear combinations of these waves, usually an integral.

Example

When you hear a musical note you will perceive it as having a particular frequency. It doesn't, and if the note has a very short duration it becomes hard to tell its* pitch. Only if its duration is long enough do you have a real chance to discern what note you're hearing. This is a reflection of the facts of Fourier transforms.

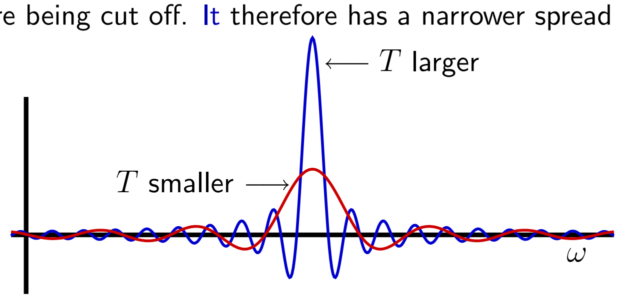
If you hear what you think of as a single note, it will not last forever. It starts and it ends. Say it lasts from $t = -T$ to $t = +T$, and in that interval it maintains the frequency ω_0.

$$f(t) = Ae^{-i\omega_0 t} \qquad (-T < t < T) \tag{15.11}$$

The frequency analysis comes from the Fourier transform.

$$g(\omega) = \int_{-T}^{T} dt\, e^{i\omega t}\, Ae^{-i\omega_0 t} = A\frac{e^{i(\omega-\omega_0)T} - e^{-i(\omega-\omega_0)T}}{i(\omega-\omega_0)} = 2A\frac{\sin(\omega-\omega_0)T}{(\omega-\omega_0)}$$

This is like the function of Eq. (15.6) except that its center is shifted. It has a peak at $\omega = \omega_0$ instead of at the origin as in that case. The width of the function g is determined by the time interval T. As T is large, g is narrow and high, with a sharp localization near ω_0. In the reverse case of a short pulse, the range of frequencies that constitute the note is spread over a wide range of frequencies, and you will find it difficult to tell by listening to it just what the main pitch is supposed to be. This figure shows the frequency spectrum for two notes having the same nominal pitch, but one of them lasts three times as long as the other before being cut off. It therefore has a narrower spread of frequencies.



Example

Though you can do these integrals numerically, and when you are dealing with real data you will have to, it's nice to have some analytic examples to play with. I've already shown, Eq. (15.7), how the Fourier transform of a Gaussian is simple, so start from there.

$$\text{If} \qquad g(\omega) = e^{-(\omega-\omega_0)^2/\sigma^2} \qquad \text{then} \qquad f(t) = \frac{\sigma}{2\sqrt{\pi}} e^{-i\omega_0 t} e^{-\sigma^2 t^2/4}$$

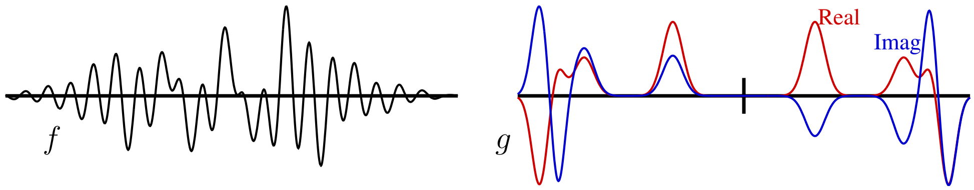
If there are several frequencies, the result is a sum.

$$g(\omega) = \sum_n A_n\, e^{-(\omega-\omega_n)^2/\sigma_n^2} \qquad \Longleftrightarrow \qquad f(t) = \sum_n A_n \frac{\sigma_n}{2\sqrt{\pi}} e^{-i\omega_n t} e^{-\sigma_n^2 t^2/4}$$

