B.Sc. EXAMINATION

MAS 347 Mathematical Aspects of Cosmology

xxx, xx May 2008 xx:xx-xx:xx
Duration: 2 hours

SOLUTIONS

1. *Show that the quantity inverse to the Hubble constant, H_0^{-1}, has the dimensions of time and calculate its value, expressing the result in years. Using the Hubble law show that this quantity in order of magnitude is equal to the age of the Universe. Show that the quantity $\frac{3H_0^2}{8\pi G}$ has the dimensions of density and calculate its value, expressing the results in kg m^{-3}. Explain briefly why this quantity corresponds to the critical density required for the Universe to recollapse.*

Solution A1[seen similar]

$[H_0] =$ km s^{-1} Mpc^{-1}has units $LT^{-1}L^{-1} = T^{-1}$.

[1/10]

$$H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1} = \frac{(10^5 h \text{ m s}^{-1})}{3.1 \times 10^{16} \times 10^6 \text{ m})} = 3.2 \times 10^{-18} h \text{ s}^{-1},$$

hence

$$H_0^{-1} = 3.1 \times 10^{17} h^{-1} \text{ s} \approx 10^{10} h^{-1} \text{ years}.$$

[1/10]

Then taking into account that

$$h = 0,7,$$

we obtain finally

$$H_0^{-1} \approx 1.4 \times 10^{10} \text{ years}.$$

[2/10]

$\frac{H_0^2}{G}$ has units $T^{-2}(M^{-1}L^3T^{-2})^{-1} = ML^{-3}$,which is dimensions of density.

[1/10]

Then

$$\rho_{cr} = \frac{3H_0^2}{8\pi G} = \frac{3 \times (3.2 \times 10^{-18} h)^2}{8 \times 3.1 \times 6.9 \times 10^{-11}} \approx 2 \times 10^{-26} h^2 \text{ kg m}^{-3}.$$

[3/10]

Then taking into account that

$$h = 0.7,$$

we obtain finally

$$\rho_{cr} \approx 5 \times 10^{-27} \text{ kg m}^{-3}.$$

[1/10]

This density is called critical because the future destiny of the Universe depends crucially on its present density, ρ_0: if $\rho_0 \leq \rho_{crit}$ the Universe will expand for ever,

if $\rho_0 > \rho_{crit}$ the Universe at some moment in future will star to contract and will end in the Big Crunch.

[1/10]

2. *Use the Friedmann equation for a spatially curved Universe to find the present value of the scale factor R_0 in terms of the present Hubble constant, H_0, and the density parameter, Ω_0. Show that in the case of the spatially flat Universe, R_0 is arbitrary.*

Solution A2[seen similar]

From the Friedmann equation we have

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G\rho}{3} - \frac{kc^2}{R^2},$$

hence

$$H^2 = \frac{8\pi G}{3}\rho_{crit}\Omega - \frac{kc^2}{R^2} = \frac{8\pi G}{3}\frac{3H^2}{8\pi G}\Omega - \frac{kc^2}{R^2} = H^2\Omega - \frac{kc^2}{R^2},$$

then

$$\frac{kc^2}{R^2} - H^2(\Omega - 1),$$

and

$$kc^2 = R^2 H^2(\Omega - 1).$$

[4/10]

Thus if $\Omega \neq 1$

$$R = \frac{c}{H}\sqrt{\frac{k}{\Omega - 1}}.$$

[2/10]

Taking into account that

$$\begin{aligned} \Omega_0 &> 1, \quad \text{if } k = 1, \\ \Omega_0 &< 1, \quad \text{if } k = -1, \\ \Omega_0 &= 1, \quad \text{if } k = 0, \end{aligned}$$

[1/10]

we have

$$R_0 = \tfrac{c}{H_0\sqrt{\Omega_0 - 1}}, \quad \text{if } k = 1,$$

[1/10]

$$R_0 = \tfrac{c}{H_0\sqrt{1 - \Omega_0}}, \quad \text{if } k = -1.$$

[1/10]

If $k = 0$ and $\Omega_0 = 1$ the equation

$$kc^2 = R^2 H^2(\Omega - 1)$$

is trivial like

$$0 = 0,$$

valid for an arbitrary R_0.

