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B. Sc. Examination 2009

MTH6123 Mathematical Aspects of Cosmology. Solutions. 2009

Duration: 2 hours

Date and time:

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Section A: Each question carries 10 marks.

Question 1 *Show that the density parameter Ω_0 (see rubric) is dimensionless. Explain briefly how the value of this parameter can be obtained from direct observations and why dark matter problem is relevant for the determination of Ω_0.*

Solution 1 *[seen similar]*

Taking into account that

$$\Omega_0 = \frac{\rho_0}{\rho_{crit}} = \frac{8\pi G\rho_0}{3H_0^2},$$

and

$$[G] = [M][L][T]^{-2}[M]^{-2}[L]^2 = [M]^{-1}[L]^3[T]^{-2};$$
$$[\rho_0] = [M][L]^{-3};$$
$$[H_0] = [L][T]^{-1}[L]^{-1} = [T]^{-1};$$

we have

$$[\Omega_0] = [M]^{-1}[L]^3[T]^{-2}[M][L]^{-3}[T]^2 = [M]^{-1+1}[L]^{3-3}[T]^{-2+2} = [M]^0[L]^0[T]^0,$$

i.e. Ω_0 is dimensionless. [4]

The value of Ω_0 is determined by ρ_0 and H_0. Both these two values are cosmological parameters and are obtained from direct observations:

(i) H_0 is obtained from spectroscopic measurements of Hubble velocities and measurements of distances to relevant cosmological objects. The measurements of the velocities are based on the Doppler effect; the measurements of distances are based on "standard candles" in Astronomy. [2]

(ii) The value of ρ_0 is determined from measurements of masses and distances within astronomical objects and counting the number of such objects in a given volume. The masses are determined from dispersions of random velocities or rotational velocity, which also obtained with the help of the Doppler effect. [2]

Such measurements show that gravitating masses exceed luminous masses, i.e. indicate that there is dark mass within astronomical objects. [2]

Question 2 *The Hubble radius is determined as*

$$R_H = \frac{c}{H_0},$$

where H_0 is the Hubble parameter. It is given that according to some cosmological model with $k = -1$ the present scale factor, R_0, is twice larger than R_H. Use the Friedman equation to find the density parameter corresponding to a such cosmological model.

Solution 2 *[seen similar]*

From the Friedman equation with $k = -1$ taken at the present moment, we have

$$\dot{R}_0^2 = \frac{8\pi G \rho_0 R_0^2}{3} + c^2.$$

[2]

Taking into account that

$$\dot{R}_0 = H_0 R_0$$

and

$$\rho_0 = \Omega_0 \rho_{crit} = \frac{3H_0^2}{8\pi G},$$

[2]

we obtain

$$H_0^2 R_0^2 (1 - \Omega_0) = c^2,$$

[2]

hence

$$R_0 = \frac{c}{H_0\sqrt{1-\Omega_0}} = \frac{R_H}{\sqrt{1-\Omega_0}}.$$

[2]

Since $R_0 = 2R_H$ we obtain that

$$\sqrt{1-\Omega_0} = \frac{1}{2} \text{ and } \Omega_0 = 1 - \frac{1}{4} = \frac{3}{4}.$$

[2]

Question 3 *Assume that the contribution of some low massive Jupiter-like dark objects of mass m to the average density of the Universe is 1% of the critical density. Estimate the average distance between these objects at the present time,d. You can assume that the Hubble parameter H_0 is equal to $70 kms^{-1} Mpc^{-1}$ and $m = 10^{-3} M_\odot$. What was the distance between such objects at the moment corresponding to the redshift $z = 9$?*

Solution 3 *[unseen]*

If there are N objects in a volume V the volume per object $v \sim d^3$ is

$$v = \frac{V}{N}.$$

On the other hand the average density of such objects is

$$\rho = \frac{M}{V} = \frac{mN}{V} = \frac{m}{v} = \frac{m}{d^3},$$

hence

$$d = \left(\frac{m}{\rho}\right)^{1/3}.$$

[4]

