I. COURSE WORKS

Time table *Last updated 7.11.11*

Course work	is handed out	is due to
CW1	Lecture 2	Lecture 4
CW2	Lecture 4	Lecture 6
CW3	Lecture 6	Lecture 7
CW4	Lecture 7	Lecture 9
CW5	Lecture 9	Lecture 11
CW6	Lecture 11	two weeks after Lecture 11

A. Course work 1

Q1

a) Formulate the equivalence principle and explain what is the difference in interpretation of this principle in Newtonian theory and in General relativity.

b) Explain the similarity between an "actual" gravitational field and a non-inertial reference system. Give the definition of a locally Galilean coordinate system.

c) Explain why an "actual" gravitational field cannot be eliminated by any transformation of coordinates over all space-time.

d) Show that in a uniformly rotating system of coordinates $x^{'}$, $y^{'}$, $z^{'}$, such that

$$x = x^{'}\cos\Omega t - y^{'}\sin\Omega t, \quad y = x^{'}\sin\Omega t + y^{'}\cos\Omega t, \quad z = z^{'}, \tag{A.1}$$

the interval ds has the following form:

$$ds^2 = g_{ik}dx^i dx^k = g^{'}_{ik}dx^{'i}dx^{'k} =$$

$$= [c^2 - \Omega^2(x^{'2} + y^{'2})]dt^2 - dx^{'2} - dy^{'2} - dz^{'2} + 2\Omega y^{'}dx'dt - 2\Omega x^{'}dy'dt. \tag{A.2}$$

Q2

a) Formulate the covariance principle and explain the relationship between this principle and the principle of equivalence.

b) Give the definition of a contravariant vector in terms of the transformation of curvilinear coordinates.

c) Give the definition of a covariant vector in terms of the transformation of curvilinear coordinates.

d) What is the mixed tensor of the second rank in terms of the transformation of curvilinear coordinates (you can assume that a mixed tensor of the second rank is transformed as a product of covariant and contrvariant vectors).

e) Explain why the principle of covariance implies that all physical equations should contain only tensors.

Q3

a) Prove that the metric tensor is symmetric. Give a rigorous proof that the interval is a scalar.

b) Give the definition of the reciprocal tensors of the second rank. What is the contravariant metric tensor g^{ik}.

c) Show that in an arbitrary non-inertial frame

$$g^{ik} = S^i_{(0)0}S^k_{(0)0} - S^i_{(0)1}S^k_{(0)1} - S^i_{(0)2}S^k_{(0)2} - S^i_{(0)3}S^k_{(0)3}, \tag{A.3}$$

where $S^i_{(0)k}$ is the transformation matrix from locally inertial frame of reference (galilean frame) to this non-inertial frame.

d) Demonstrate how using the reciprocal contravariant metric tensor g^{ik} and the covariant metric tensor g_{ik} you can form contravariant tensor from covariant tensors and vice versa.

Q4

a) In the local Galilean frame $x^i_{[G]}$ of reference a mixed tensor of the second rank, C^i_k has the only one non-vanishing component, $C^0_{0[G]} = 1$, and all other components are equal to zero. Write down all components of this mixed tensor in arbitrary frame of reference. Express your result in terms of transformation matrix.

b) In the non-rotating system of Cartesian coordinates $(x,\ y,\ z)$ the only non-vanishing component of some tensor A^i_k is $A^1_1 = 1$ and all other components vanish. Using coordinate transformation from Cartesian to the uniformly rotating cylindrical coordinates $(r,\ \theta,\ \phi)$

$$x = r\cos(\theta + \Omega t), \quad y = r\sin(\theta + \Omega t), \quad z = Z, \tag{A.4}$$

show that in the latter coordinates

$$A^{'1}_0 = -\frac{r\Omega}{2c}\sin 2(\theta + \Omega t). \tag{A.5}$$

B. Course work 2

Q1

a) Motivate the necessity to introduce parallel translation for proper differentiation of tensors and explain the geometrical and physical meaning of the Christoffel symbols.

b) List all physical and geometrical arguments, you know, to demonstrate that the Christoffel symbols do not form a tensor.

Q2

a) Write down the covariant derivative of the mixed tensor of the second rank in terms of Christoffel symbols.

b) Explain why for the derivation of physical equations in the presence of a gravitational field one can simply replace partial derivatives by covariant derivatives. Take any physical equation by your own choice and write down it in the presence of gravitational field.

Q3

Show by straightforward calculations that

$$\Gamma^i_{ki} = \frac{1}{2g}\frac{\partial g}{\partial x^k} = \frac{\partial \ln\sqrt{-g}}{\partial x^k}. \tag{B.1}$$

You can use here without proof that the differential of g can be expressed as

$$dg = gg^{ik}dg_{ik} = -gg_{ik}dg^{ik}. \tag{B.2}$$

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Q4

The four-velocity and the four-momentum of a particle of mass m in a gravitational field are defined as

$$u^i = \frac{dx^i}{ds}, \quad p^i = mcu^i. \tag{B.3}$$

a) Show that $u_i u^i = 1$ and $p_i p^i = m^2c^2$.

b) Show that in a static gravitational field with metric interval $ds^2 = g_{00}(dx^0)^2 + g_{\alpha\beta}dx^\alpha dx^\beta$, the energy of the particle, $E = mc^2u_0$, is given by

$$E = \frac{mc^2\sqrt{g_{00}}}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{B.4}$$

where

$$v = \frac{c\sqrt{-g_{\alpha\beta}dx^\alpha dx^\beta}}{\sqrt{g_{00}}dx^0}. \tag{B.5}$$

C. Course work 3

Q1

Using the locally-inertial coordinate system prove that the Riemann tensor has the following symmetry properties:

a) $R_{iklm} = -R_{kilm} = -R_{ikml}$.

b) $R_{iklm} = R_{lmik}$.

c) $R_{iklm} + R_{imkl} + R_{ilmk} = 0$.

