D. Solutions to course work 2

Q1. *[30 Marks]*

(a) Starting with the Newtonian deceleration equation

$$\ddot{R} = -\frac{4\pi G\rho}{3}R,$$

derive the Friedman equation in the form

$$\dot{R}^2 = \frac{8\pi G\rho}{3}R^2 - kc^2,$$

where ρ is homogeneous density and k is a constant. [10 Marks]
Solution: Using the mass-conservation equation

$$\rho = \rho_0 \left(\frac{R_0}{R}\right)^3, \tag{D.1}$$

we can re-write acceleration equation as

$$\ddot{R} = -\frac{4\pi G\rho_0 R_0^3}{3R^2}. \tag{D.2}$$

Then multiplying (D.2) by $\dot{R}$ we obtain

$$\dot{R}\ddot{R} = -\dot{R}\frac{4\pi G\rho_0 R_0^3}{3R^2}. \tag{D.3}$$

Taking into account that

$$\dot{R}\ddot{R} = \frac{1}{2}\left(\dot{R}^2\right)^{\cdot} \quad \text{and} \quad \frac{\dot{R}}{R^2} = -\left(\frac{1}{R}\right)^{\cdot}, \tag{D.4}$$

we have

$$\frac{\dot{R}^2}{2} - \frac{4\pi G\rho_0 R_0^3}{3R} = C, \tag{D.5}$$

where C is an arbitrary constant of integration. Introducing another constant

$$k = -\frac{2C}{c^2} \tag{D.6}$$

and using again (D.1), we can rewrite (D.5) in standard form of FE.
b) *Show that the following relationships*

$$\Omega(t) = 2q(t), \quad \textit{and} \quad kc^2 = R(t)^2 H(t)^2 (\Omega(t) - 1)$$

between deceleration

$$q(t) = -\frac{R(t)\ddot{R}(t)}{\dot{R}^2(t)},$$

the dimensionless density

$$\Omega(t) = \frac{8\pi G\rho(t)}{3H^2(t)},$$

the Hubble constant $H(t)$ and the scale factor $R(t)$ are valid for any moment of time (not only at the present moment).[15 Marks]
Solution: From the definition of $q(t)$, from acceleration equation and from the definition of $H(t)$, we have

$$q(t) = -\frac{\ddot{R}(t)R(t)}{\dot{R}^2(t)} = \frac{4\pi G\rho(t)R^2}{3\dot{R}^2} = \frac{4\pi G\rho(t)}{3H^2(t)} = \frac{\rho(t)}{2\rho_{cr}(t)} = \frac{\Omega(t)}{2}. \tag{D.7}$$

Substituting

$$\dot{R}(t) = H(t)R(t) \tag{D.8}$$

into FE, we obtain

$$H^2(t)R^2(t) = \frac{8\pi G\rho(t)R^2(t)}{3} - kc^2, \tag{D.9}$$

hence

$$kc^2 = R^2(t)H^2(t)\left(\frac{8\pi G\rho(t)}{3H^2(t)} - 1\right) = R^2(t)H^2(t)\left(\Omega(t) - 1\right). \tag{D.10}$$

Q2. *[25 Marks]*

a) Show that in the Newtonian cosmological model with k=+1 the scale factor, $R(t)$, as a function of time can be presented in the following parametric form

$$R = a(1 - \cos\eta), \quad t = \frac{a}{c}(\eta - \sin\eta),$$

where a is a constant and the parameter η goes from 0 (the Big Bang) to 2π (the Big Crunch).[10 Marks]

Solution: As the function $R(t)$ is given in the parametric form (I D)

$$\dot{R}(t) = \frac{\frac{dR}{d\eta}}{\frac{dt}{d\eta}} = \frac{a\sin\eta}{\frac{a}{c}(1-\cos\eta)} = \frac{c\sin\eta}{1-\cos\eta}. \tag{D.11}$$

Substituting (D.11), $k = 1$ and (D.1) into (**??**) we obtain

$$\frac{c^2\sin^2\eta}{(1-\cos\eta)^2} - \frac{8\pi G\rho_0 R_0^3}{3a(1-\cos\eta)} + c^2 = 0. \tag{D.12}$$

This is valid if

$$\frac{8\pi G\rho_0 R_0^3}{3ac^2} = \frac{1-\cos^2\eta}{1-\cos\eta} + (1-\cos\eta) = 1 + \cos\eta + 1 - \cos\eta = 2, \tag{D.13}$$

i.e.

$$a = \frac{4\pi G\rho_0 R_0^3}{3c^2}. \tag{D.14}$$

b) *Calculate the maximal value of the scale factor R in terms of the main cosmological parameters.[10 Marks]*