At the present moment

$$\rho = 0.01\rho_{crit} = 10^{-28} \ \text{kg} \ \text{m}^{-3}.$$

(see rubruc) [1]

Hence

$$d \approx \left(\frac{10^{-3} \times 1.99 \times 10^{30} \ kg}{10^{-28} \ \text{kg} \ \text{m}^{-3}}\right),$$

Hence

$$d \approx \left(\frac{10^{-3} \times 1.99 \times 10^{30} \ kg}{10^{-28} \ \text{kg} \ \text{m}^{-3}}\right) \approx \left(2 \times 10^{55}\right)^{1/3} \ \text{m} \approx 3 \times 10^{18} \ \text{m} \approx 100 \ \text{pc}.$$

[2]

Taking into account that the density of massive objects depends on z as

$$\rho(z) \propto (1+z)^3,$$

we can say that

$$d(z) \propto \rho^{-1/3}(z) \propto (1+z)^{-1},$$

hence at the moment corresponding to the redshift $z = 9$

$$d(z = 9) \approx \frac{100 \ \text{pc}}{10} = 10 \ \text{pc}.$$

[3]

Question 4 *A cosmological model describes the early Universe which contains a perfect fluid with equation of state $p = \alpha\rho c^2$. Using the energy conservation and acceleration equations (see rubric) show that*

$$\frac{\rho(R)}{\rho_0} = \left(\frac{R}{R_0}\right)^{-3(1+\alpha)}.$$

Express α in terms of acceleration parameter q.

Solution 4 *[unseen]*

From conservation of energy we have

$$d(\rho c^2 R^3) = -\alpha\rho c^2 d(R^3),$$

hence

$$\frac{d\rho(R)}{\rho} = -3(1+\alpha)\frac{dR}{R}$$

and

$$\rho \propto R^{-3(1+\alpha)}.$$

[2]

The fact that we consider the early Universe means that we can neglect the curvature term in the Friedman equation. Thus from the Friedman equation we have

$$\dot{R} = \sqrt{\frac{8\pi G\rho}{3}}R \propto R^{1-\frac{3}{2}(1+\alpha)} = R^{-\frac{3\alpha+1}{2}}.$$

[2]

This means that

$$dt \propto R^{\frac{3\alpha+1}{2}}dR$$

and

$$t \propto R^{\frac{3\alpha+1}{2}+1} = R^{\frac{3(1+\alpha)}{2}}.$$

Hence

$$R \propto t^{\frac{2}{3(1+\alpha)}}$$

and

$$R = R_0\left(\frac{t}{t_0}\right)^{\frac{2}{3(1+\alpha)}}.$$

[2]

Then the deceleration parameter is

$$q = -\frac{\ddot{R}R}{\dot{R}^2} = -\frac{2}{3(1+\alpha)}\left(\frac{2}{3(1+\alpha)} - 1\right)\left(\frac{2}{3(1+\alpha)}\right)^{-2} = \frac{3(1+\alpha)}{2} - 1.$$

[2]

Finally

$$\alpha = \frac{2}{3}(q - \frac{1}{2}).$$

[2]

Question 5 *Consider a spatially flat cosmological model containing dark energy with equation of state $\alpha = -1/2$ and radiation with density parameter $\Omega_{0(r)}$. Using formula for dependence of density on scale factor from the previous question, show that the ratio of the total pressure P to the total energy density ρc^2 depends on redshift as*

$$\frac{P}{\rho c^2} = \frac{\frac{1}{3}\Omega_{0(r)}(1+z)^4 - \frac{1}{2}(1-\Omega_{0(r)})(1+z)^{3/2}}{\Omega_{0(r)}(1+z)^4 + (1-\Omega_{0(r)})(1+z)^{3/2}}.$$

Find $\Omega_{0(r)}$ if it is given that according to a such model the Universe started expand with acceleration at $z = 1/4$.

