Extrasolar Planets and Astrophysical Discs
Problem Set 4: Solutions

February 2010

Problem 1

The total luminosity L_ν of the disc at frequency ν can be calculated by integrating the flux F_ν at each point on the surface of the disc over the entire surface area. The luminosity at frequency ν from a thin ring of radius R and thickness $\mathrm{d}R$ on one face of the disc is $2\pi R \mathrm{d}R\, F_\nu$. Integrating this over radius, the total luminosity per unit frequency interval from one face is

$$\frac{1}{2}L_\nu \;=\; \int_{R_{in}}^{R_{out}} 2\pi\, R\, F_\nu \mathrm{d}R \;=\; \left(\frac{2\pi}{c}\right)^2 h\nu^3 \int_{R_{in}}^{R_{out}} \frac{R\,\mathrm{d}R}{\exp(h\nu/kT(R)) - 1} \quad \text{on subs. for } F_\nu$$

$$= \left(\frac{2\pi}{c}\right)^2 h\nu^3 \int_{R_{in}}^{R_c} \frac{R\,\mathrm{d}R}{\exp(h\nu/kT(R)) - 1} \;+\; \left(\frac{2\pi}{c}\right)^2 h\nu^3 \int_{R_c}^{R_{out}} \frac{R\,\mathrm{d}R}{\exp(h\nu/kT(R)) - 1}$$

on dividing the integration into the two temperature regimes.
Use the substitution $x \;\equiv\; \dfrac{h\nu}{kT}$ and perform the integration over x.

Differentiating, $\quad \mathrm{d}x \;=\; -\,\dfrac{h\nu}{kT^2}\,\mathrm{d}T \;=\; -\,\dfrac{h\nu}{kT^2}\,\dfrac{\mathrm{d}T}{\mathrm{d}R}\,\mathrm{d}R\,, \quad$ which gives,

$$R\,\mathrm{d}R \;=\; -\,R\,\frac{kT^2}{h\nu}\,\frac{\mathrm{d}R}{\mathrm{d}T}\,\mathrm{d}x \;=\; -\,\frac{k}{h\nu}\,R^2\,T\,\frac{\mathrm{d}\ln R}{\mathrm{d}\ln T}\,\mathrm{d}x$$

For $R_{in} \le R < R_c$, $\quad T \;=\; T_c\left(\dfrac{R_c}{R}\right)^{\frac{1}{2}}, \quad$ so $R \;=\; R_c\,T_c^2\,T^{-2}\,.$

$$\therefore \quad \ln R \;=\; \ln(R_c T_c^2) - 2\ln T \qquad \therefore \quad \frac{\mathrm{d}\ln R}{\mathrm{d}\ln T} \;=\; -\,2$$

$$\therefore \;\; R\,\mathrm{d}R \;=\; -\,\frac{k}{h\nu}\,R_c^2\,\frac{T_c^4}{T^4}\,T\,\frac{\mathrm{d}\ln R}{\mathrm{d}\ln T}\,\mathrm{d}x \;=\; 2\,\frac{k}{h\nu}\,R_c^2\,\frac{T_c^4}{T^3}\,\mathrm{d}x$$

$$=\; 2\,\frac{k}{h\nu}\,R_c^2\,T_c^4\,\left(\frac{k}{h\nu}\right)^3 x^3\,\mathrm{d}x \;=\; 2\,R_c^2\,T_c^4\,\left(\frac{k}{h\nu}\right)^4 x^3\,\mathrm{d}x$$

For $R_{in} \le R < R_c$, $\quad T \;=\; T_c\left(\dfrac{R_c}{R}\right)^{\frac{3}{4}}, \quad$ so $R \;=\; R_c\,T_c^{4/3}\,T^{-4/3}\,.$

$$\therefore \quad \ln R \;=\; \ln(R_c T_c^{4/3}) - \frac{4}{3}\ln T \qquad \therefore \quad \frac{\mathrm{d}\ln R}{\mathrm{d}\ln T} \;=\; -\,\frac{4}{3}$$

$$\therefore \quad R\,\mathrm{d}R = \frac{4}{3}\,\frac{k}{h\nu}\,R_c^2\,\frac{T_c^{8/3}}{T^{5/3}}\,\mathrm{d}x \;=\; \frac{4}{3}\,\frac{k}{h\nu}\,R_c^2\,T_c^{8/3}\,\left(\frac{k}{h\nu}\right)^{5/3} x^{5/3}\,\mathrm{d}x$$

$$\therefore \quad R\,\mathrm{d}R \;=\; \frac{4}{3}\,R_c^2\,T_c^{8/3}\,\left(\frac{k}{h\nu}\right)^{8/3}\,x^{5/3}\,\mathrm{d}x$$