[1/10]

[Next question overleaf.]

3. *Assume that a small fraction of the dark matter density corresponding to the density parameter $\Omega_x \ll 1$ in a spatially flat Universe can be explained by hypothetical primordial black holes of mass $M = 50M_\odot$. Assuming that the average distance between these objects at the present time is 10kpc, estimate Ω_x. Given that the density parameter of dark energy in the form of the Λ-term at the present moment is $\Omega_\Lambda \approx 0.7$, find the redshift corresponding to the moment of time when the density of the primordial black holes was equal to the density of dark energy.*

Solution A3[unseen]

The number density of the objects at the present moment is

$$n = \frac{\Omega_x \rho_{cr}}{M},$$

[1/10]

the average distance between these objects is determined from

$$d^3 n \approx 1,$$

hence

$$n = d^{-3},$$

[1/10]

hence

$$\Omega_x = \frac{M}{\rho_{cr} d^3} = \frac{50 \times 2.0 \times 10^{30}\text{kg}}{5 \times 10^{-27}\text{ kg m}^{-3}} \times 10^3 \times 10^9 \times 3^3 \times 10^{48}\text{m}^3 \approx 0.7 \times 10^{-2+2+30+27-3-9-48} \approx 7 \times 10^{-4}$$

[3/10]

The density of non-relativistic objects depends on z as

$$\rho(z) = \Omega_x \rho_{cr} (1+z)^3,$$

[1/10]

while the density of dark energy, ρ_{DE}, in the form of the Λ-term does not depend on z

$$\rho_{DE} = \Omega_\Lambda \rho_{cr},$$

[1/10]

hence the moment when

$$\rho(z) = \rho_{DE}$$

 [This question continues overleaf . . .]

corresponds to the redshift

$$z + 1 \approx \left(\frac{0.7}{0.7 \times 10^{-3}}\right)^{1/3} \approx 10,$$

hence

$$z = 9.$$

[3/10]

[Next question overleaf.]

4. *Consider a spatially flat Universe containing dark energy with equation of state $\alpha = -1$ and dust with dimensionless density Ω_d. Show that the deceleration parameter q depends on redshift z as follows*

$$q = \frac{(1+z)^3 - \gamma}{2(1+z)^3 + \gamma},$$

where

$$\gamma = 2\left(\Omega_d^{-1} - 1\right).$$

Assuming that $\Omega_d \approx 0.25$, estimate z when the Universe started to expand with acceleration.

Solution A4[unseen]

For spatially flat Universe

$$\Omega_{\text{dark energy}} + \Omega_d = 1, \;\; \Omega_{\text{dark energy}} = 1 - \Omega_d.$$

[1/10]

According to Friedmann equation for spatially flat universe

$$H^2 = \frac{8\pi G\rho}{3}.$$

[1/10]

From the acceleration equation we obtain the following expression for deceleration parameter

$$q = -\frac{R\ddot{R}}{\dot{R}^2} = \frac{4\pi G}{3H^2}\left(\rho + \frac{3p}{c^2}\right) = \frac{1}{2}\left(1 + \frac{3P}{\rho c^2}\right),$$

[1/10]

where pressure

$$p = \alpha\Omega_{\text{de}}\rho_{cr} = -(1-\Omega_d)\rho_{cr},$$

[1/10]

and

$$\rho = \Omega_d\rho_{cr}\left(\frac{R_0}{R}\right)^3 + (1-\Omega_d).$$

[1/10]

Taking into account that

$$\frac{R_0}{R} = 1 + z,$$

[1/10]

we have

$$q = \frac{1}{2}\left(1 - \frac{3(1-\Omega_d)}{1-\Omega_d+\Omega_d(1+z)^3}\right) = \frac{(1+z)^3-\gamma}{2(1+z)^3+\gamma}.$$

[2/10]

The Universe started to expand with acceleration when $q = 0$, i.e.

$$z = \gamma^{1/3} - 1 = \left[2((0.25)^{-1}-1)\right]^{1/3} - 1 = 6^{1/3} - 1 \approx 0.82.$$

[2/10]

[Next question overleaf.]