Solution 5 *[unseen]*

The fact that the dark energy and radiation do not interact with each other means that we can write down the conservation of energy equation for dark energy and radiation separately. [1]

For radiation we have

$$\rho_{(r)} \propto R^{-4} \propto (1+z)^4.$$

Thus

$$\rho_{(r)} = \rho_{0(r)}(1+z)^4 = \Omega_{0(r)}\rho_{crit}(1+z)^4$$

and

$$P_{(r)} = \alpha_{(r)}c^2\rho_{(r)} = \frac{\Omega_{0(r)}\rho_{crit}c^2}{3}(1+z)^4.$$

[2]

For dark energy we have

$$\rho_{(de)} \propto R^{-4} \propto (1+z)^4.$$

Thus

$$\rho_{(de)} = \rho_{0(de)}(1+z)^{3/2} = \Omega_{0(de)}\rho_{crit}(1+z)^{3/2}$$

and

$$P_{(de)} = \alpha_{(de)}c^2\rho_{(de)} = -\frac{\Omega_{0(de)\rho_{crit}}}{2}(1+z)^{3/2}.$$

[2]

Taking into account that

$$\Omega_0 = \Omega_{0(r)} + \Omega_{0(de)} = 1, \quad \rho = \rho_{(r)} + \rho_{(de)}, \quad P = P_{(r)} + P_{(de)},$$

we finally obtain the required formulae for $P/\rho c^2$. [1]

The Universe starts to expand with acceleration at the moment when, $\rho c^2 + 3P$, changes sign. From the previous eq. we have

$$\left[\Omega_{0(r)}(1+z)^4 + (1-\Omega_{0(r)})(1+z)^{3/2} + \Omega_{0(r)}(1+z)^4 - \frac{3}{2}(1-\Omega_{0(r)})(1+z)^{3/2}\right] = 0.$$

[2]

Hence

$$\Omega_{0(r)} = \frac{\frac{1}{2}(1+z)^{3/2}}{2(1+z)^4 + \frac{1}{2}(1+z)^{3/2}} = \frac{1}{1+4(1+z)^{5/2}} = \frac{1}{1+4(1+\frac{1}{4})^{5/2}} = \frac{1}{1+\frac{25\sqrt{5}}{8}} \approx 0.125.$$

[2]

Section B: Each question carries 25 marks.

Question 6 *Assume that the Universe with* $\Lambda = 0$ *is open* ($k = -1$) *and contains only dust. The evolution of the scale factor in this case is given in the following parametric form*

$$R(\eta) = \frac{a}{2}(\cosh\eta - 1), \quad t(\eta) = \frac{a}{2c}(\sinh\eta - \eta),$$

where η *is a variable which runs from 0 to* ∞ *and* a *is some constant.*

(a) *Using the Friedman equation express* a *in terms of the Hubble and density parameters.* [17]

Solution 6.a *[seen similar]*

Let us put this parametric dependence of R on t into the Friedman equation to find out under what conditions such a dependence is the solution of this equation. [1]

From the mass conservation equation we have

$$\rho R^3 = \rho_0 R_0^3,$$

hence

$$\rho R^2 = AR^{-1},$$

where

$$A = \rho_0 R_0^3$$

is a constant. [2]

Then taking into account that

$$\frac{dR}{dt} = \frac{\frac{dR}{d\eta}}{\frac{dt}{d\eta}} = \frac{\frac{a}{2}(\sinh\eta}{\frac{a}{2c}(\cosh\eta - 1)} = c\frac{\sinh\eta}{\cosh\eta - 1}.$$

[3]

Substituting the above expression for ρR^2 and this derivative to the Friedman equation for $k = -1$ we obtain

$$c^2\frac{\sinh^2\eta}{(\cosh\eta - 1)^2} = \frac{16\pi GA}{3a(\cosh\eta - 1)} + c^2.$$

[3]

Using the well known identity

$$\cosh^2\eta - \sinh^2\eta = 1, \text{ i.e. } \sinh^2\eta = \cosh^2\eta - 1 = (\cosh\eta - 1)(\cosh\eta + 1),$$

[1]

we obtain

$$c^2\left[\frac{(\cosh\eta+1)}{(\cosh\eta-1)}-1\right]=\frac{16\pi GA}{3a}\frac{1}{\cosh\eta-1}.$$