Substituting for $R\,\mathrm{d}R$ and $\mathrm{d}\ln R/\mathrm{d}\ln T$, the luminosity per unit frequency of one side of the disc becomes

$$\begin{aligned}\frac{1}{2}\,L_\nu \;&=\; \left(\frac{2\pi}{c}\right)^2 h\nu^3 \int_{x_{in}}^{x_c} 2\,R_c^2\,T_c^4\,\left(\frac{k}{h\nu}\right)^4 x^3\,\frac{\mathrm{d}x}{\exp(x)-1} \;+\\ &\qquad \left(\frac{2\pi}{c}\right)^2 h\nu^3 \int_{x_c}^{x_{out}} \frac{4}{3}\,R_c^2\,T_c^{8/3}\,\left(\frac{k}{h\nu}\right)^{8/3} x^{5/3}\,\frac{\mathrm{d}x}{\exp(x)-1}\end{aligned}$$

where $x_{in} = h\nu/kT(R_{in})$, $x_c = h\nu/kT(R_c)$ and $x_{out} = h\nu/kT(R_{out})$.

$$\begin{aligned}\frac{1}{2}\,L_\nu \;&=\; 2\left(\frac{2\pi}{c}\right)^2 k^4\,h^{-3}\,R_c^2\,T_c^4\,\nu^{-1}\int_{x_{in}}^{x_c}\frac{x^3}{\mathrm{e}^x-1}\,\mathrm{d}x \;+\\ &\qquad \frac{4}{3}\left(\frac{2\pi}{c}\right)^2 k^{8/3}\,h^{-5/3}\,R_c^2\,T_c^{8/3}\,\nu^{1/3}\int_{x_c}^{x_{out}}\frac{x^{5/3}}{\mathrm{e}^x-1}\,\mathrm{d}x\;.\end{aligned}$$

So, $$\frac{1}{2}\,L_\nu \;=\; A_1\,\nu^{-1} \;+\; A_2\,\nu^{1/3}$$

where $$A_1 \;\equiv\; 2\left(\frac{2\pi}{c}\right)^2 k^4\,h^{-3}\,R_c^2\,T_c^4\int_{x_{in}}^{x_c}\frac{x^3}{\mathrm{e}^x-1}\,\mathrm{d}x$$

and $$A_2 \;\equiv\; \frac{4}{3}\left(\frac{2\pi}{c}\right)^2 k^{8/3}\,h^{-5/3}\,R_c^2\,T_c^{8/3}\int_{x_c}^{x_{out}}\frac{x^{5/3}}{\mathrm{e}^x-1}\,\mathrm{d}x\;.$$

A_1 and A_2 are weak functions of frequency ν through the limits of the integrals (because $x_{in} = h\nu/kT(R_{in})$, etc, are functions of ν). However, we can use $A_1 \simeq$ constant and $A_2 \simeq$ constant to a first approximation.
The ν^{-1} term is contributed by the hot inner part of the disc ($R_{in} < R \leq R_c$) which emits predominantly at high frequencies. The $\nu^{1/3}$ term is contributed by the cooler outer part of the disc ($R_c < R \leq R_{out}$) which emits predominantly at low frequencies.

In the cataclysmic variable system, the temperature behaviour for radial distances $R > 5 \times 10^7$ m shows the $T \propto R^{-3/4}$ relation that is expected for steady-state Keplerian discs where the disc is heated by viscous dissipation. The inner part of the disc has a different $T \propto R^{-1/2}$ behaviour.
Therefore, the power of the radiation over the radius range $1 \times 10^7\,\mathrm{m} \leq R < 5 \times 10^7\,\mathrm{m}$ due to the $T \propto R^{-1/2}$ temperature dependence will be different to what it would be if the standard $T \propto R^{-3/4}$ dependence applied there. To calculate the deficit we need to calculate the power in this radius range produced by the $R^{-1/2}$ and $R^{-3/4}$ relations, and calculate the difference.
The total luminosity emitted by the disc between radii R_{in} and R_c will be

$$L \;=\; 2\int_{R_{in}}^{R_c} 2\pi R\sigma T^4 \,\mathrm{d}R \;=\; 4\pi\sigma \int_{R_{in}}^{R_c} RT^4 \,\mathrm{d}R$$

integrated over all frequencies (from both sides of the disc).
The temperature over the radius range from $R_{in} = 1 \times 10^7$ m to $R_c = 5 \times 10^7$ m is actually given by

$$T \;=\; 2 \times 10^4 \left(\frac{5 \times 10^7\ \mathrm{m}}{R}\right)^{1/2} \ \mathrm{K}\ .$$

The luminosity is therefore

$$L \;=\; 4\pi\sigma \int_{R_{in}}^{R_c} R\left(2 \times 10^4\right)^4 \left(\frac{5 \times 10^7}{R}\right)^2 \,\mathrm{d}R$$

$$= \; 4\pi\sigma \left(2 \times 10^4\right)^4 \left(5 \times 10^7\right)^2 \int_{R_{in}}^{R_c} R^{-1} \,\mathrm{d}R \;=\; 4\pi\sigma \left(2 \times 10^4\right)^4 \left(5 \times 10^7\right)^2 \Big[\ln R\Big]_{R_{in}}^{R_c}$$

$$= \; 4\pi\sigma \left(2 \times 10^4\right)^4 \left(5 \times 10^7\right)^2 \ln\left(\frac{5 \times 10^7}{1 \times 10^7}\right) \;=\; 4.59 \times 10^{26}\ \mathrm{W}\ .$$