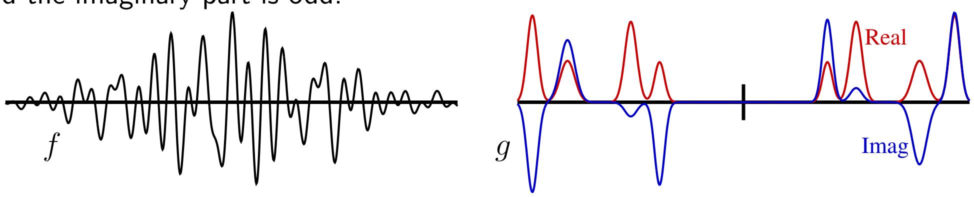
In a more common circumstance you will have the time series, $f(t)$, and will want to obtain the frequency decomposition, $g(\omega)$, though for this example I worked backwards. The function of time is real, but

* Think of a hemisemidemiquaver played at tempo prestissimo.

the transformed function g is complex. Because f is real, it follows that g satisfies $g(-\omega) = g^*(\omega)$. See problem 15.13.



This example has four main peaks in the frequency spectrum. The real part of g is an even function and the imaginary part is odd.



This is another example with four main peaks.

In either case, if you simply look at the function of time on the left it isn't obvious what sort of frequencies are present. That's why there are standard, well-developed computer programs to do the Fourier analysis.

15.4 Derivatives

There are a few simple, but important relations involving differentiation. What is the Fourier transform of the derivative of a function? Do some partial integration.

$$\mathcal{F}(\dot{f}) = \int dt\, e^{i\omega t}\,\frac{df}{dt} = e^{i\omega t} f(t)\Big|_{-\infty}^{\infty} - i\omega \int dt\, e^{i\omega t} f(t) = -i\omega \mathcal{F}(f) \tag{15.12}$$

Here I've introduced the occasionally useful notation that $\mathcal{F}(f)$ is the Fourier transform of f. The boundary terms in the partial integration will go to zero if you assume that the function f approaches zero at infinity.

The n^{th} time derivative simply give you more factors: $(-i\omega)^n$ on the transformed function.

15.5 Green's Functions

This technique showed up in the chapter on ordinary differential equations, section 4.6, as a method to solve the forced harmonic oscillator. In that instance I said that you can look at a force as a succession of impulses, as if you're looking at the atomic level and visualizing a force as many tiny collisions by atoms. Here I'll get to the same sort of result as an application of transform methods. The basic technique is to Fourier transform everything in sight.

The damped, forced harmonic oscillator differential equation is

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0(t) \tag{15.13}$$

Multiply by $e^{i\omega t}$ and integrate over all time. You do the transforms of the derivatives by partial integration as in Eq. (15.12).

$$\int_{-\infty}^{\infty} dt\, e^{i\omega t}\ [\text{Eq. (15.13)}] = -m\omega^2\tilde{x} - ib\omega\tilde{x} + k\tilde{x} = \tilde{F}_0, \quad \text{where} \quad \tilde{x}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} x(t)$$

This is an algebraic equation that is easy to solve for the function $\tilde{x}(\omega)$.

$$\tilde{x}(\omega) = \frac{\tilde{F}_0(\omega)}{-m\omega^2 - ib\omega + k}$$

Now use the inverse transform to recover the function $x(t)$.

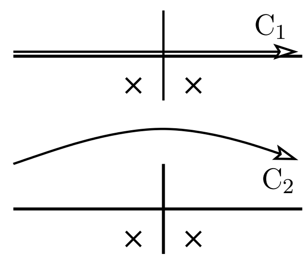
$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t}\tilde{x}(\omega) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \frac{\tilde{F}_0(\omega)}{-m\omega^2 - ib\omega + k} \\ &= \int \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{-m\omega^2 - ib\omega + k} \int dt'\, F_0(t')e^{i\omega t'} \\ &= \int dt'\, F_0(t') \int \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{-m\omega^2 - ib\omega + k} e^{i\omega t'} \end{aligned} \tag{15.14}$$

In the last line I interchanged the order of integration, and in the preceding line I had to be sure to use another symbol t' in the second integral, not t. Now do the ω integral.

