L. Solutions to course work 6

Q1. *30 Marks*

a) What is the proper area of a sphere centered at the origin. 11 Marks
Solution: The proper area of a sphere is

$$S = \int\int dl_\theta dl_\phi, \tag{L.1}$$

where dl_θ and dl_ϕ should be expressed in terms of properly specified ds. Working with the metric in χ-form, we have

$$dl_\theta = \sqrt{-ds^2|_{dt=d\chi=d\phi=0}} = \frac{R\sin A\chi}{A}d\theta, \tag{L.2}$$

where θ runs from 0 to π,and

$$dl_\phi = \sqrt{-ds^2|_{dt=d\chi=d\theta=0}} = \frac{R\sin A\chi}{A}\sin\theta d\phi, \tag{L.3}$$

where ϕ runs from 0 to 2π. Hence

$$S = \int_0^\pi\int_0^{2\pi}\left(\frac{R\sin A\chi}{A}\right)^2\sin\theta d\theta d\phi =$$

$$\left(\frac{R\sin A\chi}{A}\right)^2\cdot 2\pi(-\cos\theta)|_0^\pi = \frac{4\pi R^2\sin^2 A\chi}{A^2}. \tag{L.4}$$

b) Express your result, first, in terms of χ and then in terms of σ. 8 Marks
Solution: If we want to find the area of a sphere of a given lagrangian radius χ_s or σ_s we also should use the FLRW metric. Working with the metric in σ-form, we have

$$dl_\theta = \sqrt{-ds^2|_{dt=d\sigma=d\phi=0}} = \sigma R d\theta, \tag{L.5}$$

and correspondingly

$$dl_\phi = \sqrt{-ds^2|_{dt=d\chi=d\theta=0}} = \sigma R\sin\theta d\phi, \tag{L.6}$$

hence

$$S = 4\pi\sigma^2R^2, \tag{L.7}$$

which is the same as before if one expresses σ in terms of χ.
c) For a closed Universe, one can scale radial coordinate r so that A=1. Show that the total volume of such a Universe is

$$V = 2\pi^2R_0^3.$$

small*11 Marks*
Solution: The proper volume of a sphere is

$$V = \int\int\int dl_\theta dl_\phi dl_\chi, \tag{L.8}$$

where

$$dl_\chi = \sqrt{-ds^2|_{dt=d\theta=d\phi=0}} = Rd\chi. \tag{L.9}$$

If $A = 1$ (i.e. $k = 1$) χ runs from 0 (the first zero of $\sin\chi$)to π (the second zero of $\sin\chi$). Thus, putting $R = R_0$ we obtain the total volume of the Universe at the present moment

$$V = 4\pi R_0^3 \int_0^\pi d\chi \sin^2\chi =$$

$$= 4\pi R_0^3 \int_0^\pi d\chi \frac{1-\cos 2\chi}{2} = 2\pi R_0^3(\pi - \frac{1}{2}(\sin 2\chi)|_0^\pi = 2\pi^2 R_0^3. \tag{L.10}$$

Q2. small*20 Marks*

a) In a zero-pressure $\Omega_0 = 1$ Friedman model, show that the current physical distance to an object with redshift z is

$$r(z) = r_H[1-(1+z)^{-1/2}],$$

where r_H is the current particle horizon size.

Solution: For $\Omega_0 = 1$ the curvature parameter $k = 0$ and we have

$$ds^2 = c^2dt^2 - R^2(t)[d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{L.11}$$

For light we always put $ds = 0$. For radially propagating light we should put $d\theta = 0$ and $d\phi = 0$. Hence

$$cdt = -Rd\chi, \tag{L.12}$$

the sign $-$ corresponds to light propagating from outside toward the origin of coordinates where an observer is located. Then

$$c\int_{t_e}^{t_0} \frac{dt}{R(t)} = -\chi|_{\chi=\chi_e}^{\chi=0}, \tag{L.13}$$

where o corresponds to observer and e corresponds to emitter. Thus

$$\chi_e = c\int_{t_e}^{t_0} \frac{dt}{R(t)}. \tag{L.14}$$

To calculate physical or proper distance from the emitter to the observer we should multiply this by present scale factor. Thus,

$$r = R_0\chi_e = R_0 c\int_{t_e}^{t_o} \frac{dt}{R(t)} = R_0 c[\int_0^{t_0} \frac{dt}{R(t)} - \int_0^{t_e} \frac{dt}{R(t)}]. \tag{L.15}$$