We see that the above parametric dependence is indeed a solution of the Friedman equation if

$$\frac{16\pi GA}{3a}=c^2\left[(\cosh\eta+1)-(\cosh\eta-1)\right]=2c^2,$$

hence

$$a=\frac{8\pi GA}{3c^2}=\frac{8\pi G\rho_0 R_0^3}{3c^2}.$$

[3]

From the Friedman equation with $k=-1$ taken at the present moment, we have

$$H_0^2R_0^2=\frac{8\pi G\Omega_0 3H_0^2R_0^2}{8\pi G\times 3}+c^2$$

or

$$\frac{H_0^2R_0^2}{c^2}(1-\Omega_0)=1,$$

hence

$$R_0=\frac{c}{H_0\sqrt{1-\Omega_0}}.$$

[2]

Finally

$$a=\Omega_0H_0^2c^{-2}c^3H_0^{-3}(1-\Omega_0)^{-3/2}=\frac{c\Omega_0}{H_0(1-\Omega_0)^{3/2}}.$$

[2]

(b) *At some moment of time t_* corresponding to $\eta = \eta_*$ the density of the Universe is equal to ρ_*. Show that the moment of time, t_γ, when the density of the Universe is equal to $\gamma\rho_*$, is*

$$t_\gamma \approx t_* \gamma^{-1/3}.$$

Estimate the ratio $\rho(\eta = 10)/\rho(\eta = 20)$. [Hint: Take into account that for such values of η one can approximate $\cosh\eta$ *and* $\sinh\eta$ *by $e^\eta/2 \gg \eta \gg e^{-\eta}/2$.]* [8]

Solution 6.b *[unseen]*

From the mass conservation equation we have

$$\gamma = \frac{\rho(\eta_\gamma)}{\rho(\eta_*)} = \left(\frac{R(\eta_\gamma)}{R(\eta_*)}\right)^{-3} = \left(\frac{\cosh\eta_\gamma - 1}{\cosh\eta_* - 1}\right)^{-3} \approx e^{3(\eta_* - \eta_\gamma)},$$

[2]

hence

$$\ln\gamma = 3(\eta_* - \eta_\gamma)$$

and

$$\eta_\gamma = \eta_* - \frac{1}{3}\ln\gamma.$$

[2]

Substituting this expression into

$$t = \frac{a}{2c}(\sinh\eta - \eta),$$

we have

$$t_\gamma = \frac{a}{2c}(\sinh\eta_\gamma - \eta_\gamma) \approx \frac{a}{2c} \times \frac{e^{\eta_\gamma}}{2} = \frac{a}{4c}e^{\eta_* - \frac{1}{3}\ln\gamma} = \frac{a}{4c}e^{\eta_*}\gamma^{-1/3} \approx t_*\gamma^{-1/3}.$$

[2]

Using the same approximation we can see that

$$\rho(\eta = 10)/\rho(\eta = 20) \approx e^{3(20-10)} = e^{30} \approx 10^{13}.$$

[2]

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Question 7 (a) *Give the definition of a covariant tensor of the second rank, A_{ik}, and a mixed tensor of the fourth rank, B^i_{klm}. In the local Galilean frame $x^i_{[G]}$ of reference a mixed tensor of the fourth rank, B^i_{klm}, has only one non-vanishing component, $B^0_{000[G]} = 1$, and all other components are equal to zero. Write down all components of this mixed tensor in an arbitrary frame of reference, x^i, in terms of the transformation matrices $S^l_{m[G]} = \frac{\partial x^l}{\partial x^m_{[G]}}$ and $\tilde{S}^l_{m[G]} = \frac{\partial x^l_{[G]}}{\partial x^m}$.* [10]

Solution 7.a *[seen similar]*

The covariant tensor of the second rank is the object containing $4^2 = 16$ components A_{ik} which in the course of an arbitrary transformation from one frame of reference, x'^n, to another, x^m, are transformed according to the following transformation law:

$$A_{ik} = \tilde{S}^v_i \tilde{S}^u_k A'_{vu},$$

where

$$\tilde{S}^l_m = \frac{\partial x'^l}{\partial x^m}.$$

[2]