5. *According to some cosmological model of the early Universe the scale factor evolves as*

$$R \propto t^{\beta},$$

where β *is a constant. The equation of state at that epoch is* $p = \alpha\rho c^2$*. Express* β *in terms of* α *and find the range of* α *corresponding to the expansion with acceleration.*

Solution A5[seen similar]

For equation of state $p = \alpha\rho c^2$ from energy conservation we have

$$\frac{d(c^2\rho R^3)}{dt} = -\alpha\rho c^2\frac{d(R^3)}{dt}, \quad \dot{\rho}R^3 + 3\rho R^2\dot{R} = -3\alpha\rho R^2\dot{R},$$

hence

$$\frac{\dot{\rho}}{\rho} = -3(1+\alpha)\frac{\dot{R}}{R},$$

and

$$\rho \propto R^{-3(1+\alpha)}.$$

[2/10]

From the Friedmann equation

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho \propto R^{-3(1+\alpha)},$$

or

$$\dot{R} \propto R^{1-\frac{3(1+\alpha)}{2}}, \quad R^{\frac{3(1+\alpha)}{2}-1}dR \propto dt, \quad R^{\frac{3(1+\alpha)}{2}} \propto t, \quad R \propto t^{\frac{2}{3(1+\alpha)}}.$$

[3/10]

Finally

$$\beta = \frac{2}{3(1+\alpha)}, \quad 1+\alpha = \frac{2}{3\beta}, \quad \alpha = \frac{2}{3\beta} - 1.$$

[2/10]

By definition

$$q = -\frac{\ddot{R}R}{\dot{R}^2},$$

then

$$R \propto t^{\beta}, \quad \dot{R} \propto \beta t^{\beta-1}, \quad \ddot{R} \propto \beta(\beta-1)t^{\beta-2},$$

hence

$$q = -\frac{\beta(\beta-1)t^{\beta-2}t^{\beta}}{\beta^2 t^{2(\beta-1)}} = -\frac{(\beta-1)}{\beta}.$$

[2/10]

If $\beta > 1$ we have $q < 0$, hence $\ddot{R} > 0$, which corresponds to the expansion with acceleration. Thus from $\beta > 1$ we have $\alpha < -1/3$.

[1/10]

SECTION B

1. *Assume that the Universe with $\Lambda = 0$ is closed ($k = 1$) and contains only dust. The evolution of the scale factor in this case is given in the following parametric form*

$$R(\eta) = \frac{\beta}{2}(1 - \cos\eta), \quad t(\eta) = \frac{\beta}{2c}(\eta - \sin\eta),$$

where η is a variable which runs from 0 to 2π and β is some constant.

(a) []
Using the Friedman equation and the acceleration equation, show that

$$\beta = \frac{2cq_0}{H_0(2q_0 - 1)^{3/2}},$$

where q_0 is the deceleration parameter at the present moment.

Solution B1a[seen similar]

The energy conservation equation in the case $\alpha = 0$, which means $p = \alpha\rho c^2 = 0$, gives $d\rho R^3 = 0$, hence $\rho = \rho_0 (R_0/R)^3$.

[1/10]

Substituting this result to the Friedman equation we have $\dot{R}^2 = c^2 (\gamma/R - 1)$, where $\gamma = 8\pi\rho_0 R_0^3/3c^2$.