Then taking into account that for dust (pressure is equal to zero)

$$R(t) = R_0\left(\frac{t}{to}\right)^{2/3}, \tag{L.16}$$

we have

$$r = ct_0^{2/3}\left(\int_0^{t_o} t^{-2/3}dt - \int_0^{t_e} t^{-2/3}dt\right) = 3ct_0^{2/3}\left(t_o^{1/3}dt - t_e^{1/3}dt\right) = 3ct_o\left[1-\left(\frac{t_e}{t_o}\right)^{1/3}\right]. \tag{L.17}$$

Then taking into account the definition of redshift

$$1+z = \frac{R_0}{R} = \left(\frac{t}{to}\right)^{-2/3}, \tag{L.18}$$

we have

$$\left(\frac{t_e}{t_o}\right)^{1/3} = (1+z_e)^{-1/2}. \tag{L.19}$$

To calculate the cosmological horizon r_H we should just put in these expressions $t_e = 0$, thus

$$r_H = 3ct_o. \tag{L.20}$$

Finally,

$$r(z) = r_H[1 - (1+z)^{-1/2}]. \tag{L.21}$$

b) Deduce that objects at the particle horizon have an infinite redshift. 5 Marks

Solution: Taking into account that

$$1 + z \propto t^{-2/3}, \tag{L.22}$$

we can say that light emitted at the beginning of the expansion of the Universe, i.e. at $t = 0$, indeed has infinite redshift.

Q3. *25 Marks*

a) The energy flux received per unit area from a source of bolometric luminosity P at redshift z was shown in the lectures to be $\frac{P}{d_L^2}$, where d_L is the "luminosity distance". Show that

$$d_L = (1+z)R_0 \frac{\sin A\chi}{A}.$$

8 Marks

Solution: The expansion of the Universe results the following two very important and relevant to this section effects: i) The energy of each arriving photon, $h\nu$, drops by a factor $1 + z$. Indeed

$$h\nu_o = \frac{h}{T_o} = \frac{ch}{\lambda_o} = \frac{ch}{\lambda_e(1+z)} = \frac{h\nu_e}{1+z}. \tag{L.23}$$

ii) The rate of photon arrival which is inverse proportional to time interval between arrivals of two subsequent photons also drops by another factor $1 + z$. Thus we can expect that the flux, i.e. the energy per unit area and per unit time, measured from the source located at a point with lagrangian coordinate χ should be inverse proportional to $(1 + z)^2$ and equal to

$$F = \frac{L}{(1+z)^2 S(\chi_e)}, \tag{L.24}$$

where L is the luminosity of the source, i.e. the energy emitted by the source per unit time, and $S(\chi)$ is a sphere with the center in location of the emitter and passing through the observer with location corresponding to lagrangian coordinate χ:

$$S(\chi) = 4\pi R_0^2 \frac{\sin^2 A\chi}{A^2}. \tag{L.25}$$

If the Universe were stationary and spatially flat the flux would be equal to

$$F = \frac{L}{4\pi d^2} = \frac{P}{d^2}. \tag{L.26}$$

We can use eq. (L.26) as a definition of a distance called the Luminosity Distance d_L. From (L.24) and (**??**) we obtain

$$d_L = (1+z)\sqrt{\frac{S(\chi)}{4\pi}} = R_0(1+z)\frac{\sin A\chi}{A}. \tag{L.27}$$

b) Show that

$$R_0 \frac{\sin A\chi}{A} = \frac{2c}{H_0\Omega_0^2(1+z)}\left[\Omega_0 z + (\Omega_0 - 2)(\sqrt{\Omega_0 z + 1} - 1)\right].$$

