Queen Mary
University of London

# B.Sc. EXAMINATION

## MAS 347 Mathematical Aspects of Cosmology

4 June 2008 10:00-12:00
Duration: 2 hours

*This paper has two Sections and you should attempt both Sections. Please read carefully the instructions at the beginning of each Section. The use of an electronic calculator is permitted but no graph-plotting facilities may be used. Please state on your answer book the name and type of machine used.*

*You are not permitted to start reading this question paper until instructed to do so by an invigilator.*

### Physical Constants

| | | |
|---|---|---|
| Gravitational constant | $G$ | $6.7 \times 10^{-11}$ N m$^2$ kg$^{-2}$ |
| Speed of light | $c$ | $3 \times 10^8$ m s$^{-1}$ |
| Solar mass | $M_\odot$ | $2.0 \times 10^{30}$kg |
| Gravitational radius of Sun | $r_{g\odot}$ | 3 km |
| Hubble constant | $H_0$ | 70 km s$^{-1}$Mpc$^{-1}$ |
| Hubble radius | $c/H_0$ | $6 \times 10^3$ Mpc |

1 pc $= 3.1 \times 10^{16}$ m

### NOTATION

Three-dimensional tensor indices are denoted by Greek letters $\alpha, \beta, \gamma, ...$ and take on the values 1, 2, 3.

Four-dimensional tensor indices are denoted by Latin letters $i, k, l, ...$ and take on the values 0, 1, 2, 3.

The metric signature $(+---)$ is used.

Partial derivatives are denoted by ",".

Covariant derivatives are denoted by ";".

 

**USEFUL FORMULAS.**

## Cosmology

$$ds^2 = c^2 dt^2 - R^2(t)\left[d\chi^2 + \frac{\sin^2(\sqrt{k}\chi)}{k}(d\theta^2 + \sin^2\theta d\phi^2)\right] \quad \text{(Robertson–Walker metric)},$$

$$\ddot{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)R + \frac{\Lambda R}{3} \quad \text{(acceleration equation)}$$

$$q = -\frac{\ddot{R}R}{\dot{R}^2} \quad \text{deceleration parameter}$$

$$\dot{R}^2 + kc^2 = \frac{8\pi G}{3}\rho R^2 + \frac{\Lambda R^2}{3} \quad \text{(Friedmann equation)}$$

$$d(\rho c^2 V) = -pdV \quad \text{(energy conservation equation)}$$

$$\rho_{crit} = \frac{3H_0^2}{8\pi G} = 0.92 \times 10^{-26} \text{ kg m}^{-3} \quad \text{(critical density)}$$

## General Relativity

**Minkowski metric:**

$$ds^2 = \eta_{ik} dx^i dx^k = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

**Covariant derivatives:**

$$A^i_{;k} = A^i_{,k} + \Gamma^i_{km} A^m, \quad A_{i;k} = A_{i,k} - \Gamma^m_{ik} A_m, \text{ where } \Gamma^i_{kn} \text{ are Christoffel symbols}$$

**Christoffel symbols:**

$$\Gamma^i_{kl} = \frac{1}{2} g^{im}\left(g_{mk,l} + g_{ml,k} - g_{kl,m}\right)$$

**Geodesic equation:**

$$\frac{du^i}{ds} + \Gamma^i_{kn} u^k u^n = 0,$$

where

$u^i = dx^i/ds$ is the 4-velocity along the geodesic.

**Riemann tensor:**

$$A^i_{;k;l} - A^i_{;l;k} = -A^m R^i_{mkl}, \quad \text{where} \quad R^i_{klm} = g^{in} R_{nklm},$$

$$R^i_{klm} = \Gamma^i_{km,l} - \Gamma^i_{kl,m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl}.$$

**Symmetry properties of the Riemann tensor:**

$$R_{iklm} = -R_{kilm} = -R_{ikml}, \quad R_{iklm} = R_{lmik}.$$

**Bianchi identity:**

$$R^{n}_{ikl;m} + R^{n}_{imk;l} + R^{n}_{ilm;k} = 0.$$

**Ricci tensor:**

$$R_{ik} = g^{lm} R_{limk} = R^{m}_{imk}.$$

**Scalar curvature:**

$$R = g^{il} g^{km} R_{iklm} = g^{ik} R_{ik} = R^{i}_{i}.$$

**Einstein equations:**

$$R^{i}_{k} - \frac{1}{2}\delta^{i}_{k} R = \frac{8\pi G}{c^4} T^{i}_{k},$$

where $T^{i}_{k}$ is the Stress-Energy tensor.

**Gravitational radius**:

$$r_g = 2GM/c^2 = 3(M/M_\odot)\ \mathrm{km}, \quad \text{where } M_\odot \text{ is the mass of the Sun.}$$

## SECTION A

*Each question carries 10 marks. Attempt ALL questions.*

**1.** Show that the quantity inverse to the Hubble constant, $H_0^{-1}$, has the dimensions of time and calculate its value, expressing the result in years. Using the Hubble law show that this quantity in order of magnitude is equal to the age of the Universe. Show that the quantity $\frac{3H_0^2}{8\pi G}$ has the dimensions of density and calculate its value, expressing the results in kg m$^{-3}$. Explain briefly why this quantity corresponds to the critical density required for the Universe to recollapse.

