J. Solutions to course work 5

CW5

Q1. *30 Marks*

a) Derive the Friedman-Lemetre-Robertson-Walker metric for spatially flat Universe. 15 Marks
Solution: By considering the geometry of isotropic three-dimensional space as the geometry on a isotropic hypersurface in a fictitious four-dimensional space. Such a space is a hypersphere

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = a^2, \tag{J.1}$$

and the element of length on it is

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2. \tag{J.2}$$

Using spherical coordinates

$$x_1 = r\sin\theta\cos\phi, \quad x_2 = r\sin\theta\sin\phi, \quad x_3 = r\cos\theta, \tag{J.3}$$

we can easily eliminate the fictitious coordinate x_4:

$$r^2 + x_4^2 = a^2, \tag{J.4}$$

then differentiating we obtain

$$2rdr + 2x_4dx_4 = 0 \tag{J.5}$$

and

$$dx_4 = -\frac{rdr}{x_4} = -\frac{rdr}{\sqrt{a^2 - r^2}}. \tag{J.6}$$

Hence

$$dl^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + \frac{r^2dr^2}{a^2 - r^2} = dr^2\left(1 + \frac{r^2}{a^2 - r^2}\right) + r^2(d\theta^2 + \sin^2\theta d\phi^2) =$$

$$= dr^2\left(\frac{a^2 - r^2 + r^2}{a^2 - r^2}\right) + r^2(d\theta^2 + \sin^2\theta d\phi^2) = \frac{dr^2}{1 - \frac{r^2}{a^2}} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{J.7}$$

For spatially flat Universe

$$a \to \infty \tag{J.8}$$

and we have

$$dl^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{J.9}$$

Then introducing lagrangian coordinate χ defined by

$$r = R(t)\chi \tag{J.10}$$

and taking into account that

$$ds^2 = c^2dt^2 - dl^2 \tag{J.11}$$

we finally obtain

$$ds^2 = c^2dt^2 - R^2(t)[d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{J.12}$$

b) Derive the Friedman-Lemetre-Robertson-Walker metric in the case of constant negative curvature of the three-dimensional space. 15 Marks

Solution: In the case of constant negative curvature of the three-dimensional space a should be replaced by ia and we obtain (see Solution 1a)

$$ds^2 = c^2dt^2 - dl^2 = c^2dt^2 - \frac{dr^2}{1 - \frac{r^2}{(ia)^2}} - r^2(d\theta^2 + \sin^2\theta d\phi^2) = c^2dt^2 - \frac{dr^2}{1 + \frac{r^2}{a^2}} - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{J.13}$$

Then introducing lagrangian coordinate χ defined by

$$r = a(t)\sinh\chi \tag{J.14}$$

we finally obtain

$$ds^2 = c^2dt^2 - a^2(t)[\frac{d(\sinh\chi)^2}{1+\sinh^2\chi} + \sinh^2\chi^2(d\theta^2 + \sin^2\theta d\phi^2)] = c^2dt^2 - a^2(t)[\frac{\cosh^2\chi d\chi^2}{\cosh^2\chi} + \sinh^2\chi^2(d\theta^2 + \sin^2\theta d\phi^2)] =$$

$$= c^2dt^2 - a^2(t)[d\chi^2 + \sinh^2\chi^2(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{J.15}$$

Q2. *30 Marks*

a) Show that a positive Λ - term corresponds to a repulsive force whose strength is proportional to distance. Show that when Λ - term dominates

$$R \sim exp[(\Lambda/3)^{1/2}t].$$

20 Marks

Solution: The Λ - term is additional term in the following version of the EFES.

$$R^i_k - \frac{1}{2}\delta^i_k R - \Lambda\delta^i_k = \frac{8\pi G}{c^4}T^i_k, \tag{J.16}$$

where T^i_k is the Stress-Energy tensor and Λ is a constant introduced by Einstein. Let us first show that in this case the acceleration equation for scale factor $a(t)$ can be written as

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)a + \frac{\Lambda a}{3}, \tag{J.17}$$

As we know from Chapter 19

$$R^0_0 = -\frac{3\ddot{a}}{ac^2}. \tag{J.18}$$

Taking $i = 0$ and $k = 0$ in Eq.(J.16), we obtain

$$-\frac{3\ddot{a}}{ac^2} - \frac{1}{2}R - \Lambda = \frac{8\pi G}{c^4}T^0_0 = \frac{8\pi G}{c^4}\epsilon,. \tag{J.19}$$