[1/10]

Then we calculate $\dot{R}^2$, using the parametric solution:

$$\dot{R}^2 = \left[\frac{d\left[\frac{\beta}{2}(1 - \cos\eta)\right]}{d\left[\frac{\beta}{2c}(\eta - \sin\eta)\right]}\right]^2 = c^2 \frac{\sin^2\eta}{(1 - \cos\eta)^2} = c^2 \frac{1 + \cos\eta}{1 - \cos\eta}.$$

[2/10]

Putting this into the Friedman equation we have

$$c^2 \frac{1 + \cos\eta}{1 - \cos\eta} = c^2 \left(\frac{2\gamma}{\beta(1 - \cos\eta)} - 1\right), \quad 1 + \cos\eta = \frac{2\gamma}{\beta} - 1 + \cos\eta,$$

[2/10]

so we see that this parametric solution does satisfies the Friedman equation, if $\beta = \gamma = 8\pi G\rho_0 R_0^3/3c^2$.

[1/10]

From the Friedman equation, taken at the moment t_0 we have

$$H_0^2 R_0^2 = H_0^2 R_0^2 \Omega_0 - c^2,$$

[1/10]

so we can express R_0 in terms of H_0 and Ω_0 as

$$R_0 = \frac{c}{H_0\sqrt{\Omega_0 - 1}}.$$

[2/10]

Then substituting this to the formula for β we have

$$\beta = \frac{c\Omega_0}{H_0(\Omega_0 - 1)^{3/2}}.$$

[2/10]

From the acceleration equation

$$q_0 = -\frac{R_0\ddot{R}_0}{(\dot{R}_0)^2} = \frac{4\pi G\rho_0}{3H_0^2} = \frac{\Omega_0}{2},$$

[2/10]

then substituting

$$\Omega_0 = 2q_0$$

[1/10]

into expression for βwe, finally, obtain

$$\beta = \frac{2cq_0}{H_0(2q_0 - 1)^{3/2}}.$$

[2/10]

(b) [8Marks]
Present time dependence of the Hubble constant in parametric form, using the above expression for β.

Solution B1b[unseen]

$$H = \frac{\dot{R}}{R} = \frac{\frac{\beta}{2}\sin\eta}{\frac{\beta}{2c}(1-\cos\eta)\frac{\beta}{2}(1-\cos\eta)} = \frac{2c}{\beta}\frac{\sin\eta}{(1-\cos\eta)^2} = \frac{(2q_0-1)^{3/2}\sin\eta}{q_0(1-\cos\eta)^2}.$$

[4/10]

Finally, time dependence $H(t)$ has the following parametric form:

$$H(\eta) = \frac{(2q_0-1)^{3/2}}{q_0}\frac{\sin\eta}{(1-\cos\eta)^2},$$

[2/10]

$$t(\eta) = \frac{q_0}{H_0(2q_0-1)^{3/2}}(\eta - \sin\eta).$$

[2/10]

2. (a) [15 Marks] *Show that all covariant derivatives of metric tensor are equal to zero. Assuming that the Cristoffel symbols are symmetric with respect to low indices, i.e.* $\Gamma^n_{ik} = \Gamma^n_{ki}$, *show that*

$$\Gamma^n_{ik} = \frac{1}{2} g^{nm} \left(g_{mi,k} + g_{mk,i} - g_{ik,m} \right).$$

SOLUTION B2a

$$DA_i = g_{ik} DA^k$$

[1/10]

$$DA_i = D(g_{ik}A^k) = g_{ik}DA^k + A^k Dg_{ik},$$

[1/10]

hence

$$g_{ik}DA^k = g_{ik}DA^k + A^k Dg_{ik},$$

[1/10]

which obviously means that

$$A^k Dg_{ik} = 0.$$

[1/10]

Taking into account that A^k is arbitrary vector, we conclude that

$$Dg_{ik} = 0.$$

[1/10]

Then taking into account that

$$Dg_{ik} = g_{ik;m} dx^m = 0$$

[1/10]

for arbitrary infinitesimally small vector dx^m we have

$$g_{ik;m} = 0.$$

[1/10]

Introducing useful notation

$$\Gamma_{k,\, il} = g_{km} \Gamma^m_{il},$$

 [This question continues overleaf . . .]