The mixed tensor of the fourth rank with one contravariant and three covariant indices is the object containing $4^4 = 256$ components B^i_{klm} which in the course of an arbitrary transformation from one frame of reference, x'^n, to another, x^m, are transformed according to the following transformation law:

$$B^i_{klm} = S^i_p \tilde{S}^v_k \tilde{S}^u_l \tilde{S}^w_m B^p_{vuw(G)},$$

where

$$S^l_m = \frac{\partial x^l}{\partial x'^m}$$

[2]

The law of transformation for the tensor B^i_{klm} from local Galilean to an arbitrary frame of reference is

$$B^i_{klm} = S^i_{p(G)} \tilde{S}^v_{k(G)} \tilde{S}^u_{l(G)} \tilde{S}^w_{m(G)} B^p_{vuw(G)}.$$

[2]

As given

$$B^p_{vuw(G)} = \delta^p_0 \delta^0_v \delta^0_u \delta^0_w,$$

[2]

hence

$$B^i_{klm} = S^i_{p(G)} \tilde{S}^v_{k(G)} \tilde{S}^u_{l(G)} \tilde{S}^w_{m(G)} \delta^p_0 \delta^0_v \delta^0_u \delta^0_w = S^i_{0(G)} \tilde{S}^0_{k(G)} \tilde{S}^0_{l(G)} \tilde{S}^0_{m(G)}.$$

[2]

(b) *Using the EFEs and Bianchi identity (see rubric) show that the stress-energy tensor satisfies conservation law $T^i_{k:i} = 0$.* [15]

Solution 7.b *[seen similar]*

After contracting the Bianchi identity

$$R^i_{klm;n} + R^i_{knl;m} + R^i_{kmn;l} = 0$$

over indices i and n (taking summation $i = n$) we obtain

$$R^i_{klm;i} + R^i_{kil;m} + R^i_{kmi;l} = 0.$$

[1]

According to the definition of Ricci tensor

$$R^i_{kil} = R_{kl},$$

the second term can be rewritten as

$$R^i_{kil;m} = R_{kl;m}.$$

[2]

Taking into account that the Riemann tensor is antisymmetric with respect permutations of indices within the same pair

$$R^i_{kmi} = -R^i_{kim} = -R_{km},$$

the third term can be rewritten as

$$R^i_{kmi;l} = -R_{km;l}.$$

The first term can be rewritten as

$$R^i_{klm;i} = g^{ip} R_{pklm;i},$$

then taking mentioned above permutation twice we can rewrite the first term as

$$R^i_{klm;i} = g^{ip} R_{pklm;i} = -g^{ip} R_{kplm;i} = g^{ip} R_{kpml;i}.$$

After all these manipulations we have

$$g^{ip} R_{kpml;i} + R_{kl;m} - R_{km;l} = 0.$$

[2]

Then multiplying by g^{km} and taking into account that all covariant derivatives of the metric tensor are equal to zero, we have

$$g^{km} g^{ip} R_{kpml;i} + g^{km} R_{kl;m} - g^{km} R_{km;l} = \left(g^{km} g^{ip} R_{kpml}\right)_{;i} + \left(g^{km} R_{kl}\right)_{;m} - \left(g^{km} R_{km}\right)_{;l} = 0.$$

[2]

In the first term the expression in brackets can be simplified as

$$g^{km}g^{ip}R_{kpml} = g^{ip}R_{pl} = R^i_l.$$

In the second term the expression in brackets can be simplified as

$$g^{km}R_{kl} = R^m_l.$$

[2]

According to the definition of scalar curvature

$$R = g^{km}R_{km},$$

the third term can be simplified as

$$\left(g^{km}R_{km}\right)_{;l} = R_{;l} = R_{,l}.$$

[2]

Thus

$$R^i_{l;i} + R^m_{l;m} - R_{,l} = 0,$$

replacing in the second term index of summation m by i we finally obtain

$$2R^i_{l;i} - R_{,l} = 0, \quad \text{or} \quad R^i_{l;i} - \frac{1}{2}R_{,l} = 0.$$

[2]