Q2

a) Show that

$$R_{ik} = \Gamma^l_{ik,l} - \Gamma^l_{il,k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}.$$

b) Using a locally-inertial coordinate system prove the Bianchi identity:

$$R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0.$$

c) Using the Bianchi identity, prove that the Ricci tensor and the scalar curvature $R = g^{ik}R_{ik}$ satisfy the following identity:

$$R^l_{m;l} = \frac{1}{2}R_m.$$

Q3

a) Using the Einstein equations in the form

$$R^i_k = \frac{8\pi G}{c^4}\left(T^i_k - \frac{1}{2}\delta^i_k T\right),$$

where G is the gravitational constant, prove that the energy-momentum tensor of matter T^i_k satisfies the conservation law $T^k_{i;k} = 0$.

b) In the limiting case of a weak gravitational field described by a Newtonian potential ϕ we can write

$$g_{00} = 1 + \frac{2\phi}{c^2}, \quad g_{0\alpha} = 0 \text{ and } g_{\alpha\beta} = -\delta_{\alpha\beta},$$

where $\alpha, \beta = 1, 2, 3$. Consider the $(0, 0)$ - component of EFEs to show that in this case

$$\triangle\phi = 4\pi G\mu,$$

where μ is the density of matter. [Hint: in the non-relativistic case $T^k_i = \mu c^2 u_i u^k$, $u^\alpha = 0$ and $u^0 = u_0 = 1$.]

D. Course work 4

Q1

Prove that the determinant of the metric tensor, $g = |g_{ik}|$, is negative in all frames of reference.

Q2

Prove the following identity:

$$d \ln \sqrt{-g} = \frac{1}{2} g^{ik} dg_{ik} = -\frac{1}{2} g_{ik} dg^{ik}.$$

Q3

Let ϕ is an arbitrary scalar field. Prove that

$$g^{ik} \phi_{;k;i} = \frac{1}{\sqrt{-g}} \left(\sqrt{-g} g^{ik} \phi_{,k} \right)_{,i}.$$

Q4

Let A_{ik} is a symmetric tensor. Prove that

$$A^{k}_{i;k} = \frac{1}{\sqrt{-g}} \left(\sqrt{-g} A^{k}_{i} \right)_{,k} - \frac{1}{2} g_{kl,i} A^{kl}.$$

E. Course work 5

Q1

a) A spherically symmetric gravitational field in vacuum is given by the Schwarzschild metric. Using this metric show that

$$\Gamma^0_{10} = -\Gamma^1_{11} = \frac{r_g}{2r^2}\left(1 - \frac{r_g}{r}\right)^{-1},$$

and

$$\Gamma^1_{00} = \frac{r_g}{2r^2}\left(1 - \frac{r_g}{r}\right).$$

b) Show that the time component of the geodesic equation has the following form:

$$\frac{d^2t}{d\tau^2} + \frac{r_g}{r^2}\left(1 - \frac{r_g}{r}\right)^{-1}\frac{dt}{d\tau}\frac{dr}{d\tau} = 0,$$

where τ is proper time ($ds = cd\tau$). Solve this equation for a particle which is falling radially towards a black hole. Show that

$$\frac{dt}{d\tau}\left(1 - \frac{r_g}{r}\right) = 1.$$

c) Show that when the particle in b) has zero velocity at infinity, then

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{c^2 r_g}{r}.$$

Find τ and t as functions of r and sketch them on the same graph. Explain why $t \to \infty$ when $r \to r_g$.

Q2

A rotating black hole is described by the Kerr metric.

$$ds^2 = (1 - \frac{r_g r}{\rho^2})dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta)\sin^2\theta d\phi^2 +$$

$$\frac{2r_g r a}{\rho^2}\sin^2\theta d\phi dt,$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - r_g r + a^2$, and $a = \frac{J}{mc}$, where J is the angular momentum of the black hole.

a) Explain why the location of the event horizon, r_{hor}, is given by $g_{rr} = \infty$ and show that

$$r_{hor} = \frac{r_g}{2} + \sqrt{(\frac{r_g}{2})^2 - a^2}.$$

b) Compare this expression with the expression for the radius of the event horizon in the case of a non-rotating black hole.

Q3

a) What is the limit of stationarity and what is the ergosphere?
b) Explain why the location of the sphere corresponding to the limit of stationarity, r_{ls}, is determined by $g_{tt} = 0$, and show then that

$$r_{ls} = \frac{r_g}{2} + \sqrt{(\frac{r_g}{2})^2 - a^2 \cos^2 \theta}.$$

c) Sketch the rotating black hole as projected on i) the equatorial plane, $\theta = \pi/2$, and ii) the perpendicular plane, $\phi = 0$ (indicate the event horizon, the limit of stationarity and the ergosphere).

Q4

a) Explain qualitatively why it is possible to extract the energy of a rotating black hole despite the fact that no signal can escape outside from within the black hole horizon.
b) Show that the circle $r = r_{hor}$ and $\theta = \pi/2$, is the world line of a photon moving around the rotating black hole with angular velocity

$$\Omega_{hor} = \frac{a}{r_g r_{hor}}.$$

(Hint: put $dr = 0$, $d\theta = 0$ and $d\phi = \Omega_{hor} dt$ into ds and show that $ds = 0$.)

F. Course work 6

Q1.

a) A binary system consists of two neutron stars of the same mass M. The orbital period of the system is P. Using Newtonian mechanics, estimate to an order of magnitude the separation between the neutron stars, r, and the fractional relativistic corrections to the orbital motion.

b) Evaluate the relativistic corrections if $P = 8$ min and $M = 1.5M_{\odot}$. Compare your estimate with relativistic effects in the solar system. It is known that the perihelion shift of Mercury is 43" per century. What analogous shift can you expect in the case of the binary system of neutron stars? (Hint: The relativistic shift per one orbital period is of order r_g/r, where r_g is gravitational radius of the neutron star.)