we have

$$g_{ik;\,l} = \frac{\partial g_{ik}}{\partial x^l} - g_{mk}\Gamma^m_{il} - g_{im}\Gamma^m_{kl} = \frac{\partial g_{ik}}{\partial x^l} - \Gamma_{k,\,il} - \Gamma_{i,\,kl} = 0.$$

[2/10]

Permuting the indices i, k and l twice as

$$i \to k, \quad k \to l, \quad l \to i,$$

[1/10]

we have

$$\frac{\partial g_{ik}}{\partial x^l} = \Gamma_{k,\,il} + \Gamma_{i,\,kl}, \quad \frac{\partial g_{li}}{\partial x^k} = \Gamma_{i,\,kl} + \Gamma_{l,\,ik} \text{ and } -\frac{\partial g_{kl}}{\partial x^i} = -\Gamma_{l,\,ki} - \Gamma_{k,\,li}.$$

[2/10]Taking into account that

$$\Gamma_{k,\,il} = \Gamma_{k,\,li},$$

[1/10]

after summation of these three equation we have

$$g_{ik,l} + g_{li,k} - g_{kl,i} = 2\Gamma_{i,\,kl},$$

[1/10]

and finally

$$\Gamma^i_{kl} = \frac{1}{2} g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right).$$

[1/10]

(b) [10 Marks] *Using the Bianchi identity show that*

$$R^k_{i;k} - \frac{1}{2}\delta^k_i R_{,k} = 0.$$

Explain briefly why this pure geometrical identity is so important for physics.

Solution B2b[Seen similar]

Contracting the Bianchi identity on the pairs of indices ik and ln we have

$$g^{ik}(R^l_{ikl;m} + R^l_{imk;l} + R^l_{ilm;k}) = g^{ik}g^{lp}(R_{pikl;m} + R_{pimk;l} + R_{pilm;k}) =$$

[1/10]

$$g^{ik}g^{lp}(-R_{ipkl;m} - R_{ipmk;l} - R_{iplm;k}) =$$

[1/10]

By symmetry properties of the Riemann tensor

$$g^{ik}g^{lp}(R_{pikl;m} + R_{pimk;l} + R_{pilm;k}) =$$

[1/10]

$$g^{ik}g^{lp}(-R_{klip;m} - R_{mkip;l} - R_{lmip;k}) = g^{ik}g^{lp}(-R_{klip;m} + R_{kmip;l} - R_{lmip;k}) = 0$$

[2/10]

By the definition of the Ricci tensor

$$-g^{lp}(R_{lp;m} - R_{mp;l} + g^{ik}R_{mi;k}) = -R, m + R^l_{m;l} + R^k_{m;k} = -R_{,m} + 2R^l_{m;l} = 0,$$

[1/10]

thus

$$R^l_{m;l} = \frac{1}{2}R_{,m}.$$

[1/10]

Taking into account the EFEs, From

$$R^l_{m;l} - \frac{1}{2}R_{,m} = 0$$

follows that

$$T^l_{m:l} = 0,$$

which is the local law for conservation of energy.

[2/10]

[Next question overleaf.]

3. *Consider a sphere in a Robertson-Walker model with comoving coordinate $\chi = \chi_s$.*

(a) [8 Marks] *Verify that the substitution*

$$\sigma = A^{-1} \sin A\chi$$

turns the metric

$$ds^2 = -c^2 dt^2 + R(t)^2 [d\chi^2 + (A^{-1} \sin A\chi)^2 (d\theta^2 + \sin^2 \theta d\phi^2)]$$

into the form

$$ds^2 = -c^2 dt^2 + R(t)^2 [(1 - A^2\sigma^2)^{-1} d\sigma^2 + \sigma^2 (d\theta^2 + \sin^2 \theta d\phi^2)].$$

Solution B3a[seen similar]

Differentiating

$$\sigma = A^{-1} \sin A\chi$$

we have

$$d\sigma = A^{-1} \cos A\chi \cdot A \cdot d\chi = \cos A\chi d\chi,$$

[2/10]

hence

$$d\chi = \frac{d\sigma}{\cos A\chi} = \frac{d\sigma}{\sqrt{1 - \sin^2 A\chi}} = \frac{d\sigma}{\sqrt{1 - \sigma^2 A^2}}.$$

