B. Solutions to course work 1

Q1 *[25 Marks]*

A gravitationally bound system of mass M and radius R has a characteristic velocity

$$V \sim \sqrt{\frac{GM}{R}}.$$

[This neglects a numerical factor which depends upon the precise distribution of the matter and on whether the speed is associated with rotational or random motion.] Use this formula to estimate the following:

(a) The velocity dispersion of stars in a globular cluster, assumed to have a mass of $10^6 M_\odot$ and a radius of 10 pc. [9 Marks]

Solution: Let us work in MKS units and start with the following useful normalization:

$$v = \sqrt{\frac{GM}{R}} = \left(\frac{6.7\times10^{-11}\times2\times10^{30}}{3.1\times10^{16}}\right)^{\frac{1}{2}}\left(\frac{M}{M_\odot}\right)^{\frac{1}{2}}\left(\frac{R}{\text{pc}}\right)^{-\frac{1}{2}}\text{m s}^{-1} =$$

$$= 0.066\left(\frac{M}{M_\odot}\right)^{\frac{1}{2}}\left(\frac{R}{\text{pc}}\right)^{-\frac{1}{2}}\text{km s}^{-1}. \quad \text{(B.1)}$$

[extra 5 Marks]

If $M = 10^6 M_\odot$ and $R = 10$ pc,

$$v \approx 0.07\times10^3\times0.3 \approx 20\text{km s}^{-1}. \quad \text{(B.2)}$$

(b) The rotation velocity of stars in a spiral galaxy, assumed to have a mass of $10^{11}M_\odot$ and a radius of 10 kpc. [8 Marks]

Solution: If $M = 10^{11}M_\odot$ and $R = 10$ kpc,

$$v \approx 0.07\times310^5\times10^{-2} \approx 200\text{km s}^{-1}. \quad \text{(B.3)}$$

(c) The velocity dispersion of galaxies in a rich cluster, assumed to have a mass of $10^{14}M_\odot$ and a radius of 3 Mpc. [8 Marks]

Solution: If $M = 10^{14}M_\odot$ and $R = 3$Mpc,

$$v \approx 0.07\times10^7\times6\times10^{-4} \approx 400\text{km s}^{-1}. \quad \text{(B.4)}$$

Q2 *[30 Marks]*

Although structures larger than clusters cannot be regarded as gravitationally bound, groups and clusters of galaxies will have peculiar motions due to the gravitational attraction of nearby superclusters.

(a) Explain why, over the age of the Universe t_0, a supercluster of mass M and distance D will induce a velocity in the Local Group of roughly

$$V \sim \frac{GMt_0}{D^2},$$

where D is the distance between the supercluster and the Local Group. [20 Marks]

Solution: Mass M at distance D induces acceleration

$$a = \frac{GM}{D^2}, \quad \text{(B.5)}$$

hence velocity after time equal to the age of the Universe is

$$v = \frac{GM}{D^2}t_0. \quad \text{(B.6)}$$

This order of magnitude estimate neglects the expansion of the Universe (so that the distance D is nearly the same during time t_0), expansion reduces v by small factor.

(b) *If $t_0 = 10^{10}$ y, calculate the velocity induced by the Great Attractor, assumed to have a mass of $10^{17} M_\odot$ and a distance of 100 Mpc. [10 Marks]*

Solution: Taking $M = 10^{17} M_\odot$ and $D = 100$ Mpc we have

$$v = \frac{(6.7 \times 10^{-11})(2 \times 10^{30} \times 10^{17})}{(10^8 \times 3.1 \times 10^{16})^2} \times 3 \times 10^{17} \text{ m s}^{-1} \approx 450 \text{ km s}^{-1}. \tag{B.7}$$

(Compare with velocity with respect to CMB).

Q3 *[45 Marks]*

One usually accounts for the uncertainty in the Hubble parameter by writing it as

$$H_0 = 100h kms^{-1} Mpc^{-1},$$

where the dimensionless number h lies between 1/2 and 1.

(a) Show that the quantity H_0^{-1} has the dimensions of time and calculate this in terms of h, expressing the result in years. [As shown later, this is roughly the age of the Universe in the Big Bang picture.] [10 Marks]

Solution: $[H_0] =$ km s^{-1} Mpc^{-1} has units $LT^{-1}L^{-1} = T^{-1}$.

$$h_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1} = \frac{(10^5 h \text{ m s}^{-1})}{3.1 \times 10^{16} \times 10^6 \text{ m})} = 3.2 \times 10^{-18} h \text{ s}^{-1}. \tag{B.8}$$

Hence

$$H_0^{-1} = 3.1 \times 10^{17} h^{-1} \text{ s} \approx 10^{10} h^{-1} \text{ years}. \tag{B.9}$$

(b) Show that the quantity $3H_0^2/(8\pi G)$ has the dimensions of density and calculate this in terms of h, expressing the results in kgm^{-3}. [As shown later, this corresponds to the critical density required for the Universe to recollapse.] [10 Marks]

Solution: $\frac{H_0^2}{G}$ has units $T^{-2}(M^{-1}L^3T^{-2})^{-1} = ML^{-3}$, which is dimensions of density. Then

$$\frac{3H_0^2}{8\pi G} = \frac{3 \times (3.2 \times 10^{-18} h)^2}{8 \times 3.1 \times 6.9 \times 10^{-11}} \approx 2 \times 10^{-26} h^2 \text{ kg m}^{-3}. \tag{B.10}$$

(c) Galaxies and their constituent baryons (mainly protons) have roughly a tenth of the critical density. If a typical galaxy has a mass of $10^{11} M_\odot$, infer the average distance between galaxies. [15 Marks]

Solution:

$$n_{gal} = \frac{0.1\rho_{crit}}{M_{gal}} = \frac{0.1 \times 2 \times 10^{-26} h^2}{2 \times 10^{30} \times 10^{11}} \approx 10^{-68} h^2 \text{ m}^{-3}, \tag{B.11}$$

hence separation between galaxies

$$d_{gal} \approx n_{gal}^{-1/3} \approx 5 \times 10^{22} h^{-2/3} m \approx 2h^{-2/3} \text{ Mpc}. \tag{B.12}$$

(d) If all the baryons in the Universe were spread out uniformly (instead of being clumped into galaxies), what would be the average separation between them? [10 Marks]

Solution:

$$n_{baryon} = \frac{0.1\rho_{crit}}{m_{baryon}} = \frac{0.1 \times 2 \times 10^{-26} h^2}{1.7 \times 10^{-27} \times 10^{11}} \approx 1.1 \times h^2 \text{ m}^{-3}, \tag{B.13}$$

hence separation between baryons

$$d_{baryon} \approx n_{baryon}^{-1/3} \approx h^{-2/3} \text{ m}^{-3}. \tag{B.14}$$