Queen Mary

UNIVERSITY OF LONDON

MSc Examination

ASTM001 Solar System **Duration 3h**

Friday, 30 May 2003 18.15 – 21.15

Calculators are NOT permitted in this examination.

You may attempt any number of questions. Full marks will be obtained by providing complete answers to about THREE questions.

continued overleaf ...

1. A natural satellite moves under the gravitational attraction of a planet and its resulting path is an ellipse with the planet at one focus. The relationship between the satellite's radial distance, r from an origin O at the centre of the planet and its true anomaly, f, is given in polar coordinates by,

$$r = \frac{a(1-e^2)}{1+e\cos f}$$

where a and e are the semi-major axis and eccentricity of satellite's orbit. The same orbital path can be described by the equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

where $b = a\sqrt{1-e^2}$ is the semi-minor axis of the ellipse, (x, y) are the coordinates of the satellite in a frame with origin, O', at the centre of the ellipse (midway between the two foci) and the x-axis lies along the line joining the two foci.

(a) Draw a diagram to illustrate the relationship between the polar coordinate system with origin O and the cartesian coordinate system with origin O'. Sketch a circle of radius a centred on the origin O' and use it to illustrate the relationship between f and the eccentric anomaly, E. Derive expressions for $r\cos f$ and $r\sin f$, and hence show that

$$r = a(1-e\cos E).$$

(11 marks)

(b) Substitute the result from part (a) in the equation

$$\dot{r} = \frac{na}{r}\sqrt{a^2e^2-(r-a)^2}$$

where n is the mean motion of the satellite, and hence solve it to derive Kepler's equation,

$$M = E - e\sin E$$

where $M = n(t-\tau)$ is the mean anomaly and τ is a constant. (9 marks)

(c) Use an iterative method of the form

$$E_{i+1} = M + e\sin E_i$$

with $E_0 = M$ in order to derive a series solution for Kepler's equation to express E as a function of M including terms up to and including $\mathcal{O}(e^2)$. (8 marks)

(d) Another satellite is discovered orbiting around the same planet. It has negligible mass and it is observed to have an orbital period of 6 days with a semi-major axis that is 6 times larger than that of the first satellite. Ignoring any perturbations from the second satellite, show that the mean anomaly of the first satellite must be increasing at a rate of one degree every $40\sqrt{6}$ seconds. (5 marks)

2. In the planar, circular restricted three-body problem the equations of motion of the massless test particle in the frame rotating with unit angular velocity are given by

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x}$$
$$\ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y}$$

where the test particle has rectangular coordinates (x, y) in a frame where the x-axis is directed along the line joining the two masses, and

$$U = \frac{1}{2}(x^2 + y^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}$$

with $\mu_1 = m_1/(m_1 + m_2)$, $\mu_2 = m_2/(m_1 + m_2)$ and it is assumed that $m_2 < m_1$. The square of the distances from the particle to the masses m_1 and m_2 are given by $r_1^2 = (x + \mu_2)^2 + y^2$, $r_2^2 = (x - \mu_1)^2 + y^2$ respectively. In this system the unit of distance is taken to be the constant separation of the two masses.

(a) Derive expressions for $\partial U/\partial x$ and $\partial U/\partial y$ using the definitions of U, r_1 and r_2 given above. Hence show that the equations of motion have equilibrium solutions at the points $r_1 = r_2 = 1$. Draw a sketch showing the location of these two equilibrium points in relation to the two masses.

(11 marks)

(b) From the definitions of r_1 and r_2, show that

$$\mu_1 r_1^2 + \mu_2 r_2^2 = x^2 + y^2 + \mu_1 \mu_2$$

and hence derive an expression for U that is an explicit function of r_1 and r_2 only.

(7 marks)

(c) An equilibrium point is located close to m_2 along the line joining the two masses such that $r_1 + r_2 = 1$. Using the new expression for U from part (b), or otherwise, show that at this equlibrium point r_2 must satisfy the equation

$$\frac{\mu_2}{\mu_1} = \frac{r_2^3(r_2^2 - 3r_2 + 3)}{(1 - r_2)^3(1 + r_2 + r_2^2)}.$$

and hence show that this equilibrium point is located at an approximate distance $\alpha = (\mu_2/3\mu_1)^{\frac{1}{3}}$ from the mass m_2, where it is assumed that $m_2 \ll m_1$.

(10 marks)

(d) A small spherical satellite of mass m_2 and physical radius R is in a circular orbit and in synchronous rotation about a planet of mass m_1. Consider a test particle lying on the equator of the satellite along the line joining the satellite and the planet centres on the planet side of the satellite. Show that the orbital radius at which the sum of the excess gravitational or centrifugal force on the particle due to tidal shear and the centrifugal force due to the satellite's rotation is balanced by the gravitational attraction of the satellite is given by R/α.