[2/10]

Substituting $d\chi$ into the metric in the χ-form, we obtain the metric in σ-form:

$$ds^2 = -c^2 dt^2 + R(t)^2 [(1 - A^2\sigma^2)^{-1} d\sigma^2 + \sigma^2 (d\theta^2 + \sin^2 \theta d\phi^2)].$$

[4/10]

(b) [17 Marks] *What is the proper area and volume of a sphere centred at the origin. Express your result, first, in terms of χ and then in terms of σ. For a closed Universe, one can scale radial coordinate r so that A=1. Show that the total volume of such a Universe is*

$$V = 2\pi^2 R_0^3.$$

Solution B3b

The proper area of a sphere is

$$S = \int \int dl_\theta dl_\phi.$$

 [This question continues overleaf . . .]

[1/10]

Working with the metric in χ-form, we have

$$dl_\theta = ds|_{dt=d\chi=d\phi=0} = \frac{R\sin A\chi}{A}d\theta,$$

where θ runs from 0 to π,

[1/10]

and

$$dl_\phi = ds|_{dt=d\chi=d\theta=0} = \frac{R\sin A\chi}{A}\sin\theta d\phi,$$

where ϕ runs from 0 to 2π.

[1/10]

Hence

$$S = \int_0^\pi \int_0^{2\pi} \left(\frac{R\sin A\chi}{A}\right)^2 \sin\theta d\theta d\phi = \left(\frac{R\sin A\chi}{A}\right)^2 \cdot 2\pi(-\cos\theta)|_0^\pi = \frac{4\pi R^2\sin^2 A\chi}{A^2}.$$

[3/10]Working with the metric in σ-form, we have

$$dl_\theta = ds|_{dt=d\sigma=d\phi=0} = \sigma R d\theta,$$

[1/10]

and correspondingly

$$dl_\phi = ds|_{dt=d\chi=d\theta=0} = \sigma R\sin\theta d\phi,$$

[1/10]

hence

$$S = 4\pi\sigma^2 R^2,$$

[2/10]

which is the same as before if on expresses σ in terms of χ.

[1/10]

The proper volume of a sphere is

$$V = \int\int\int dl_\theta dl_\phi dl_\chi,$$

[1/10]

where

$$dl_\chi = ds|_{dt=d\theta=d\phi=0} = R d\chi,$$

[1/10]

for $A = 1$ χ runs from 0 (the first zero of $\sin\chi$)to π (the second zero of $\sin\chi$). Thus, putting $R = R_0$

$$V = 4\pi R_0^3 \int_0^\pi d\chi \sin^2\chi =$$

$$= 4\pi R_0^3 \int_0^\pi d\chi \frac{1-\cos 2\chi}{2} = 2\pi R_0^3(\pi - \frac{1}{2}(\sin 2\chi)|_0^\pi = 2\pi^2 R_0^3.$$

[4/10]

[Next question overleaf.]

4. (a) 15 Marks]

Derive the equation for the evolution of small density perturbations, $\delta = (\rho^{'} - \rho)/\rho$ after decoupling to show that

$$\ddot{\delta} + (4/3t)\dot{\delta} - (2/3t^2)\delta = 0.$$

(Hint: Take into account that $\rho' R'^3 = \rho R^3$.) Solve this equation using the trial solution $\delta \propto t^m$ to obtain the two modes of perturbations: $\delta = A(t/t_0)^{m_1} + B(t/t_0)^{m_2}$.

B4a. Solution[seen similar]

Starting from

$$\ddot{R} = -\frac{4\pi G\rho R}{3},$$

perturb R and ρ: $R^{'} = R(1+h)$, and $\rho^{'} = \rho(1+\delta)$.