Q2.

The quadrupole formula for the metric perturbation associated with gravitational waves is given by

$$h_{\alpha\beta} = -\frac{2G}{3c^4R}\frac{d^2D_{\alpha\beta}}{dt^2}(t - R/c),$$

where R is the distance to the source of the gravitational waves and

$$D_{\alpha\beta} = \int (3x_\alpha x_\beta - r^2\delta_{\alpha\beta})dM$$

is the quadrupole tensor of the source. Consider a mass m moving along circular orbit around the black hole of mass M, assuming that $m \ll M$.

a) Show that all the amplitudes $h_{\alpha\beta}$ of gravitational wave, emitted by such system, are periodic functions of time with $\omega = 2\omega_0$, where $\omega_0 = 2\pi/T$, and T is the orbital period.

b) Show that, to an order of magnitude (omitting the indices α and β)

$$h \approx \frac{r_g}{R}\left(\frac{R_g\omega}{c}\right)^{2/3},$$

where r_g is the gravitational radius of the mass m and R_g is the gravitational radius of the black hole.

Q3.

The future LISA mission will be able to detect gravitational waves with $h > 10^{-23}$, if $10^{-4}Hz < \omega < 3 \cdot 10^{-3}Hz$. From what distance will it be possible to detect gravitational radiation from the binary system, containing the black hole of mass $m = 3M_{\odot}$, moving along a circular orbit with radius $r = 10^4R_g$ around the massive black hole of mass $M = 10^3M_{\odot}$?

II. SOLUTIONS TO COURSE WORKS

A. Solutions to course work 1

Q1

a) Formulate the equivalence principle and explain what is the difference in interpretation of this principle in Newtonian theory and in General relativity.
Solution: The equivalence principle states that a uniform gravitational field is equivalent, i.e. is not distinguishable from a uniform acceleration.
It is explained within frame of newtonian theory just by the following "coincidence": inertial mass m_{in} is equal to gravitational mass m_{gr}.
The GR gives a very simple and natural explanation of the Principle of Equivalence: in curved space-time all bodies move along geodesics, that is why their world lines are the same in given gravitational field.

b) Explain the similarity between an "actual" gravitational field and a non-inertial reference system. Give the definition of a locally Galilean coordinate system.
Solution: The similarity is that for a given gravitational and at a given event one can always choose such a non-inertial frame that the motion of particles or propagation of light with respect to this non-inertial frame in the vicinity of that event (i.e. **locally**) is the same as in the given "actual" gravitational field.
The local galilean frame of reference (the freely falling frame of reference) is a such frame in which

$$g_{ik} \to \eta_{ik} \equiv \mathrm{diag}(1, -1, -1, -1). \tag{A.1}$$

c) *Explain why an "actual" gravitational field cannot be eliminated by any transformation of coordinates over all space-time.*
Solution: In general "actual" gravitational field is non-uniform and one should produce different transformations to the local galilean frame of reference at different events (points) of space-time.

d) *Show that in a uniformly rotating system of coordinates $x^{'}$, $y^{'}$, $z^{'}$, such that*

$$x = x^{'} \cos \Omega t - y^{'} \sin \Omega t, \quad y = x^{'} \sin \Omega t + y^{'} \cos \Omega t, \quad z = z^{'},$$

the interval ds has the following form:

$$ds^2 = g_{ik} dx^i dx^k = g^{'}_{ik} dx^{'i} dx^{'k} =$$
$$= [c^2 - \Omega^2 (x^{'2} + y^{'2})] dt^2 - dx^{'2} - dy^{'2} - dz^{'2} + 2\Omega y^{'} dx' dt - 2\Omega x^{'} dy' dt.$$

Solution: The transformation should be back rotation around $z^{'}$-axis:

$$t^{'} = t, \quad z^{'} = z, \quad x^{'} = x \cos \phi + y \sin \phi, \quad y^{'} = x sin\phi - y \cos \phi, \tag{A.2}$$

where $\phi = -\Omega t$.
Then

$$x^{'2} + y^{'2} = (x \cos \phi + y \sin \phi)^2 + (x \sin \phi - y \cos \phi)^2 = x^2 + y^2, \tag{A.3}$$

and

$$dt^{'} = dt, \; dz^{'} = dz, \quad dx^{'} = dx \cos \phi + dy \sin \phi + (x \sin \phi - y \cos \phi) \Omega dt,$$

$$dy^{'} = dx \sin \phi - dy \cos \phi - (x \cos \phi + y \sin \phi) \Omega dt. \tag{A.4}$$

Then

$$dx^{'2} + dy^{'2} = (dx \cos \phi + dy \sin \phi) - (x \sin \phi + y \cos \phi)(\Omega dt)^2 + (-dx \sin \phi - dy \cos \phi - (x \cos \phi - y \sin \phi) \Omega dt)^2 =$$

$$= dx^2 + dy^2 + (x^2 + y^2)(\Omega dt)^2 - 2\Omega dt (y dx - x dy), \tag{A.5}$$

$$y^{'}dx^{'} - x^{'}dy^{'} = (-x\sin\phi - y\cos\phi)(dx\cos\phi - dy\sin\phi - (x\sin\phi + y\cos\phi)\Omega dt) -$$

$$-(x\cos\phi - y\sin\phi)(-dx\sin\phi - dy\cos\phi - (x\cos\phi - y\sin\phi)\Omega dt) =$$

$$= (x^2 + y^2)\Omega dt - (ydx - xdy). \tag{A.6}$$

Thus

$$ds^2 = [c^2 - \Omega^2(x^2 + y^2)]dt^2 - dx^2 - dy^2 - dz^2 - \Omega^2(x^2 + y^2)dt^2 +$$