**2.** Use the Friedmann equation for a spatially curved Universe to find the present value of the scale factor $R_0$ in terms of the present Hubble constant, $H_0$, and the density parameter, $\Omega_0$. Show that in the case of the spatially flat Universe, $R_0$ is arbitrary.

**3.** Assume that a small fraction of the dark matter density corresponding to the density parameter $\Omega_x \ll 1$ in a spatially flat Universe can be explained by hypothetical primordial black holes of mass $M = 50M_\odot$. Assuming that the average distance between these objects at the present time is 10kpc, estimate $\Omega_x$. Given that the density parameter of dark energy in the form of the $\Lambda$-term at the present moment is $\Omega_\Lambda \approx 0.7$, find the redshift corresponding to the moment of time when the density of the primordial black holes was equal to the density of dark energy.

**4.** Consider a spatially flat Universe containing dark energy with equation of state $\alpha = -1$ and dust with dimensionless density $\Omega_d$. Show that the deceleration parameter $q$ depends on redshift $z$ as follows

$$q = \frac{(1+z)^3 - \gamma}{2(1+z)^3 + \gamma},$$

where

$$\gamma = 2\left(\Omega_d^{-1} - 1\right).$$

Assuming that $\Omega_d \approx 0.25$, estimate $z$ when the Universe started to expand with acceleration.

**5.** According to some cosmological model of the early Universe the scale factor evolves as

$$R \propto t^\beta,$$

where $\beta$ is a constant. The equation of state at that epoch is $p = \alpha\rho c^2$. Express $\beta$ in terms of $\alpha$ and find the range of $\alpha$ corresponding to the expansion with acceleration.

## SECTION B

*Each question carries 25 marks. You may attempt all questions but, except for a bare pass, only marks for the best TWO questions will be counted.*

**1.** Assume that the Universe with $\Lambda = 0$ is closed ($k = 1$) and contains only dust. The evolution of the scale factor in this case is given in the following parametric form

$$R(\eta) = \frac{\beta}{2}(1 - \cos\eta), \quad t(\eta) = \frac{\beta}{2c}(\eta - \sin\eta),$$

where $\eta$ is a variable which runs from 0 to $2\pi$ and $\beta$ is some constant.

(a) **[17 Marks]**
Using the Friedman equation and the acceleration equation, show that

$$\beta = \frac{2cq_0}{H_0(2q_0 - 1)^{3/2}},$$

where $q_0$ is the deceleration parameter at the present moment.

(b) **[8 Marks]**
Derive the time dependence of the Hubble constant in parametric form, using the above expression for $\beta$.

**2.** (a) **[15 Marks]** Show that all covariant derivatives of metric tensor are equal to zero. Assuming that the Cristoffel symbols are symmetric with respect to low indices, i.e. $\Gamma^n_{ik} = \Gamma^n_{ki}$, show that

$$\Gamma^n_{ik} = \frac{1}{2}g^{nm}\left(g_{mi,k} + g_{mk,i} - g_{ik,m}\right).$$

(b) **[10 Marks]** Using the Bianchi identity, show that

$$R^k_{i;k} - \frac{1}{2}\delta^k_i R_{,k} = 0.$$

Explain briefly why this pure geometrical identity is so important for physics.

**3.** Consider a sphere in a Robertson-Walker model with comoving coordinate $\chi = \chi_s$.



*[This question continues overleaf . . . ]*

(a) [**8 Marks**] Verify that the substitution

$$\sigma = A^{-1} \sin A\chi,$$

where $A$ is a constant, turns the metric

$$ds^2 = -c^2dt^2 + R(t)^2[d\chi^2 + (A^{-1} \sin A\chi)^2(d\theta^2 + \sin^2\theta d\phi^2)]$$

into the form

$$ds^2 = -c^2dt^2 + R(t)^2[(1 - A^2\sigma^2)^{-1}d\sigma^2 + \sigma^2(d\theta^2 + \sin^2\theta d\phi^2)].$$

(b) [**17 Marks**] What is the proper area and volume of a sphere centred at the origin. Express your result, first, in terms of $\chi$ and then in terms of $\sigma$. For a closed Universe, one can scale radial coordinate $r$ so that A=1. Show that the total volume of a closed Universe is

$$V = 2\pi^2 R_0^3.$$

**4.** (a) [**15 Marks**]

Derive the equation for the evolution of small density perturbations, $\delta = (\rho' - \rho)/\rho$ after decoupling to show that

$$\ddot{\delta} + (4/3t)\dot{\delta} - (2/3t^2)\delta = 0.$$

(*Hint: Take into account that* $\rho' R'^3 = \rho R^3$.) Solve this equation using the trial solution $\delta \propto t^m$ to obtain the two modes of perturbations:

$$\delta = A(t/t_0)^{m_1} + B(t/t_0)^{m_2},$$

where $A$ and $B$ are arbitrary constants.

(b) [**10 Marks**]

According to the COBE observations of the Microwave Background anisotropy, the amplitude of the density perturbations at the moment of decoupling is about $10^{-5}$. Assuming that the first objects were formed at a redshift $z = 9$, estimate the two arbitrary constants, $A$ and $B$, in your solution for the density perturbations. (You may assume that the redshift at the moment of decoupling is $z = 999$).