G. Course work 4

CW4: due to 11.03.10:

Q1. *25 Marks*

Using a locally-inertial coordinate system prove that the Riemann tensor has the following symmetry properties:
a) Permutation within a pair of indices

$$R_{iklm} = -R_{kilm} = -R_{ikml}. \tag{G.1}$$

b) Permutation of pairs

$$R_{iklm} = R_{lmik}. \tag{G.2}$$

Q2. *25 Marks*

a) Prove that

$$R_{iklm} + R_{imkl} + R_{ilmk} = 0. \tag{G.3}$$

b) Using a locally-inertial coordinate system prove the Bianchi identity:

$$R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0. \tag{G.4}$$

Q3. *25 Marks*

a) Give the definition of the Ricci tensor R_{ik} and prove that

$$R_{ik} = \Gamma^l_{ik,l} - \Gamma^l_{il,k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}. \tag{G.5}$$

b) Using the Bianchi identity, prove that the Ricci tensor and the scalar curvature, $R = g^{ik}R_{ik}$, satisfy the following identity:

$$R^l_{m;l} = \frac{1}{2}R_{,m}. \tag{G.6}$$

Q4. *25 Marks*

a) Starting from the Einstein equations in the form

$$R_{ik} = \frac{8\pi G}{c^4}\left(T_{ik} - \frac{1}{2}g_{ik}T\right), \tag{G.7}$$

prove that

$$T^i_k = \frac{c^4}{8\pi G}\left(R^i_k - \frac{1}{2}\delta^i_k R\right), \tag{G.8}$$

where δ^i_k is the unit diagonal four-tensor.
b) What can you say about the nature of gravitational field, for which $R_{ik} = 0$, while R_{ikln} is not equal to zero?
c) Prove that the energy-momentum tensor of matter T^i_k satisfies the conservation law $T^k_{i;k} = 0$.