$$+2\Omega dt(ydx - xdy) + 2\Omega^2(x^2 + y^2)]dt^2 + 2\Omega dt(xdy - ydx) =$$

$$= c^2dt^2 - dx^2 - dy^2 - dz^2. \tag{A.7}$$

Q2

a) Formulate the covariance principle and explain the relationship between this principle and the principle of equivalence.
Solution: The Principle of Covariance which says: the shape of all physical equations should be the same in an arbitrary frame of reference.
The Principle of Covariance is more general and more mathematical version of the Principle of Equivalence.
b) Give the definition of a contravariant vector in terms of the transformation of curvilinear coordinates.
Solution: The Contravariant four-vector is the combination of four quantities (components) A^i, which are transformed like differentials of coordinates:

$$A^i = \frac{\partial x^i}{\partial x'^k}A'^k = S^i_k A'^k. \tag{A.8}$$

c) Give the definition of a covariant vector in terms of the transformation of curvilinear coordinates.
Solution: The Covariant four-vector is the combination of four quantities (components) A_i, which are transformed like components of the gradient of a scalar field:

$$A_i = \frac{\partial x'^k}{\partial x^i}A'_k = \tilde{S}^k_i A'_k. \tag{A.9}$$

d) What is the mixed tensor of the second rank in terms of the transformation of curvilinear coordinates (you can assume that a mixed tensor of the second rank is transformed as a product of covariant and contrvariant vectors).
Solution: Mixed tensor of the 2 rank has $4^2 = 16$ components and 2 indices, 1 contravariant and 1 covariant. Corresponding transformation law is

$$A^i_k = S^i_n \tilde{S}^m_k A'^n_m, \tag{A.10}$$

we see 2 transformation matrices in the transformation law. It really looks like transformation of the product of two vectors

$$B^i C_k = \left(S^i_n B'^n\right)\left(\tilde{S}^m_k C'_m\right) = S^i_n \tilde{S}^m_k B'^n C'_m. \tag{A.11}$$

e) Explain why the principle of covariance implies that all physical equations should contain only tensors.
Solution: The Principle of Covariance predetermines the mathematical structure of General Relativity: all equations should contain tensors only, because tensors are objects which are transformed properly in the course of coordinate transformations from one frame of reference to another keeping the shape of any physical equation being unchanged.

Q3

a) Prove that the metric tensor is symmetric. Give a rigorous proof that the interval is a scalar.
Solution:

$$ds^2 = g_{ik}dx^i dx^k = \frac{1}{2}(g_{ik}dx^i dx^k + g_{ik}dx^i dx^k) = \frac{1}{2}(g_{ki}dx^k dx^i + g_{ik}dx^i dx^k) = \frac{1}{2}(g_{ki} + g_{ik})dx^i dx^k =$$

$$= \tilde{g}_{ik}dx^i dx^k, \tag{A.12}$$

where

$$\tilde{g}_{ik} = \frac{1}{2}(g_{ki} + g_{ik}), \tag{A.13}$$

which is obviously a symmetric one. Then we just drop " ˜ ".

$$ds^2 = g_{ik}dx^i dx^k = (\tilde{S}_i^n \tilde{S}_k^m g'_{nm})(S_p^i dx'^p)(S_w^k dx'^w) = (\tilde{S}_i^n S_p^i)(\tilde{S}_k^m S_w^k)(g'_{nm}dx'^p dx'^w) =$$

$$= \delta_p^n \delta_w^m (g'_{nm}dx'^p dx'^w) = g'_{pw}dx'^p dx'^w = g'_{ik}dx'^i dx'^k = ds'^2, \tag{A.14}$$

hence $ds = ds'$ which means that ds is a scalar.
b) Give the definition of the reciprocal tensors of the second rank. What is the contravariant metric tensor g^{ik}.
Solution: Two tensors A_{ik} and B^{ik} are called reciprocal to each other if

$$A_{ik}B^{kl} = \delta_i^l. \tag{A.15}$$

A contravariant metric tensor g^{ik} is reciprocal to the covariant metric tensor g_{ik}:

$$g_{ik}g^{kl} = \delta_i^l. \tag{A.16}$$

c) Show that in an arbitrary non-inertial frame

$$g^{ik} = S^i_{(0)0}S^k_{(0)0} - S^i_{(0)1}S^k_{(0)1} - S^i_{(0)2}S^k_{(0)2} - S^i_{(0)3}S^k_{(0)3},$$

where $S^i_{(0)k}$ is the transformation matrix from locally inertial frame of reference (galilean frame) to this non-inertial frame.
Solution: Let $S^i_{(0)k}$ to be the transformation matrix from locally inertial frame of reference (galilean frame) to an arbitrary non-inertial frame, let us denote it as . In the galilean frame of reference

$$g^{ik} = \eta^{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \eta^{ik} \equiv \mathrm{diag}(1, -1, -1, -1), \tag{A.17}$$

hence

$$g^{ik} = S^i_{(0)n}S^k_{(0)m}\eta^{lm} = S^i_{(0)0}S^k_{(0)0} - S^i_{(0)1}S^k_{(0)1} - S^i_{(0)2}S^k_{(0)2} - S^i_{(0)3}S^k_{(0)3}. \tag{A.18}$$

d) Demonstrate how using the reciprocal contravariant metric tensor g^{ik} and the covariant metric tensor g_{ik} you can form contravariant tensor from covariant tensors and vice versa.
Solution: With the help of the metric tensor and its reciprocal we can form contravariant tensors from covariant tensors and vice versa, for example:

$$A^i = g^{ik}A_k, \quad A_i = g_{ik}A^k, \tag{A.19}$$

in other words we can rise and descend indices as we like, like a kind of juggling with indices. We can say that contravariant, covariant and mixed tensors can be considered as different representations of the same geometrical object.

Q4

a) In the local Galilean frame $x^i_{[G]}$ of reference a mixed tensor of the second rank, C^i_k has the only one non-vanishing component, $C^0_{0[G]} = 1$, and all other components are equal to zero. Write down all components of this mixed tensor in arbitrary frame of reference. Express your result in terms of transformation matrix.