Using the EFEs in the mixed form

$$R^i_k - \frac{1}{2}\delta^i_k R = \frac{8\pi G}{c^4}T^i_k,$$

we obtain

$$T^i_{k;i} = \frac{c^4}{8\pi G}\left(R^i_k - \frac{1}{2}\delta^i_k R\right)_{;i} = \frac{c^4}{8\pi G}\left(R^i_{k;i} - \frac{1}{2}\delta^i_k R_{,i}\right) = \frac{c^4}{8\pi G}\left(R^i_{k;i} - \frac{1}{2}R_{,k}\right) = 0.$$

[2]

Question 8 (a) *A spherical galaxy of diameter D has redshift z and apparent angular diameter $\Delta\theta$. Using the Robertson-Walker model with co-moving co-ordinate χ, find the physical distance to this galaxy and show that*

$$\Delta\theta = \frac{D\sqrt{k}(1+z)}{R_0 \sin(\sqrt{k}\chi)},$$

where R_0 is a scale factor at the present moment. [7]

Solution 8.a *[seen similar]*

From the Robertson-Walker metric the radial distance is

$$r = \int_0^{\chi_s} ds|_{dt=0,\, d\theta=0,\, d\phi=0} = R\int_0^{\chi_s} d\chi = R\chi_s.$$

Then taking into account that

$$R(z) = R_0/(1+z),$$

we obtain

$$r = R_0(1+z)^{-1}\chi_s.$$

[2]

From the Robertson-Walker metric the circumference is

$$C = \int_0^{2\pi} ds|_{dt=0,\, d\theta=0,\, d\chi=0} = R\frac{\sin(\sqrt{k}\chi)}{\sqrt{k}}\int_0^{2\pi} d\phi = 2\pi R\frac{\sin(\sqrt{k}\chi)}{\sqrt{k}}.$$

[2]

Taking into account that $\Delta\theta \propto D$ we obtain

$$\Delta\theta = 2\pi\frac{D}{C} = \frac{2\pi D}{2\pi R\frac{\sin(\sqrt{k}\chi}{\sqrt{k}}} = \frac{D\sqrt{k}}{R\sin(\sqrt{k}\chi)} = \frac{D\sqrt{k}(1+z)}{R_0\sin(\sqrt{k}\chi)}.$$

[3]

(b) *Consider radially propagating photons to determine an integral relationship between z and χ. Then assuming that equation of state parameter $\alpha = 0$ and $k = 0$, use the formula for $\Delta\theta$ from the previous sub-question to find $\Delta\theta$ as a function of z only. Show that the function $\Delta\theta(z)$ is not monotonic even for spatially flat Universe. Give very brief qualitative explanation of such effect. Find z at which this function attains its minimum. .* [18]

Solution 8.b *[50 % seen similar, 50 % unseen]*

For radially propagating photons

$$ds = 0, \;\; d\theta = 0 \;\text{ and }\; d\phi = 0.$$

From the Robertson-Walker metric we have

$$(cdt)^2 - (Rd\chi)^2 = 0 \;\text{ or }\; d\chi = \pm cdt/R.$$

[2]

Choosing sign "-" corresponding to photons propagating inward we have

$$\chi = -c\int_{t_0}^{t_z} dt/R,$$

where t_z is the moment of emission determined by z. Then

$$\chi = -c\int_{R_0}^{R_z} \frac{dR}{HR^2} = c\int_{R_z}^{R_0} \frac{dR}{HR^2}.$$

From the Friedman equation we have

$$H^2R^2 = H_0^2\Omega_0 R_0^3 R^{-1} - kc^2.$$

[3]

For the present time

$$H_0^2R_0^2 = H_0^2\Omega_0 R_0^2 - kc^2,$$

hence

$$HR = H_0R_0\sqrt{\Omega_0\frac{R_0}{R} + (1-\Omega_0)}.$$

[2]

Then taking into account again that

$$R(z) = R_0/(1+z)$$

and

$$dR/R = -dz/(1+z),$$

we have

$$\chi = \frac{c}{H_0R_0}\int_0^z \frac{dx}{(1+x)\sqrt{1+\Omega_0 x}}.$$

[2]