On other hand if we produce summation $i = k$ in (J.16), we obtain

$$R - \frac{1}{2}\times 4R - \Lambda\times 4 = \frac{8\pi G}{c^4}T = \frac{8\pi G}{c^4}(\epsilon - 3P) = \frac{8\pi G}{c^2}(\rho - \frac{3P}{c^2}), \tag{J.20}$$

hence

$$-R - 4\Lambda = \frac{8\pi G}{c^2}(\rho - \frac{3P}{c^2}) \tag{J.21}$$

and

$$-\frac{3\ddot{a}}{ac^2} - \frac{1}{2}[-4\Lambda - \frac{8\pi G}{c^2}(\rho - \frac{3P}{c^2})] - \Lambda = \frac{8\pi G}{c^2}\rho. \tag{J.22}$$

Finally

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)a + \frac{\Lambda a}{3}. \tag{J.23}$$

One can see that if $\Lambda > 0$ the Λ - term corresponds to a repulsive force whose strength is proportional to distance. When this term dominates we can re-write Eq (J.23) as

$$\ddot{a} = \frac{\Lambda a}{3}. \tag{J.24}$$

Trying solution

$$a = Ce^{\lambda t} \tag{J.25}$$

corresponds to

$$\lambda = \sqrt{\frac{\Lambda}{3}}. \tag{J.26}$$

b) Verify that the substitution

$$\sigma = A^{-1}\sin A\chi$$

turns the metric

$$ds^2 = -c^2dt^2 + R(t)^2[d\chi^2 + (A^{-1}\sin A\chi)^2(d\theta^2 + \sin^2\theta d\phi^2)]$$

into the form

$$ds^2 = -c^2dt^2 + R(t)^2[(1 - A^2\sigma^2)^{-1}d\sigma^2 + \sigma^2(d\theta^2 + \sin^2\theta d\phi^2)].$$

10 Marks

Solution: Differentiating

$$\sigma = A^{-1}\sin A\chi \tag{J.27}$$

we have

$$d\sigma = A^{-1}A\cos A\chi d\chi = \cos A\chi d\chi, \tag{J.28}$$

hence

$$d\chi = \frac{d\sigma}{\cos A\chi}. \tag{J.29}$$

Substituting this into the above expression for ds^2 we obtain

$$ds^2 = -c^2dt^2 + R(t)^2[\frac{d\sigma^2}{\cos^2 A\chi} + \sigma^2(d\theta^2 + \sin^2\theta d\phi^2)] = -c^2dt^2 + R(t)^2[(1 - \sin^2 A\chi)^{-1}d\sigma^2 + \sigma^2(d\theta^2 + \sin^2\theta d\phi^2)] =$$

$$= -c^2dt^2 + R(t)^2[(1 - A^2\sigma^2)^{-1}d\sigma^2 + \sigma^2(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{J.30}$$

Q3. *40 Marks*

a) Starting from the first law of thermodynamics (in cosmological context it means the change of energy is equal to work done by forces of pressure)

$$dE = -pdV,$$

where E is total energy in some volume V and p is pressure and ρ is mass density, derive the energy conservation equation

$$\dot{\rho} = -(3\dot{R}/R)(\rho + p/c^2).$$

10 Marks

Solution: Taking into account that

$$E = \rho c^2 V, \tag{J.31}$$

after differentiation we have

$$dE = d(\rho c^2 V) = c^2 d\rho V + c^2 \rho dV. \tag{J.32}$$

Substituting this into the first law of thermodynamics, we have

$$d\rho V + \rho dV = -\frac{p}{c^2} dV, \tag{J.33}$$

hence

$$d\rho V = -(\rho + \frac{p}{c^2}) dV \tag{J.34}$$

and

$$\dot{\rho} = -(\dot{V}/V)(\rho + p/c^2). \tag{J.35}$$

Taking into account that

$$V \sim R^3, \tag{J.36}$$

we finally obtain

$$\dot{\rho} = -(3\dot{R}/R)(\rho + p/c^2). \tag{J.37}$$

b) Show that the Einstein equation

$$(3\ddot{R}/R) = -4\pi G(\rho + \frac{3p}{c^2})$$

can be combined with the energy conservation equation and integrated to give the Friedman equation