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{-m\omega^2 - ib\omega + k} e^{i\omega t'} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{-m\omega^2 - ib\omega + k} \tag{15.15}$$

To do this, use contour integration. The singularities of the integrand are at the roots of the denominator, $-m\omega^2 - ib\omega + k = 0$. They are

$$\omega = \frac{-ib \pm \sqrt{-b^2 + 4km}}{2m} = \omega_\pm$$

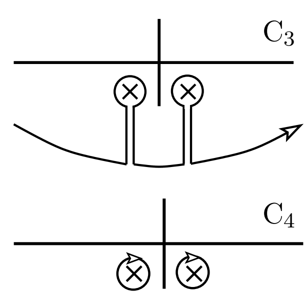


Both of these poles are in the lower half complex plane. The contour integral C_1 is along the real axis, and now I have to decide where to push the contour in order to use the residue theorem. This will be governed by the exponential, $e^{-i\omega(t-t')}$.

First take the case $t < t'$, then $e^{-i\omega(t-t')}$ is of the form $e^{+i\omega}$, so in the complex ω-plane its behavior in the $\pm i$ directions is as a decaying exponential toward $+i$ ($\propto e^{-|\omega|}$). It is a rising exponential toward $-i$ ($\propto e^{+|\omega|}$). This means that pushing the contour C_1 up toward C_2 and beyond will make this integral go to zero. I've crossed no singularities, so that means that Eq. (15.15) *is* zero for $t < t'$.

Next, the case that $t > t'$. Now $e^{-i\omega(t-t')}$ is of the form $e^{-i\omega}$, so its behavior is reversed from that of the preceding paragraph. It dies off rapidly toward $-i\infty$ and rises in the opposite direction. That means that I must push the contour in the opposite direction, down to C_3 and to C_4. Because of the decaying exponential, the large arc of the contour that is pushed down to $-i\infty$ gives zero for its integral; the two lines that parallel the i-axis cancel each other; only the two residues remain.

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{-m\omega^2 - ib\omega + k} = -2\pi i \sum_{\omega_\pm} \text{Res} \tag{15.16}$$



The denominator in Eq. (15.15) is $-m(\omega-\omega_+)(\omega-\omega_-)$. Use this form to compute the residues. Leave the $1/2\pi$ aside for the moment and you have

$$\frac{e^{-i\omega(t-t')}}{-m\omega^2-ib\omega+k}=\frac{e^{-i\omega(t-t')}}{-m(\omega-\omega_+)(\omega-\omega_-)}$$

The residues of this at $\omega_\pm$ are the coefficients of these first order poles.

$$\text{at } \omega_+: \qquad \frac{e^{-i\omega_+(t-t')}}{-m(\omega_+-\omega_-)} \qquad\qquad \text{and at } \omega_-: \qquad \frac{e^{-i\omega_-(t-t')}}{-m(\omega_--\omega_+)}$$

The explicit values of $\omega_\pm$ are

$$\omega_+=\frac{-ib+\sqrt{-b^2+4km}}{2m} \qquad \text{and} \qquad \omega_-=\frac{-ib-\sqrt{-b^2+4km}}{2m}$$

$$\text{Let} \qquad \omega'=\frac{\sqrt{-b^2+4km}}{2m} \qquad \text{and} \qquad \gamma=\frac{b}{2m}$$