(5 marks)

3. The asteroid Hilda orbits at the 3:2 interior mean motion resonance with the planet Jupiter with successive conjunctions of the planet and the asteroid occurring at the asteroid's perihelion. Jupiter can be assumed to be moving in a circular orbit in the same plane as Hilda.

(a) Show that the ratio of the perihelion distance, r_p, to the aphelion distance, r_a of Hilda can be written as

$$\frac{r_\mathrm{p}}{r_\mathrm{a}} = 1 + 2\sum_{i=1}^{\infty}(-1)^i e^i$$

where $e \approx 0.2$ is the eccentricity of Hilda's orbit. Sketch the orbits of Hilda and Jupiter in the inertial, non-rotating frame marking the location of the asteroid's perihelion and aphelion. Indicate possible locations of both objects ensuring that these are consistent with the nature of the resonance. (8 marks)

(b) Sketch the approximate path of Hilda in a frame rotating at a rate equal to the mean motion of Jupiter. Indicate on your plot all locations where Hilda is at the perihelion or aphelion of its orbit. (4 marks)

(c) The angular momentum per unit mass of an object orbiting the Sun is given by $h = na^2\sqrt{1-e^2}$, where n is the mean motion and a is the semi-major axis. Use this to obtain an expression for the angular velocity of the planet as a function of the true anomaly f. By deriving an expression for the angular velocity of Hilda at its aphelion and equating it to the mean motion of Jupiter, show that Hilda would be instantaneously stationary at one or more locations in the rotating frame provided its eccentricity e satisfies the equation

$$4e^3 + 12e^2 + 21e - 5 = 0\,.$$

(11 marks)

(d) A detailed analytical study of the motion of the Hilda–Jupiter system requires an expansion of the disturbing function involving the eccentricities (e and e') and inclinations (I and I') of both objects to the second degree. This implies a total of eight cosine arguments in the expansion of the disturbing function: two are associated with the 3:2 resonance and six are associated with the 6:4 resonance. Use your knowledge of the properties of the disturbing function to write down each cosine argument, stating clearly the angles involved and the form of the term in eccentricity and inclination associated with each argument. What is the relationship, if any, between the coefficients of the angles in the argument and the powers of eccentricity and inclination in the term?

(10 marks)

4. Images of the outer part of Saturn's main rings show a series of features associated with first-order resonances between ring material and the moon Prometheus orbiting beyond the edge of the ring system. Each resonance has a resonant argument of the form $\varphi = j\lambda' + (1-j)\lambda - \varpi$ where λ' is the mean longitude of Prometheus, λ is the mean longitude of the ring particle, ϖ is the longitude of pericentre of the ring particle, and j is an integer.

   (a) Use the supplied sheet of Lagrange's planetary equations to show that to lowest order the ring particle's eccentricity, e varies according to the equation

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -\frac{m'}{M} n \frac{a}{a'} A \sin\varphi$$

   where n is the mean motion of the particle, a is its semimajor axis, a' is the semi-major axis of Prometheus, A is a constant, m' is the mass of Prometheus, and M is the mass of Saturn.

   (11 marks)

   (b) With suitable, stated approximations integrate the equation for $\mathrm{d}e/\mathrm{d}t$ to show that the forced eccentricity associated with this resonance is given by

$$e_{\mathrm{f}} = \frac{2(a^2/a')(m'/M)|A|}{3(j-1)|a - a_{\mathrm{res}}|}$$

   where a_{res} is the semimajor axis of the exact resonance.

   (11 marks)

   (c) Provide a qualitative explanation of how a satellite in a gap can produce waves on adjacent rings without the effects of resonance. Use this to explain the phenomena seen in Saturn's Encke gap.

   (11 marks)

End of Examination

C.D. Murray

LAGRANGE'S EQUATIONS

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{-3}{a^2}\frac{\partial R}{\partial \lambda}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{-\sqrt{1-e^2}}{na^2 e}(1-\sqrt{1-e^2})\frac{\partial R}{\partial \lambda} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \varpi}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{-\tan\frac{1}{2}I}{na^2\sqrt{1-e^2}}\left(\frac{\partial R}{\partial \lambda} + \frac{\partial R}{\partial \varpi}\right) - \frac{1}{na^2\sqrt{1-e^2}\sin I}\frac{\partial R}{\partial \Omega}$$

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = \frac{-2}{na}\frac{\partial R}{\partial a} + \frac{\sqrt{1-e^2}(1-\sqrt{1-e^2})}{na^2 e}\frac{\partial R}{\partial e} + \frac{\tan\frac{1}{2}I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial I}$$

$$\frac{\mathrm{d}\varpi}{\mathrm{d}t} = \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e} + \frac{\tan\frac{1}{2}I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial I}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{1}{na^2\sqrt{1-e^2}\sin I}\frac{\partial R}{\partial I}$$

where R is the disturbing function, ϵ is the mean longitude at epoch, and

$$\lambda = \int n\,\mathrm{d}t + \epsilon$$