Solution: The cosmological parameters by definition can be obtained from direct observations. H_0 and Ω_0 are cosmological parameters. R_{max} corresponds to $\cos\eta = -1$, i.e. $\eta = \pi$. Hence

$$R_{max} = 2a = \frac{8\pi G\rho_0 R_0^3}{3c^2}. \tag{D.15}$$

We can express R_0 in terms of H_0 and Ω_0: indeed, taking into account that

$$H_0^2 R_0^2 = \frac{8\pi G\rho_0 R_0^2}{3} - c^2, \tag{D.16}$$

[see eq. (D.9) taken at the present moment] we have

$$R_0^2 = \frac{c^2}{\frac{8\pi G\rho_0}{3} - H_0^2} = \frac{c^2}{H_0^2\left(\frac{8\pi G\rho_0}{3H_0^2} - 1\right)} = \frac{c^2}{H_0^2(\Omega_0 - 1)}, \quad \text{thus} \quad R_0 = \frac{c}{H_0\sqrt{\Omega_0 - 1}}. \tag{D.17}$$

Then

$$R_{max} = \frac{8\pi G\rho_0 R_0^3}{3c^2} = \frac{8\pi G\Omega_0\rho_{cr} R_0^3}{3c^2} = \frac{8\pi G\Omega_0 \times 3H_0^2 R_0^3}{8\pi G \times 3c^2} = \frac{\Omega_0 H_0^2}{c^2}\frac{c^3}{H_0^3(\Omega_0 - 1)^{3/2}} = \frac{c\Omega_0}{H_0(\Omega_0 - 1)^{3/2}}. \quad \text{(D.18)}$$

Q3. *[15 Marks]*

Find in a parametric form the dependence of the Hubble constant, $H(t)$, on time in Newtonian cosmological model with $k = 1$.
Solution: As follows from (I D)

$$H(\eta) = \frac{\dot{R}(t)}{R(t)} = \frac{c\sin\eta}{a(1-\cos\eta)^2}, \quad \text{(D.19)}$$

hence H can be presented as function of time in the following parametric form:

$$H(\eta) = \frac{c\sin\eta}{a(1-\cos\eta)^2},$$

$$t(\eta) = \frac{a}{c}(\eta - \sin\eta). \quad \text{(D.20)}$$

Q4. *[30 Marks]*

a) *Motivate that the fate of the Universe can be predicted on basis of observations.[10 Marks]*
Solution: From eq. (D.9) taken at the present time

$$kc^2 = H_0^2 R_0^2(\Omega_0 - 1), \quad \text{(D.21)}$$

follows that
(i) if $k = 1$, which means that the fate of the Universe in the future is the Big Crunch, $\Omega_0 > 1$;
(ii) if $k = 0$, which means the the fate of the Universe in the future is to expand for ever as $R \propto t^{2/3}$, $\Omega_0 = 1$;
(iii) if $k = -1$, which means the the fate of the Universe in the future is to expand for ever, asymptotically as $R \propto t$ (i.e. asymptotically without acceleration), $\Omega_0 < 1$;
Taking into account that Ω_0 can be determined directly from observation we can conclude that the fate of the Universe can be predicted on basis of observations.

b) *Find the Hubble constant, $H(t)$, as a function of time in Newtonian cosmological model with $k = 0$.[15 Marks]*
Solution: If $k = 0$

$$\dot{R}^2 = \frac{8\pi G\rho_0 R_0^3}{3R}, \text{ hence } \dot{R} \equiv \frac{dR}{dt} = \left(\frac{8\pi G\rho_0 R_0^3}{3}\right)^{1/2} R^{-1/2} = AR^{-1/2}, \text{ where } A = \left(\frac{8\pi G\rho_0 R_0^3}{3}\right)^{1/2} \quad \text{(D.22)}$$

Thus

$$R^{1/2}dR = Adt \text{ and } d\left(\frac{2}{3}R^{3/2}\right) = Adt, \textbf{ hence } \frac{2}{3}R^{3/2} = At + B, \quad \text{(D.23)}$$

where B is a constant of integration. If $R = 0$ at $t = 0$ the constant $B = 0$. Then

$$\left(\frac{R(t)}{R_0}\right)^{3/2} = \frac{t}{t_0} \quad R(t) = R_0\left(\frac{t}{t_0}\right)^{2/3}, \textbf{ hence } H(t) = \frac{\dot{R}(t)}{R(t)} = \frac{2R_0 t^{-1/3}}{3R_0 t^{2/3}\left(\frac{t}{t_0}\right)^{2/3}} = \frac{2}{3t}, \text{ finally } H(t) = \frac{2}{3t}. \text{(D.24)}$$

c) *Find the age of the Universe predicted by the Newtonian theory in this case and express your results in terms of observable cosmological parameters.[5 Marks]*
Solution: Taking (D.24) at the present moment we obtain

$$t_0 = \frac{2}{3H_0}, \quad \text{(D.25)}$$

where t_0 is obviously the present age of the Universe and H_0 is the observable cosmological parameter.