Solution:

$$C^i_k = S^i_{(0)n}\tilde{S}^m_{(0)k}C^n_{m[G]} = S^i_{(0)n}\tilde{S}^m_{(0)k}\delta^n_0\delta^0_m = S^i_{(0)0}\tilde{S}^0_{(0)k}. \tag{A.20}$$

b) In the non-rotating system of Cartesian coordinates $(x,\ y,\ z)$ the only non-vanishing component of some tensor A^i_k is $A^1_1 = 1$ and all other components vanish. Using coordinate transformation from Cartesian to the uniformly rotating cylindrical coordinates $(r,\ \theta,\ \phi)$

$$x = r\cos(\theta + \Omega t), \;\; y = r\sin(\theta + \Omega t), \;\; z = Z,$$

show that in the latter coordinates

$$A'^1_0 = -\frac{r\Omega}{2c}\sin 2(\theta + \Omega t).$$

Solution:

$$A^i_k = S^i_n\tilde{S}^m_k A'^n_m \tag{A.21}$$

Multiplying both sides of (A.22) by $S^k_0\tilde{S}^1_i$ one obtains

$$S^k_0\tilde{S}^1_i A^i_k = S^i_n\tilde{S}^1_i S^k_0\tilde{S}^m_k A'^n_m, \tag{A.22}$$

hence

$$S^k_0\tilde{S}^1_i\delta^i_1\delta^1_k = \delta^1_n\delta^m_0 A'^n_m = A'^1_0. \tag{A.23}$$

Thus

$$A'^1_0 = S^1_0\tilde{S}^1_1 = \frac{\partial x^1}{\partial x'^0}\frac{\partial x'^1}{\partial x^1} = \frac{\partial(r\cos(\theta + \Omega t))}{c\partial t}\frac{\partial r}{\partial x}. \tag{A.24}$$

Taking into account that

$$r^2 = x^2 + y^2 \;\text{ and }\; 2r\frac{\partial r}{\partial x} = 2x, \tag{A.25}$$

one finally obtains

$$A'^1_0 = [-\frac{r\Omega}{c}\sin(\theta + \Omega t)][\cos(\theta + \Omega t)] = -\frac{r\Omega}{2c}\sin 2(\theta + \Omega t). \tag{A.26}$$

B. Solutions to course work 2

Q1.

a) Motivate the necessity to introduce parallel translation for proper differentiation of tensors and explain the geometrical and physical meaning of the Christoffel symbols.

Solution: Differentiating, say, vectors we take difference between components of this vector at infinitesimally close but nevertheless different points. At different points in general case matrices of transformation are different and the differential of vector (tensor) is not a vector (tensor). In order to make the differential of a tensor be also a tensor, we should take difference of two objects at the same point. To do this we need produce a parallel translation.

The Christoffel symbols from geometrical point of view appear as coefficients in corrections to components of tensors due to the parallel translations in curved space-time (in flat space-time such corrections obviously are equal to zero). From physical point of view the Christoffel symbols represent non-inertial "forces" in non-inertial frames of reference or gravitational "forces".

b) List all physical and geometrical arguments, you know, to demonstrate that the Christoffel symbols do not form a tensor.

Solution: 1) Corrections for the parallel translations are added to not tensors to obtain tensors, hence the are not tensors (geometrical argument);

2) Gravitational field can be eliminated locally, which means that in locally inertial frame of reference the Christoffel symbols are equal to zero, while according to the definition of tensors, the tensors vanishing in one frame of reference should vanish in any other frame of reference, hence the Christoffel symbols do not form a tensor (physical argument).

Q2

a) Write down the covariant derivative of the mixed tensor of the second rank in terms of Christoffel symbols.

Solution:

$$A^i_{k;n} = A^i_{k,n} + \Gamma^i_{nm} A^m_k - \Gamma^m_{kn} A^i_m. \tag{B.1}$$

b) Explain why for the derivation of physical equations in the presence of a gravitational field one can simply replace partial derivatives by covariant derivatives. Take any physical equation by your own choice and write down it in the presence of gravitational field.

Solution: According to the principle of covariance all physical equation should have the same shape in arbitrary frames of reference, but in locally inertial frames where the Christoffel symbols are equal to zero partial derivatives are identical to covariant derivatives.

For example the energy-momentum conservation without gravity (in flat space time) and in inertial frame of reference $T^i_{k,i}$ should be replaced by $T^i_{k;i}$.

Q3

Show by straightforward calculations that

$$\Gamma^i_{ki} = \frac{1}{2g}\frac{\partial g}{\partial x^k} = \frac{\partial \ln\sqrt{-g}}{\partial x^k}.$$

You can use here without proof that the differential of g can be expressed as

$$dg = g g^{ik} dg_{ik} = -g g_{ik} dg^{ik}.$$

Solution:

$$\Gamma^i_{ki} = \frac{1}{2} g^{in}(g_{kn,i} + g_{in,k} - g_{ki;n}). \tag{B.2}$$

Replacing indices of summation in the first term in the brackets, $i \to n$ and $n \to i$ we obtain

$$\frac{1}{2}(g^{ni} g_{ki,n} + g^{in} g_{in,k} - g^{in} g_{ki;n}) = \frac{1}{2} g^{in} g_{in,k} = \frac{g-,k}{2g} = \frac{1}{2}(\ln|g|)_{,k}. \tag{B.3}$$

Taking into account that $g < 0$, i.e. $|g| = -g$, we finally obtain

$$\Gamma^i_{ki} = \frac{1}{2g}\frac{\partial g}{\partial x^k} = \frac{\partial \ln\sqrt{-g}}{\partial x^k}. \tag{B.4}$$

Q4

The four-velocity and the four-momentum of a particle of mass m in a gravitational field are defined as

$$u^i = \frac{dx^i}{ds}, \ \ p^i = mcu^i.$$

Solution:

a) Show that $u_iu^i = 1$ and $p_ip^i = m^2c^2$.