$$(\dot{R}/R)^2 - 8\pi G\rho/3 = -kc^2/R^2.$$

10 Marks

Solution: From the energy conservation equation we obtain

$$\frac{p}{c^2} = -\rho - \frac{\dot{\rho}R}{3\dot{R}}. \tag{J.38}$$

Putting this expression for p into the acceleration equation, we have

$$\ddot{R} = -\frac{4\pi GR}{3}\left(\rho + \frac{3p}{c^2}\right) = -\frac{4\pi GR}{3}\left(\rho - 3\rho - \frac{\dot{\rho}R}{\dot{R}}\right) =$$

$$= \frac{4\pi G}{3\dot{R}}\left(2\rho R\dot{R} + \dot{\rho}R^2\right) = \frac{4\pi G}{3\dot{R}}\left(\rho R^2\right)^{\cdot}, \tag{J.39}$$

then multiplying both sides of this equation by $2\dot{R}$ and taking into account that

$$2\dot{R}\ddot{R} = \left(R^2\right)^{\cdot}, \tag{J.40}$$

we obtain

$$\left(\dot{R}^2\right)^{\cdot} = \frac{8\pi G}{3}\left(\rho R^2\right)^{\cdot}, \tag{J.41}$$

hence

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - kc^2, \tag{J.42}$$

This is the Friedmann equation in the relativistic Cosmology.

c) Assume that $p = \alpha\rho c^2$, where α is equation of state parameter, show that at some times

$$R \sim t^{2/[3(1+\alpha)]}$$

and find the range of time when this equation is valid for different α, which is supposed to be $-1 \le \alpha \le 1$. 20 Marks

Solution: If we put the equation of state

$$P = \alpha\rho c^2, \tag{J.43}$$

into the energy conservation equation we obtain

$$\frac{\dot{\rho}}{\rho} = -3(1+\alpha)\frac{\dot{R}}{R}, \tag{J.44}$$

we obtain

$$(\ln\rho)^{\cdot} + 3(1+\alpha)(\ln R)^{\cdot} = [\ln\rho + 3(1+\alpha)\ln R]^{\cdot} = \left\{\ln[\rho R^{3(1+\alpha)}]\right\}^{\cdot} = 0, \tag{J.45}$$

thus

$$\ln[\rho R^{3(1+\alpha)}] = C \tag{J.46}$$

and

$$\rho R^{3(1+\alpha)} = C'. \tag{J.47}$$

Finally we obtain that

$$\rho = \rho_0 \left(\frac{R_0}{R}\right)^{3(1+\alpha)}. \tag{J.48}$$

If

$$\frac{8\pi G\rho}{3} \gg \frac{|k|c^2}{R^2}, \tag{J.49}$$

the Friedman equation is reduced to

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G\rho}{3}. \tag{J.50}$$

Substituting into this equation the expression for ρ obtained above, we have

$$\frac{\dot{R}}{R} = \left(\frac{8\pi G\rho}{3}\right)^{1/2} \left(\frac{R_0}{R}\right)^{\frac{3(1+\alpha)}{2}}; \tag{J.51}$$

we can solve this equation by the separation of variables. For that let introduce

$$x = \frac{R}{R_0}\ \beta = \frac{3(1+\alpha)}{2} \text{ and } A = \left(\frac{8\pi G\rho}{3}\right)^{1/2}. \tag{J.52}$$

In terms of x, β and A the above equation can be written as

$$\dot{x}x^{\beta-1} = A, \tag{J.53}$$

then

$$\frac{1}{\beta}dx^{\beta} = Adt, \tag{J.54}$$

hence

$$x^{\beta} = \beta At + C. \tag{J.55}$$

Taking into account that $x = 0$ at $t = 0$ we put $C = 0$. Thus

$$x^\beta \sim t, \tag{J.56}$$

and

$$R \sim x \sim t^{\frac{1}{\beta}} = t^{\frac{2}{3(1+\alpha)}}. \tag{J.57}$$

This solution is valid if

$$\frac{8\pi G\rho}{3} \gg \frac{|k|c^2}{R^2}. \tag{J.58}$$

If $k = 0$ this solution is valid for all t. If $k = \pm 1$ the LHS goes like $R^{-3(1+\alpha)}$ while the RHS goes like R^{-2}. Hence if $-3(1+\alpha) > -2$ or $\alpha < -1/3$ our solution is valid for small R and correspondingly small t, if $\alpha > -1/3$ our solution is valid for large R and correspondingly large t.