The difference that appears in the preceding equation is then

$$\omega_+-\omega_-=(\omega'-i\gamma)-(-\omega'-i\gamma)=2\omega'$$

Eq. (15.16) is then

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\cdot\frac{e^{-i\omega(t-t')}}{-m\omega^2-ib\omega+k}&=-i\left[\frac{e^{-i(\omega'-i\gamma)(t-t')}}{-2m\omega'}+\frac{e^{-i(-\omega'-i\gamma)(t-t')}}{+2m\omega'}\right]\\&=\frac{-i}{2m\omega'}e^{-\gamma(t-t')}\big[-e^{-i\omega'(t-t')}+e^{+i\omega'(t-t')}\big]\\&=\frac{1}{m\omega'}e^{-\gamma(t-t')}\sin\big(\omega'(t-t')\big)\end{aligned}$$

Put this back into Eq. (15.14) and you have

$$x(t)=\int_{-\infty}^{t}dt'\,F_0(t')G(t-t'), \qquad \text{where} \qquad G(t-t')=\frac{1}{m\omega'}e^{-\gamma(t-t')}\sin\big(\omega'(t-t')\big) \tag{15.17}$$

If you eliminate the damping term, setting $b=0$, this is exactly the same as Eq. (4.34). The integral stops at $t'=t$ because the Green's function vanishes beyond there. The motion at time t is determined by the force that was applied in the past, not the future.

Example

Apply a constant external force to a damped harmonic oscillator, starting it at time $t=0$ and keeping it on. What is the resulting motion?

$$F_0(t)=\begin{cases}0 & (t<0)\\ F_1 & (t>0)\end{cases}$$

where F_1 is a constant. The equation (15.17) says that the solution is ($t > 0$)

$$\begin{aligned} x(t) &= \int_{-\infty}^{t} dt'\, F_0(t')G(t-t') = \int_0^t dt'\, F_1 G(t-t') \\ &= F_1 \int_0^t dt' \frac{1}{m\omega'} e^{-\gamma(t-t')} \sin\big(\omega'(t-t')\big) \\ &= \frac{F_1}{2im\omega'} \int_0^t dt' e^{-\gamma(t-t')} \big[e^{i\omega'(t-t')} - e^{-i\omega'(t-t')}\big] \\ &= \frac{F_1}{2im\omega'} \left[\frac{1}{\gamma - i\omega'} e^{(-\gamma+i\omega')(t-t')} - \frac{1}{\gamma + i\omega'} e^{(-\gamma-i\omega')(t-t')}\right]_{t'=0}^{t'=t} \\ &= \frac{F_1}{2im\omega'} \left[\frac{2i\omega'}{\gamma^2 + \omega'^2} - \frac{e^{-\gamma t}}{\gamma^2 + \omega'^2} \big[2i\gamma \sin\omega' t + 2i\omega' \cos\omega' t\big]\right] \\ &= \frac{F_1}{m(\gamma^2 + \omega'^2)} \Big[1 - e^{-\gamma t}\Big[\cos\omega' t + \frac{\gamma}{\omega'} \sin\omega' t\Big]\Big] \end{aligned}$$

Check the answer. If $t = 0$ it is correct; $x(0) = 0$ as it should.
If $t \to \infty$, $x(t)$ goes to $F_1/\big(m(\gamma^2 + \omega'^2)\big)$; is *this* correct? Check it out! And maybe simplify the result in the process. Is the small time behavior correct?

15.6 Sine and Cosine Transforms

Return to the first section of this chapter and look again at the derivation of the Fourier transform. It started with the Fourier series on the interval $-L < x < L$ and used periodic boundary conditions to define which series to use. Then the limit as $L \to \infty$ led to the transform.