Solution:

$$ds^2 = g_{ik}dx^idx^k, \tag{B.5}$$

hence

$$1 = \frac{ds^2}{ds^2} = g_{ik}\frac{dx^i}{ds}\frac{dx^k}{ds} = g_{ik}u^iu^k = u_iu^i. \tag{B.6}$$

$$p_ip^i = (mcu_i)(mcu^i) = m^2c^2u_iu^i = m^2c^2. \tag{B.7}$$

b) Show that in a static gravitational field with metric interval $ds^2 = g_{00}(dx^0)^2 + g_{\alpha\beta}dx^\alpha dx^\beta$, the energy of the particle, $E = mc^2u_0$, is given by

$$E = \frac{mc^2\sqrt{g_{00}}}{\sqrt{1-\frac{v^2}{c^2}}},$$

where

$$v = \frac{c\sqrt{-g_{\alpha\beta}dx^\alpha dx^\beta}}{\sqrt{g_{00}}dx^0}.$$

Solution:

$$1 = g_{00}(\frac{dx^0}{ds})^2 + g_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = g_{00}(u^0)^2 + (\frac{dx^0}{ds})^2 - \frac{v^2g_00(dx^0)^2}{c^2ds^2} = g_{00}(u^0)^2(1-\frac{v^2}{c^2}), \tag{B.8}$$

hence

$$u^0 = \frac{1}{\sqrt{g_{00}(1-\frac{v^2}{c^2}}}. \tag{B.9}$$

Finally

$$E = mc^2u_0 = mc^2g_{00}u^0 = \frac{mc^2\sqrt{g_{00}}}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{B.10}$$

C. Solutions to course work 3

Q1

Using the locally-inertial coordinate system prove that the Riemann tensor has the following symmetry properties:

a) $R_{iklm} = -R_{kilm} = -R_{ikml}$.

b) $R_{iklm} = R_{lmik}$.

c) $R_{iklm} + R_{imkl} + R_{ilmk} = 0$.

Solution: In the local galilean frame of reference

$$R_{iklm} = \eta_{in} R^n_{klm} = \eta_{in} \left(\Gamma^n_{km,l} - \Gamma^n_{kl,m} \right) =$$

$$= \frac{1}{2} \eta_{in} \left[g^{np} (g_{kp,m} + g_{mp,k} - g_{km,p} \right]_{,l} - \frac{1}{2} \eta_{in} \left[g^{np} (g_{kp,l} + g_{lp,k} - g_{kl,p} \right]_{,m} =$$

$$= \frac{1}{2} \eta_{in} \eta^{np} \left(g_{kp,m,l} + g_{mp,k,l} - g_{km,p,l} - g_{kp,l,m} - g_{lp,k,m} + g_{kl,p,m} \right) =$$

$$= \frac{1}{2} \delta^p_i \left(g_{mp,k,l} - g_{km,p,l} - g_{lp,k,m} + g_{kl,p,m} \right) = \frac{1}{2} \left(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l} \right).$$

a)

$$R_{kilm} = \frac{1}{2} \left(g_{km,i,l} + g_{il,k,m} - g_{kl,i,m} - g_{im,k,l} \right) = -\frac{1}{2} \left(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l} \right) = -R_{iklm}.$$

$$R_{ikml} = \frac{1}{2} \left(g_{il,k,m} + g_{km,i,l} - g_{im,k,l} - g_{kl,i,m} \right) = -\frac{1}{2} \left(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l} \right) = -R_{iklm}.$$

b)

$$R_{lmik} = \frac{1}{2} \left(g_{lk,m,i} + g_{mi,l,k} - g_{li,m,k} - g_{mk,l,i} \right) = \frac{1}{2} \left(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l} \right) = R_{iklm}.$$

c)

$$R_{iklm} + R_{imkl} + R_{ilmk} = \eta_{in} \left(\Gamma^n_{km,l} - \Gamma^n_{kl,m} + \Gamma^n_{ml,k} - \Gamma^n_{mk,l} + \Gamma^n_{lk,m} - \Gamma^n_{lm,k} \right) =$$

$$\eta_{in} \left(\Gamma^n_{km,l} - \Gamma^n_{mk,l} - \Gamma^n_{kl,m} + \Gamma^n_{lk,m} + \Gamma^n_{ml,k} - \Gamma^n_{lm,k} \right) =$$

$$= \eta_{in} \left(\Gamma^n_{km,l} - \Gamma^n_{km,l} - \Gamma^n_{kl,m} + \Gamma^n_{kl,m} + \Gamma^n_{ml,k} - \Gamma^n_{ml,k} \right) = 0.$$

Q2

a) Show that

$$R_{ik} = \Gamma^l_{ik,l} - \Gamma^l_{il,k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}.$$

b) Using a locally-inertial coordinate system prove the Bianchi identity:

$$R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0.$$

c) Using the Bianchi identity, prove that the Ricci tensor and the scalar curvature $R = g^{ik}R_{ik}$ satisfy the following identity:

$$R^l_{m;l} = \frac{1}{2}R_m.$$

Solution:a) Starting from

$$R^i_{klm} = \Gamma^i_{km,l} - \Gamma^i_{kl,m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl},$$