[1/10]

Putting this in the perturbed equation

$$\ddot{R'} = -\frac{4\pi G\rho^{'} R^{'}}{3},$$

[1/10]

we obtain

$$\ddot{R}(1+h) + 2\dot{R}\dot{h} + R\ddot{h} = -\frac{4\pi G\rho(1+\delta)R(1+h}{3}.$$

[1/10]

Using unperturbed equation, we obtain linearized equation

$$\ddot{R}h + 2\dot{R}\dot{h} + R\ddot{h} = -\frac{4\pi G\rho R(\delta + h)}{3}.$$

[1/10]

To relate h and δ we use the conservation of energy equation $\rho R^3 = \rho R^3(1 + 3h)(1+\delta)$,

[1/10]

or $1 = 1 + 3h + \delta$, so $h = -\delta/3$.

[1/10]

Thus we have

$$-\ddot{R}\frac{\delta}{3} - \frac{2}{3}\dot{R} - R\frac{\ddot{\delta}}{3} = -\frac{4\pi G\rho}{3}R\frac{2}{3}\delta,$$

[1/10]

then

$$\ddot{\delta} + 2\frac{\dot{R}}{R} + \frac{\ddot{R}}{R}\delta = \frac{8\pi G\rho}{3}\delta.$$

[1/10]

For the dust-like Universe $R \sim t^{2/3}$, so

$$\frac{\dot{R}}{R} = \frac{2}{3t}, \quad \frac{\ddot{R}}{R} = \frac{2}{3}(\frac{2}{3} - 1)t^{-2} = -\frac{2}{9t^2}.$$

[1/10]

From the unperturbed equation

$$\frac{8\pi G\rho}{3} = -\frac{2\ddot{R}}{R} = \frac{4}{9t^2}.$$

[1/10]

Then

$$\ddot{\delta} + 2\frac{2}{3t} + (-\frac{2}{9} - \frac{4}{9})\frac{\delta}{t^2} = 0,$$

[1/10]

and finally

$$\ddot{\delta} + \frac{4}{3t} - \frac{2}{3t^2}\delta = 0.$$

[1/10]

Taking trial solution $\delta = At^m$, we obtain

$$m(m-1) + \frac{4m}{3} - \frac{2}{3} = 0, \quad 3m^2 + m - 2 = 0.$$

[1/10]

Solutions of this quadratic equation are

$$m = \frac{-1 \pm \sqrt{1+24}}{6} = \frac{-1 \pm 5}{6},$$

[1/10]

thus $m_+ = 2/3$ and $m_- = -1$ (growing and decaying modes). So we have

$$\delta = A_+(t/t_0)^{2/3} + A_-(t/t_0)^{-1}.$$

[1/10]

(b) [10 Marks]

According to the COBE observations of the Microwave Background anisotropy, the amplitude of the density perturbations at the moment of decoupling is about 10^{-5} *(Take redshift at this moment* $z = 999$*). Assuming that the first objects were formed at a redshift* $z = 9$*, estimate the two arbitrary constants in your solution for the density perturbations.*

 [This question continues overleaf . . .]

B4b. Solution[unseen]

Taking into account that

$$R = R_0(t/t_0)^{2/3} = R_0(1+z)^{-1},$$

[1/10]

we have

$$t/t_0 = (1+z)^{-3/2},$$

[1/10]

hence

$$\delta = A_+(1+z)^{-1} + A_-(1+z)^{3/2}.$$

[2/10]

So we have the system of equations for A_+ and A_- For $z = 9$

$$1 = A_+10^{-1} + A_-10^{3/2},$$

[1/10]

and for $z = 999$

$$10^{-5} = A_+10^{-3} + A_-10^{9/2},$$

[1/10]

So

$$A_+ = \frac{10^{9/2} - 10^{3/2-5}}{10^{9/2-1} - 10^{3/2-3}} \approx 10^{9/2-9/2+1} = 10,$$

[2/10]

$$A_- = -\frac{10}{10^{15/2}} = 10^{-13/2}.$$

[2/10]

 [End of examination paper.]