What if you know the function only for positive values of its argument? If you want to write $f(x)$ as a series when you know it only for $0 < x < L$, it doesn't make much sense to start the way I did in section 15.1. Instead, pick the boundary condition at $x = 0$ carefully because this time the boundary won't go away in the limit that $L \to \infty$. The two common choices to define the basis are

$$u(0) = 0 = u(L), \qquad \text{and} \qquad u'(0) = 0 = u'(L) \tag{15.18}$$

Start with the first, then $u_n(x) = \sin(n\pi x/L)$ for positive n. The equation (15.2) is unchanged, save for the limits.

$$f(x) = \sum_1^\infty a_n\, u_n(x), \qquad \text{and} \qquad \big\langle u_m, f\big\rangle = \Big\langle u_m, \sum_{n=1}^\infty a_n\, u_n\Big\rangle = a_m \big\langle u_m, u_m\big\rangle$$

In this basis, $\big\langle u_m, u_m\big\rangle = L/2$, so

$$f(x) = \sum_{n=1}^\infty \frac{\big\langle u_n, f\big\rangle}{\big\langle u_n, u_n\big\rangle} u_n(x) = \frac{2}{L} \sum_{n=1}^\infty \big\langle u_n, f\big\rangle u_n(x)$$

Now explicitly use the sine functions to finish the manipulation, and as in the work leading up to Eq. (15.3), denote $k_n = \pi n/L$, and the difference $\Delta k_n = \pi/L$.

$$\begin{aligned} f(x) &= \frac{2}{L} \sum_1^\infty \int_0^L dx' f(x') \sin\frac{n\pi x'}{L}\, \sin\frac{n\pi x}{L} \\ &= \frac{2}{\pi} \sum_1^\infty \sin\frac{n\pi x}{L}\, \Delta k_n \int_0^L dx' f(x') \sin n\pi x'/L \end{aligned} \tag{15.19}$$

For a given value of k, define the integral

$$g_{_L}(k) = \int_0^L dx'\, \sin(kx') f(x')$$

If the function f vanishes sufficiently fast as $x' \to \infty$, this integral will have a limit as $L \to \infty$. Call that limit $g(k)$. Look back at Eq. (15.19) and you see that for large L the last factor will be approximately $g(k_n)$, where the approximation becomes exact as $L \to \infty$. Rewrite that expression as

$$f(x) \approx \frac{2}{\pi} \sum_1^\infty \sin(k_n x) \Delta k_n\, g(k_n) \tag{15.20}$$

As $L \to \infty$, you have $\Delta k_n \to 0$, and that turns Eq. (15.20) into an integral.

$$f(x) = \frac{2}{\pi} \int_0^\infty dk\, \sin kx\, g(k), \qquad \text{where} \qquad g(k) = \int_0^\infty dx\, \sin kx\, f(x) \tag{15.21}$$

This is the Fourier Sine transform. For a parallel calculation leading to the Cosine transform, see problem 15.22, where you will find that the equations are the same except for changing sine to cosine.

$$f(x) = \frac{2}{\pi} \int_0^\infty dk\, \cos kx\, g(k), \qquad \text{where} \qquad g(k) = \int_0^\infty dx\, \cos kx\, f(x) \tag{15.22}$$

What is the sine transform of a derivative? Integrate by parts, remembering that f has to approach zero at infinity for any of this to make sense.

$$\int_0^\infty dx\, \sin kx\, f'(x) = \sin kx f(x)\Big|_0^\infty - k \int_0^\infty dx\, \cos kx\, f(x) = -k \int_0^\infty dx\, \cos kx\, f(x)$$

For the second derivative, repeat the process.

$$\int_0^\infty dx\, \sin kx\, f''(x) = k f(0) - k^2 \int_0^\infty dx\, \sin kx\, f(x) \tag{15.23}$$

15.7 Wiener-Khinchine Theorem

If a function of time represents the pressure amplitude of a sound wave or the electric field of an electromagnetic wave the power received is proportional to the amplitude squared. By Parseval's identity, the absolute square of the Fourier transform has an integral proportional to the integral of this power. This leads to the interpretation of the transform squared as some sort of power density in frequency. $|g(\omega)|^2 d\omega$ is then a power received in this frequency interval. When this energy interpretation isn't appropriate, $|g(\omega)|^2$ is called the "spectral density." A useful result appears by looking at the Fourier transform of this function.