17 Marks

Solution: From $ds = 0$ we have

$$\chi = c\int_{t_e}^{t_0} \frac{dt}{R(t)}, \tag{L.28}$$

from the Friedmann equation we have

$$\frac{dr}{dt} = \sqrt{\frac{8\pi G\rho R^2}{3} - A^2c^2}. \tag{L.29}$$

Taking into account that

$$k = A^2, \tag{L.30}$$

we have

$$dt = \frac{dR}{\sqrt{\frac{8\pi G\rho R^2}{3} - A^2c^2}}. \tag{L.31}$$

Then taking into account the definition of redshift

$$1 + z = \frac{R_0}{R}, \tag{L.32}$$

we obtain

$$R = \frac{R_0}{1+z}, \;\; dR = -\frac{R_0 dz}{(1+z)^2}. \tag{L.33}$$

For ρ we have

$$\rho = \rho_0 \frac{R_0^3}{R^3} = \rho_{cr}\Omega_0(1+z)^3 = \frac{3H_0^2}{8\pi G}\Omega_0(1+z)^3. \tag{L.34}$$

From the Friedman equation written at the present moment we have

$$H_0^2 = H_0^2\Omega_0 - \frac{A^2c^2}{R_0^2}, \tag{L.35}$$

hence

$$A = \frac{H_0R_0}{c}\sqrt{\Omega_0 - 1}. \tag{L.36}$$

Integrating over z and properly changing limits of integration in expression for χ ($t_0 \to z = 0$ and $t_e \to z$, we obtain

$$\chi = -\int_z^0 \frac{cR_0dz}{(1+z)^2R\sqrt{\frac{8\pi G}{3}\Omega_0\frac{3H_0^2}{8\pi G}\frac{R_0^3}{R} - A^2c^2}} =$$

$$= \int_0^z \frac{cR_0dz}{(1+z)^2\frac{R_0}{1+z}\sqrt{\Omega_0H_0^2R_0^3\frac{1+z}{R_0} - \frac{H_0^2(\Omega_0-1)R_0^2}{c^2}}} =$$

$$= \frac{c}{H_0R_0}\int_0^z \frac{dz}{(1+z)\sqrt{\Omega_0(1+z) - (\Omega_0 - 1)}} = \frac{c}{H_0R_0}\int_0^z \frac{dz}{(1+z)\sqrt{\Omega_0 z + 1)}}. \tag{L.37}$$

Thus

$$A\chi = \frac{H_0 R_0}{c}\sqrt{\Omega_0 - 1}\frac{c}{H_0 R_0}\int_0^z \frac{dz}{(1+z)\sqrt{\Omega_0 z + 1}} = \sqrt{\Omega_0 - 1}\int_0^z \frac{dz}{(1+z)\sqrt{\Omega_0 z + 1}}. \tag{L.38}$$

Natural substitution of variable in this case is

$$\Omega_0 z + 1 = x^2, \;\; \Omega_0 dz = 2x dx, \;\; z = \frac{x^2 - 1}{\Omega_0}, \tag{L.39}$$

after that

$$A\chi = \frac{\sqrt{\Omega_0 - 1}}{\Omega_0}\int_1^{\sqrt{\Omega_0 z + 1}} \frac{2x dx}{(\frac{x^2-1}{\Omega_0} + 1)x} = 2\sqrt{\Omega_0 - 1}\int_1^{\sqrt{\Omega_0 z + 1}} \frac{dx}{x^2 + (\Omega_0 - 1)}. \tag{L.40}$$

Final substitution of variable:

$$x = \sqrt{\Omega_0 - 1}\xi \tag{L.41}$$

and we have

$$A\chi = \frac{2\sqrt{\Omega_0 - 1}\sqrt{\Omega_0 - 1}}{\Omega_0 - 1}\int_{\sqrt{\frac{1}{\Omega_0 - 1}}}^{\sqrt{\frac{\Omega_0 z + 1}{\Omega_0 - 1}}} \frac{d\xi}{\xi^2 + 1} = 2(a - b), \tag{L.42}$$

where

$$\tan a = \sqrt{\frac{\Omega_0 z + 1}{\Omega_0 - 1}}, \;\; \tan b = \frac{1}{\sqrt{\Omega_0 - 1}}. \tag{L.43}$$