In the case $k = 0$ and $\Omega_0 = 1$ we have

$$\chi = \frac{c}{H_0 R_0} \int_0^z \frac{dx}{(1+x)^{3/2}} = \frac{2c}{H_0 R_0} \left(1 - (1+z)^{-1/2}\right),$$

and

$$\Delta\theta = \frac{DH_0}{4\pi c} \frac{(1+z)^{3/2}}{(1+z)^{1/2} - 1}.$$

[3]

When $z \to \infty$

$$\Delta\theta(z) \propto z \text{ and } \frac{d\Delta\theta(z)}{dz} > 0.$$

When $z \to 0$

$$\Delta\theta(z) \propto z^{-1} \text{ and } \frac{d\Delta\theta(z)}{dz} < 0.$$

This means that the function $\Delta\theta(z)$ is non-monotonic. [2]

This fact is explained in terms of gravitational focussing (lensing) effect of matter within the light beam. [1]

The minimum of $\Delta\theta(z)$ corresponds to

$$\frac{d\Delta\theta(z)}{dz} = 0,$$

$$\frac{d\Delta\theta(z)}{dz} \propto \frac{\frac{3}{2}(1+z)^{1/2}[(1+z)^{1/2} - 1] - (1+z)^{3/2}\frac{1}{2}(1+z)^{1/2}}{[(1+z)^{1/2} - 1]^2} =$$

$$= \frac{(1+z)^{1/2}[(1+z)^{1/2} - \frac{3}{2}]}{[(1+z)^{1/2} - 1]^2} = 0,$$

hence

$$(1+z)^{1/2} = \frac{3}{2} \quad 1 + z = \frac{9}{4} \quad z = \frac{9}{4} - 1 = \frac{5}{4} = 1.25.$$

[3]

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Question 9 *Consider a dust sphere of average density ρ' in a background flat Universe with $k = \Lambda = 0$. Consider the amplitude of the small density perturbation*

$$\triangle(z, M) = \sqrt{\left\langle \frac{\rho'(z, \vec{r}) - \rho(z)}{\rho(z)} \right\rangle^2},$$

where $\rho(R)$ is the average density of the Universe and $\langle\rangle$ means the average over volumes containing mass M. Assume that

$$\triangle(z, M) = \delta(z)F(M),$$

where $F(M)$ is determined by the power spectrum of primordial fluctuations.

(a) *Show that $\delta(z)$ as a function of redshift z is the solution of the following equation:*

$$\frac{d^2\delta}{dz^2} + \frac{2(1+z)d\delta}{dz} - \frac{3\delta}{2(1+z)^2} = 0.$$

[Hint: Show first that $(R' - R)/R = -\delta/3$.] [15]

Solution 9.a *[seen similar]*

One can find first an equation for $\delta(R)$. Starting from

$$\ddot{R} = -\frac{4\pi G\rho R}{3},$$

perturb R and ρ:

$$R^{'} = R(1 + h) \text{ and } \rho^{'} = \rho(1 + \delta).$$

To relate h and δ we use the conservation of energy equation

$$\rho R^3 = \rho R^3(1 + 3h)(1 + \delta), \text{ or } 1 = 1 + 3h + \delta, \textit{ hence } h = -\delta/3,$$

[2]

Then

$$R' = R(1 - \frac{\delta}{3}),$$

$$\dot{R}' = \dot{R}\frac{dR'}{dR},$$

$$\ddot{R}' = \ddot{R}\frac{dR'}{dR} + \dot{R}^2\frac{d^2R'}{dR^2},$$

[2]

Putting this in the perturbed equation

$$\ddot{R'} = -\frac{4\pi G\rho^{'} R^{'}}{3},$$

we have

$$\ddot{R}\frac{d}{dR}[R(1 - \frac{\delta}{3})] + \dot{R}^2\frac{d^2}{dR^2}[R(1 - \frac{\delta}{3})] = -\frac{4\pi G\rho R}{3}(1 + \delta)(1 - \frac{\delta}{3}).$$

[3]