$$\begin{aligned} \int \frac{d\omega}{2\pi} |g(\omega)|^2 e^{-i\omega t} &= \int \frac{d\omega}{2\pi} g^*(\omega) e^{-i\omega t} \int dt'\, f(t') e^{i\omega t'} \\ &= \int dt'\, f(t') \int \frac{d\omega}{2\pi} g^*(\omega) e^{i\omega t'} e^{-i\omega t} \\ &= \int dt'\, f(t') \left[\int \frac{d\omega}{2\pi} g(\omega) e^{-i\omega(t'-t)} \right]^* \\ &= \int dt'\, f(t') f(t'-t)^* \end{aligned} \tag{15.24}$$

When you're dealing with a real f, this last integral is called the autocorrelation function. It tells you in some average way how closely related a signal is to the same signal at some other time. If the signal that you are examining is just noise then what happens now will be unrelated to what happened a few milliseconds ago and this autocorrelation function will be close to zero. If there is structure in the signal then this function gives a lot of information about it.

The left side of this whole equation involves two Fourier transforms ($f \to g$, then $|g|^2$ to it's transform). The right side of this theorem seems to be easier and more direct to compute than the left, so why is this relation useful? It is because of the existence of the FFT, the "Fast Fourier Transform," an algorithm that makes the process of Fourier transforming a set of data far more efficient than doing it by straight-forward numerical integration methods — faster by factors that reach into the thousands for large data sets.

Problems

15.1 Invert the Fourier transform, g, in Eq. (15.7).

15.2 What is the Fourier transform of $e^{ik_0x - x^2/\sigma^2}$? Ans: A translation of the $k_0 = 0$ case

15.3 What is the Fourier transform of xe^{-x^2/σ^2}?

15.4 What is the square of the Fourier transform operator? That is, what is the Fourier transform of the Fourier transform?

15.5 A function is defined to be

$$f(x) = \begin{cases} 1 & (-a < x < a) \\ 0 & \text{(elsewhere)} \end{cases}$$

What is the convolution of f with itself? $(f * f)(x)$ And graph it of course. Start by graphing both $f(x')$ and the other factor that goes into the convolution integral.
Ans: $(2a - |x|)$ for $(-2a < x < +2a)$, and zero elsewhere.

15.6 Two functions are

$$f_1(x) = \begin{cases} 1 & (a < x < b) \\ 0 & \text{(elsewhere)} \end{cases} \qquad \text{and} \qquad f_2(x) = \begin{cases} 1 & (A < x < B) \\ 0 & \text{(elsewhere)} \end{cases}$$

What is the convolution of f_1 with f_2? And graph it.

15.7 Derive these properties of the convolution:
(**a**) $f * g = g * f$ (**b**) $f * (g * h) = (f * g) * h$ (**c**) $\delta(f * g) = f * \delta g + g * \delta f$ where $\delta f(t) = tf(t)$, $\delta g(t) = tg(t)$, *etc.* (**d**) What are $\delta^2(f * g)$ and $\delta^3(f * g)$?

15.8 Show that you can rewrite Eq. (15.9) as

$$\mathcal{F}(f * g) = \mathcal{F}(f) \cdot \mathcal{F}(g)$$

where the shorthand notation $\mathcal{F}(f)$ is the Fourier transform of f.

15.9 Derive Eq. (15.10) from Eq. (15.9).

15.10 What is the analog of Eq. (15.10) for two different functions? That is, relate the scalar product of two functions,

$$\langle f_1, f_2 \rangle = \int_{-\infty}^{\infty} f_1^*(x) f_2(x)$$

to their Fourier transforms. Ans: $\int g_1^*(k) g_2(k)\, dk/2\pi$

15.11 In the derivation of the harmonic oscillator Green's function, and starting with Eq. (15.15), I assumed that the oscillator is underdamped: that $b^2 < 4km$. Now assume the reverse, the overdamped case, and repeat the calculation.