(which is obtained by straightforward calculation of $A_{k;l;m} - A_{k;m;l}$), and using definition of the Ricci tensor $R_{ik} = R^l_{ilk}$, one obtains

$$R_{ik} = R^n_{ink} = \Gamma^n_{ik,n} - \Gamma^n_{in,k} + \Gamma^n_{pn}\Gamma^p_{ik} - \Gamma^n_{pk}\Gamma^p_{in} =$$

$$= \Gamma^l_{ik,l} - \Gamma^l_{il,k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}.$$

b) The Bianchi identity, $R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0$, in the local galilean frame of reference, where all Christoffel symbols are equal to zero, can be re-written as

$$R^n_{ikl,m} + R^n_{imk,l} + R^n_{ilm,k} = 0$$

and the Riemann tensor in this frame can be written as

$$R^i_{klm} = \Gamma^i_{km,l} - \Gamma^i_{kl,m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl} = \Gamma^i_{km,l} - \Gamma^i_{kl,m},$$

$$R^n_{ikl,m} + R^n_{imk,l} + R^n_{ilm,k} = \left(\Gamma^n_{il,k} - \Gamma^n_{ik,l}\right)_{,m} + \left(\Gamma^n_{im,l} - \Gamma^n_{il,m}\right)_{,k} + \left(\Gamma^n_{ik,m} - \Gamma^n_{im,k}\right)_{,l} =$$

$$= \Gamma^n_{il,k,m} - \Gamma^n_{ik,l,m} + \Gamma^n_{im,l,k} - \Gamma^n_{il,m,k} + \Gamma^n_{ik,m,l} - \Gamma^n_{im,k,l} =$$

$$= [\Gamma^n_{il,k,m} - \Gamma^n_{il,m,k}] + [\Gamma^n_{ik,m,l} - \Gamma^n_{ik,l,m}] + [\Gamma^n_{im,l,k} - \Gamma^n_{im,k,l}] = [0] + [0] + [0] = 0.$$

c) After contracting the Bianchi identity we obtain

$$g^{kl}R^i_{klm;i} + g^{kl}R^i_{kil;m} + g^{kl}R^i_{kmi;l} = g^{in}g^{kl}\left(R_{nklm;i} + R_{nkil;m} + R_{nkmi;l}\right) =$$

$$= g^{kl}g^{in}\left(-R_{knlm;i} + R_{nkil;m} - R_{nkim;l}\right) = -g^{in}R_{nm;i} + g^{kl}R_{kl;m} - g^{kl}R_{km;l} = -R^i_{m;i} + R_{,m} - R^l_{m;l} =$$

$$= -2R^i_{m;i} + R_{,m} = 0.$$

Q3

a) Using the Einstein equations in the form

$$R^i_k = \frac{8\pi G}{c^4}\left(T^i_k - \frac{1}{2}\delta^i_k T\right),$$

where G is the gravitational constant, prove that the energy-momentum tensor of matter T^i_k satisfies the conservation law $T^k_{i;k} = 0$.
b) In the limiting case of a weak gravitational field described by a Newtonian potential ϕ we can write

$$g_{00} = 1 + \frac{2\phi}{c^2}, \quad g_{0\alpha} = 0 \quad and \quad g_{\alpha\beta} = -\delta_{\alpha\beta},$$

where $\alpha, \beta = 1, 2, 3$. Consider the $(0, 0)$ - component of EFEs to show that in this case

$$\triangle\phi = 4\pi G\mu,$$

where μ is the density of matter. [Hint: in the non-relativistic case $T^k_i = \mu c^2 u_i u^k$, $u^\alpha = 0$ and $u^0 = u_0 = 1$.]
Solution:a) Contracting EFEs in given form, one obtains

$$R = R^i_i = \frac{8\pi G}{c^4}\left(T^i_i - \frac{1}{2}\delta^i_i T\right) = \frac{8\pi G}{c^4}\left(T - \frac{4}{2}T\right) = -\frac{8\pi G}{c^4}T.$$

Then taking covariant divergence of both sides of EFEs, one obtains

$$R^i_{k;i} = \frac{8\pi G}{c^4}\left(T^i_{k;i} - \frac{1}{2}\delta^i_k T_{,i}\right) = \frac{8\pi G}{c^4}T^i_{k;i} - \frac{4\pi G}{c^4}T_{,k} = \frac{8\pi G}{c^4}T^i_{k;i} + \frac{1}{2}R_{,k},$$

hence

$$T^i_{k;i} = \frac{c^4}{8\pi G}(R^i_{k;i} - \frac{1}{2}R_{,k}) = 0.$$

b) In small velocity approximation

$$T^k_i \approx \rho c^2 u_i u^k, \tag{C.1}$$

where ρ is the mass density, i.e., $T^0_0 = \rho c^2$ and all other components are small, i.e., $|T^0_\alpha| \ll T^0_0$ and $|T^\beta_\alpha| \ll T^0_0$. This means that $T \equiv T^i_i \approx T^0_0$.
In weak field approximation one can neglect by the non-linear part in the Ricci tensor:

$$R_{00} = R^0_0 \approx \Gamma^\alpha_{00,\alpha} = -\frac{1}{2}\eta^{\alpha\beta}g_{00,\alpha,\beta} = \frac{1}{c^2}\phi_{,\alpha,\beta}, \tag{C.2}$$

where ϕ is defined by

$$g_{00} = 1 - \frac{2\phi}{c^2}. \tag{C.3}$$

Following usual notations

$$\eta^{\alpha\beta}g_{00,\alpha,\beta} = \triangle g_{00}, \tag{C.4}$$

where $\triangle$ is the Laplace operator. From EFEs we obtain

$$R^0_0 = \frac{1}{c^2}\triangle\phi = \frac{8\pi G}{c^4}(T^0_0 - \frac{1}{2}T) \approx \frac{8\pi G}{c^4}(T^0_0 - \frac{1}{2}T^0_0) = \frac{4\pi G}{c^4}T^0_0. \tag{C.5}$$

Hence,

$$\triangle\phi = 4\pi G\rho. \tag{C.6}$$

This is the Poisson equation, hence, as one can see, in this approximation EFEs give the Newtonian gravity and ϕ is the Newtonian gravitational potential.