Taking into account unperturbed equation

$$\ddot{R} = -\frac{4\pi G\rho R}{3}$$

and unperturbed Friedman equation

$$\dot{R}^2 = \frac{8\pi G\rho R^2}{3},$$

in first order with respect to δ [1]

we obtain

$$\frac{4\pi G\rho R}{3}\left\{-\frac{d}{dR}[R(1-\frac{\delta}{3})] + 2R\frac{d^2}{dR^2}[R(1-\frac{\delta}{3})] + 1 + \delta - \frac{\delta}{3}\right\} = 0,$$

thus

$$\delta - R\frac{d\delta}{dR} - \frac{2}{3}R^2\frac{d^2\delta}{dR^2} = 0.$$

Finally

$$\frac{d^2\delta}{dR^2} + \frac{3}{2R}\frac{d\delta}{dR} - \frac{3}{2R^2}\delta = 0.$$

[3]

Then taking into account that

$$1 + z = \frac{R_0}{R},$$

we have

$$\frac{d\delta}{dR} = \frac{dz}{dR}\frac{d\delta}{dz} = -\frac{R_0}{R^2}\frac{d\delta}{dz} = -\frac{(1+z)^2}{R_0\frac{d\delta}{dz}};$$

$$\frac{d^2\delta}{dR^2} = -\frac{(1+z)^2}{R_0\frac{d}{dz}}\left(-\frac{(1+z)^2}{R_0\frac{d\delta}{dz}}\right) = \frac{(1+z)^2}{R_0^2}\frac{d}{dz}\left[(1+z)^2\frac{d\delta}{dz}\right] = \frac{(1+z)^4}{R_0^2}\frac{d^2\delta}{dz^2} + \frac{2(1+z)^3}{R_0^2}\frac{d\delta}{dz}.$$

Substituting this into equation for $\delta(R)$we obtain

$$\frac{(1+z)^4}{R_0^2}\frac{d^2\delta}{dz^2} + \frac{2(1+z)^3}{R_0^2}\frac{d\delta}{dz} - \frac{3(1+z)^3}{2R_0^2}\frac{d\delta}{dz} - \frac{3(1+z)^2\delta}{2R_0^2} = 0,$$

finally

$$\frac{d^2\delta}{dz^2} + 21 + z\frac{d\delta}{2(1+z)dz} - \frac{3(1+z)^3\delta}{2(1+z)^2} = 0.$$

[4]

 TURN OVER

(b) *Show that the general solution of this equation can be represented in terms of two independent modes, one of which is growing, while the other is decaying. Given that $\delta(z) = 10^{-5}$ at $z = 999$ and $\delta(z) = 1$ at $z = 9$, find $\delta(z)$ at $z = 99$.* [10]

Solution 9.b *[unseen]*

Taking trial solution

$$\delta = AR^m,$$

we obtain

$$m(m-1) + \frac{3m}{2} - \frac{3}{2} = 0, \;\; 2m^2 + m - 3 = 0.$$

The solutions of this quadratic equation are

$$m_{\pm} = \frac{1}{2}\left(-\frac{1}{2} \pm \sqrt{\frac{1}{6} + 4}\right) = \frac{-1 \pm 5}{4},$$

thus $m_+ = 1$ and $m_- = -\frac{3}{2}$ (growing and decaying modes). [3]

So we have

$$\delta = A_g(R/R_0) + A_d(R/R_0)^{-3/2} = A_g(1+z)^{-1} + A_d(1+z)^{3/2}.$$

[3]

Then to find A_g and A_d one should solve the coupling equations

$$10^{-5} = A_g 10^{-3} + A_d 10^{9/2}$$
$$1 = A_g 10^{-1} + A_d 10^{3/2}$$

$$A_d = \frac{10^{-2} - 10}{10^{15/2} - 10^{5/2}} \approx -3 \times 10^{-7}$$
$$A_g = 10 - A_d 10^{5/2} \approx 10$$

Hence

$$\delta \approx 10(1+z)^{-1} - 3 \times 10^{-7}(1+z)^{3/2} \approx 0.1.$$

[4]

 END OF EXAMINATION