15.12 Repeat the preceding problem, but now do the critically damped case, for which $b^2 = 4km$. Compare your result to the result that you get by taking the limit of critical damping in the preceding problem and in Eq. (15.17).

15.13 Show that if $f(t)$ is real then the Fourier transform satisfies $g(-\omega) = g^*(\omega)$. What are the properties of g if f is respectively even or odd?

15.14 Evaluate the Fourier transform of

$$f(x) = \begin{cases} A\big(a - |x|\big) & (-a < x < a) \\ 0 & \text{(otherwise)} \end{cases}$$

How do the properties of the transform vary as the parameter a varies?
Ans: $2A(1 - \cos ka)/k^2$

15.15 Evaluate the Fourier transform of $Ae^{-\alpha|x|}$. Invert the transform to verify that it takes you back to the original function. Ans: $2\alpha/(\alpha^2 + k^2)$

15.16 Given that the Fourier transform of $f(x)$ is $g(k)$, what is the Fourier transform of the the function translated a distance a to the right, $f_1(x) = f(x - a)$?

15.17 Schroedinger's equation is

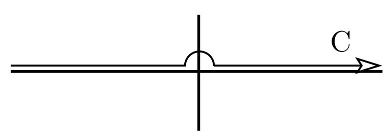
$$-i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi$$

Fourier transform the whole equation with respect to x, and find the equation for $\Phi(k,t)$, the Fourier transform of $\psi(x,t)$. The result will *not* be a differential equation.
Ans: $-i\hbar\partial\Phi(k,t)/\partial t = (\hbar^2k^2/2m)\Phi + (v * \Phi)/2\pi$

15.18 Take the Green's function solution to Eq. (15.13) as found in Eq. (15.17) and take the limit as both k and b go to zero. Verify that the resulting single integral satisfies the original second order differential equation.

15.19 (**a**) In problem 15.18 you have the result that a double integral (undoing two derivatives) can be written as a single integral. Now solve the equation

$$\frac{d^3x}{dt^3} = F(t)$$



directly, using the same method as for Eq. (15.13). You will get a pole at the origin and how do you handle this, where the contour of integration goes straight through the origin? Answer: Push the contour up as in the figure. Why? This is what's called the "retarded solution" for which the value of $x(t)$ depends on only those values of $F(t')$ in the past. If you try any other contour to define the integral you will not get this property. (And sometimes there's a reason to make another choice.)
(**b**) Pick a fairly simple F and verify that this gives the right answer.
Ans: $\frac{1}{2}\int_{-\infty}^{t} dt'\, F(t')(t - t')^2$

15.20 Repeat the preceding problem for the fourth derivative. Would you care to conjecture what $3\,^1\!/_2$ integrals might be? Then perhaps an arbitrary non-integer order?
Ans: $\frac{1}{6}\int_{-\infty}^{t} dt'\, F(t')(t - t')^3$

15.21 What is the Fourier transform of $xf(x)$? Ans: $ig'(k)$

15.22 Repeat the calculations leading to Eq. (15.21), but for the boundary conditions $u'(0) = 0 = u'(L)$, leading to the Fourier cosine transform.

15.23 For both the sine and cosine transforms, the original function $f(x)$ was defined for positive x only. Each of these transforms define an extension of f to negative x. This happens because you compute $g(k)$ and from it get an inverse transform. Nothing stops you from putting a negative value of x into the answer. What are the results?

15.24 What are the sine and cosine transforms of $e^{-\alpha x}$. In each case evaluate the inverse transform.

15.25 What is the sine transform of $f(x) = 1$ for $0 < x < L$ and $f(x) = 0$ otherwise. Evaluate the inverse transform.

15.26 Repeat the preceding calculation for the cosine transform. Graph the two transforms and compare them, including their dependence on L.

15.27 Choose any different way around the pole in problem 15.19, and compute the difference between the result with your new contour and the result with the old one. Note: Plan ahead before you start computing.