If instead the temperature is given by the

$$T \;=\; 2 \times 10^4 \left(\frac{5 \times 10^7\ \mathrm{m}}{R}\right)^{3/4} \ \mathrm{K}$$

relation over the $R_{in} = 1 \times 10^7$ m to $R_c = 5 \times 10^7$ m range in radius, the luminosity would be

$$L \;=\; 4\pi\sigma \int_{R_{in}}^{R_c} R\left(2 \times 10^4\right)^4 \left(\frac{5 \times 10^7}{R}\right)^3 \,\mathrm{d}R$$

$$= \; 4\pi\sigma \left(2 \times 10^4\right)^4 \left(5 \times 10^7\right)^3 \int_{R_{in}}^{R_c} R^{-2} \,\mathrm{d}R \;=\; 4\pi\sigma \left(2 \times 10^4\right)^4 \left(5 \times 10^7\right)^3 \Big[-R^{-1}\Big]_{R_{in}}^{R_c}$$

$$= \; 4\pi\sigma \left(2 \times 10^4\right)^4 \left(5 \times 10^7\right)^3 \left(-\frac{1}{5 \times 10^7} + \frac{1}{1 \times 10^7}\right) \;=\; 1.14 \times 10^{27}\ \mathrm{W}\ .$$

The luminosity from the $R_{in} \leq R < R_c$ region is less in the $T \propto R^{-1/2}$ case. This is a deficit of

$$1.14 \times 10^{27} - 4.59 \times 10^{26}\ \mathrm{W} \;=\; 6.8 \times 10^{26}\ \mathrm{W}\ .$$

If this luminosity deficit is emitted from the system in the form of 1 keV X-ray photons, the flux of X-ray photons is

$$\frac{6.8 \times 10^{26}\ \mathrm{W}}{10^{3} \times 1.602 \times 10^{-19}\ \mathrm{J}} \;=\; 4.3 \times 10^{43}\ \text{ X-ray photons per second.}$$

Problem 2

The first and second parts of this question were given on pages 40 to 42 of the course notes. You can find the answers in full there! The estimate of the total torque exerted on the inner part of the accretion disc by the magnetic field is

$$\mathcal{T} \;=\; \frac{4\pi}{3}\,\frac{B_z^2(R_*)}{\mu_0}\,\frac{R_*^6}{R_{min}^3} \;,$$

where $B_z(R_*)$ is the component of the magnetic flux density in the direction perpendicular to the disc at the surface of the star, R_* is the radius of the star, and μ_0 is the permeability of free space.
The inner radius R_{min} and the radius of the central star R_* are related by

$$\frac{R_{min}}{R_*} \;=\; \left(\frac{4\pi\, B_z^2(R_*)\, R_*^{5/2}}{3\,\sqrt{GM}\,\dot{m}\,\mu_0}\right)^{2/7} \;,$$

where $\dot{m}$ is the mass flow rate through the disc, and M is the mass of the central star. G is the constant of gravitation.

From page 42 of the course notes, the ratio R_{min}/R_* is given numerically by

$$\frac{R_{min}}{R_*} \;=\; 50\left(\frac{B_z(R_*)}{1\ \mathrm{Tesla}}\right)^{4/7}\left(\frac{R_*}{R_\odot}\right)^{5/7}\left(\frac{\dot{m}}{10^{-8}M_\odot \mathrm{yr}^{-1}}\right)^{-2/7}\left(\frac{M}{M_\odot}\right)^{-1/7} \;,$$

where $R_\odot$ is the radius of the Sun, and $M_\odot$ is the mass of the Sun. For the T Tauri star, putting $M = 1M_\odot$, $R_* = 1R_\odot$, $B_z(R_*) = 10^{-1}$ Tesla and $\dot{m} = 10^{-8}\ M_\odot\,\mathrm{yr}^{-1}$, we get an inner radius of the accretion disc

$$R_{min} \;=\; 13.4\ R_\odot \;.$$

If the accretion rate increases by a factor 10^4 in the FU Orionis star, it becomes $\dot{m} = 10^{-4}\ M_\odot\,\mathrm{yr}^{-1}$. Putting this into the equation, we get

$$R_{min} \;=\; 0.97\ R_\odot$$

during the outburst, if we use the same values for $B_z(R_*)$, R_* and M used for the T Tauri star. This figure is actually smaller than the radius ($R_* = 1R_\odot$) used for the star itself.
In this case the magnetic field lines are swept in towards the star by the increased mass flow and are essentially crushed against the stellar surface. The configuration is now very different since a boundary layer is generated.