Thus

$$\sin A\chi = \sin(2a - 2b) = \sin 2a \cos 2b - \cos 2a \sin 2b =$$

$$= \frac{2\tan a}{1 + \tan^2 a}\frac{1 - \tan^2 b}{1 + \tan^2 b} - \frac{2\tan b}{1 + \tan^2 b}\frac{1 - \tan^2 a}{1 + \tan^2 a}, \tag{L.44}$$

then

$$1 + \tan^2 a = 1 + \frac{\Omega_0 z + 1}{\Omega_0 - 1} = \frac{\Omega_0 - 1 + \Omega_0 z + 1}{\Omega_0 - 1} = \frac{\Omega_0(1+z)}{\Omega_0 - 1}, \tag{L.45}$$

$$1 - \tan^2 a = 1 - \frac{\Omega_0 z + 1}{\Omega_0 - 1} = \frac{\Omega_0 - 1 - \Omega_0 z - 1}{\Omega_0 - 1} = \frac{\Omega_0 - 2 - \Omega_0 z}{\Omega_0 - 1}, \tag{L.46}$$

$$1 + \tan^2 b = 1 + \frac{1}{\Omega_0 - 1} = \frac{\Omega_0 - 1 + 1}{\Omega_0 - 1} = \frac{\Omega_0}{\Omega_0 - 1}. \tag{L.47}$$

$$1 - \tan^2 b = 1 - \frac{1}{\Omega_0 - 1} = \frac{\Omega_0 - 1 - 1}{\Omega_0 - 1} = \frac{\Omega_0 - 2}{\Omega_0 - 1}. \tag{L.48}$$

Finally

$$\frac{R_0 \sin A\chi}{A} = \frac{R_0 c}{R_0 H_0 \sqrt{\Omega_0 - 1}} 2\sqrt{\Omega_0 - 1}\frac{(\Omega_0 - 2)(\sqrt{\Omega_0 z + 1} - 1) + \Omega_0 z}{\Omega_0^2(1+z)} =$$

$$= \frac{2c}{H_0 \Omega_0^2 (1+z)}\left[\Omega_0 z + (\Omega_0 - 2)(\sqrt{\Omega_0 z + 1} - 1)\right]. \tag{L.49}$$

Q4. *25 Marks*

a) The apparent angular size of an object with linear diameter D at redshift z was shown in the lectures to be

$$\theta = \frac{D(1+z)}{R_0}\frac{A}{\sin A\chi},$$

where χ is the co-moving radial coordinate. Using the Friedman equation show that for $\Omega_0 = 1$

$$\theta(z) = \frac{DH_0(1+z)^{3/2}}{2c(\sqrt{1+z}-1)}.$$

12 Marks

Solution: Using the previous results we obtain for $\Omega_0 = 1$:

$$R_0\frac{\sin A\chi}{A} = \frac{2c}{H_0(1+z)}\left[z - (\sqrt{z+1}-1)\right], \tag{L.50}$$

hence

$$\theta(z) = \frac{D(1+z)}{R_0}\frac{A}{\sin A\chi} = \frac{DH_0(1+z)^{3/2}}{2c(\sqrt{1+z}-1)}. \tag{L.51}$$

b) Prove that $\theta(z)$ is non-monotonic function and find z corresponding to the minimum of $\theta(z)$. 13 Marks

Solution: The simplest way to see that $\theta(z)$ is non-monotonic function of z is to take derivative $\frac{d\theta}{dz}$ and solve equation $\frac{d\theta}{dz} = 0$:

$$\theta \propto \frac{x^3}{x-1}, \tag{L.52}$$

where

$$x = \sqrt{1+z}, \quad \frac{dx}{dz} = \frac{1}{2\sqrt{1+z}} \neq 0, \tag{L.53}$$

$$0 = \frac{d\theta}{dz} = \frac{d\theta}{dx}\frac{dx}{dz} \propto \frac{3x^2(x-1)-x^3}{(x-1)^2} \propto (3x-3-x) = 2x-3, \tag{L.54}$$

hence

$$x = \frac{3}{2}, \quad 1+z = \frac{9}{4}, \quad z = \frac{5}{4} = 1.25. \tag{L.55}$$

This extremum is obviously minimum because when

$$z \to 0, \quad \theta(z) \to z^{-1}, \quad z \to \infty, \quad \theta(z) \to z. \tag{L.56}$$