D. Solutions to course work 4

Q1

Prove that the determinant of the metric tensor, $g = |g_{ik}|$, is negative in all frames of reference.
Solution: Taking into account that g_{ik} and g^{ik} are reciprocal, one obtains that $g \equiv det(g_{ik}) = 1/det(g^{ik})$.
We know that

$$g^{ik} = S^i_{(0)n} S^k_{(0)m} \eta^{lm}.$$

Obviously, $det(\eta^{lm} = -1$, hence

$$det(g^{ik}) = det S^i_{(0)n} \times det S^k_{(0)m} \times det(\eta^{lm}) = -S^2,$$

where S is the determinant of the transformation matrix. One can see that $g = -S^{-2} < 0$ in all frames of reference.

Q2

Prove the following identity:

$$d \ln \sqrt{-g} = \frac{1}{2} g^{ik} dg_{ik} = -\frac{1}{2} g_{ik} dg^{ik}.$$

Solution: The determinant g depends on all components g_{ik}. Calculating g with the help, say the first raw, one can write

$$g = M^{1i} g_{1i},$$

where M^{1i} are minors of the components in the first row. Obviously M^{1i} do not contain g_{1i}. Hence

$$\frac{\partial g}{\partial g_{1i}} = M^{1i}.$$

This is true for any row in determinant, thus

$$\frac{\partial g}{\partial g_{ni}} = M^{ni}.$$

Taking into account that g^{ik} is inverse matrix of g_{ik}, one can write $g^{ik} = M^{ik}/g$, i.e. $M^{ik} = gg^{ik}$. Thus

$$dg = \frac{\partial g}{\partial g_{ik}} dg_{ik} = M^{ik} dg_{ik} = gg^{ik} dg_{ik},$$

hence

$$g^{ik} dg_{ik} = \frac{dg}{g} = d \ln |g| = d \ln(-g) = 2 \ln \sqrt{-g}.$$

Then

$$g^{ik} dg_{ik} = d(g^{ik} g_{ik}) - g_{ik} dg^{ik} = d\delta^i_i - g_{ik} dg^{ik} = -g_{ik} dg^{ik}.$$

Finally

$$d \ln \sqrt{-g} = \frac{1}{2} g^{ik} dg_{ik} = -\frac{1}{2} g_{ik} dg^{ik}.$$

Q3

Let ϕ is an arbitrary scalar field. Prove that

$$g^{ik}\phi_{;k;i} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}g^{ik}\phi_{,k}\right)_{,i}.$$

Solution: Let us introduce the following covariant vector: $A_i = \phi_{,i} = \phi_{;i}$. After that

$$g^{ik}\phi_{;k;i} = g^{ik}A_{k;i} = \left(g^{ik}A_k\right)_{;i} = A^i_{;i} = A^i_{,i} + \Gamma^i_{in}A^n = A^i_{,i} + \frac{1}{2}g^{im}\left(g_{im,n} + g_{nm,i} - g_{in,m}\right)A^n =$$

$$= A^i_{,i} + \frac{1}{2}g^{im}g_{im,n}A^n + \frac{1}{2}g^{im}\left(g_{nm,i} - g_{in,m}\right)A^n.$$

Using the answer to Q2, one has

$$\frac{1}{2}g^{im}g_{im,n} = \left(\ln\sqrt{-g}\right)_{,n} = \frac{(\sqrt{-g})_{,n}}{\sqrt{-g}}.$$

And taking into account that

$$g^{im}\left(g_{nm,i} - g_{in,m}\right) = g^{im}g_{nm,i} - g^{im}g_{in,m} = g^{mi}g_{ni,m} - g^{im}g_{in,m} = g^{im}g_{in,m} - g^{im}g_{in,m} = 0,$$

we obtain

$$g^{ik}\phi_{;k;i} = A^i_{,i} + \frac{(\sqrt{-g})_{,n}}{\sqrt{-g}}A^n == \frac{1}{\sqrt{-g}}\left(\sqrt{-g}A^n\right)_{,n} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}g^{ni}A_i\right)_{,n} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}g^{ni}\phi_{,i}\right)_{,n} =$$

$$= \frac{1}{\sqrt{-g}}\left(\sqrt{-g}g^{ik}\phi_{,k}\right)_{,i}.$$

Q4

Let A_{ik} is a symmetric tensor. Prove that

$$A^k_{i;k} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}A^k_i\right)_{,k} - \frac{1}{2}g_{kl,i}A^{kl}.$$

Solution:

$$A^k_{i;k} = A^k_{i,k} + \Gamma^k_{km}A^m_i - \Gamma^m_{ik}A^k_m = A^k_{i,k} + \frac{1}{2}g^{kn}\left(g_{kn,m} + g_{mn,k} - g_{km,n}\right)A^m_i - \frac{1}{2}g^{mn}\left(g_{in,k} + g_{kn,i} - g_{ik,n}\right)A^k_m =$$

$$= A^m_{i,m} + \frac{g_{,m}}{2g}A^m_i + \frac{1}{2}g^{kn}\left(g_{mn,k} - g_{km,n}\right)A^m_i - \frac{1}{2}g^{mn}A^k_m g_{kn,i} - \frac{1}{2}g^{mn}A^k_m\left(g_{in,k} - g_{ik,n}\right) =$$

$$= A^m_{i,m} + \frac{(\sqrt{-g})_{,m}}{\sqrt{-g}}A^m_i + \frac{1}{2}A^m_i\left(g^{kn}g_{mn,k} - g^{kn}g_{km,n}\right) - \frac{1}{2}A^{kn}g_{kn,i} - \frac{1}{2}A^{kn}\left(g_{in,k} - g_{ik,n}\right) = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}A^k_i\right)_{,k} - \frac{1}{2}g_{kl,i}A^{kl}.$$

E. Solutions to course work 5

Q1

a) A spherically symmetric gravitational field in vacuum is given by the Schwarzschild metric. Using this metric show that

$$\Gamma^0_{10} = -\Gamma^1_{11} = \frac{r_g}{2r^2}\left(1-\frac{r_g}{r}\right)^{-1},$$

and

$$\Gamma^1_{00} = \frac{r_g}{2r^2}\left(1-\frac{r_g}{r}\right).$$

Solution:

b) Show that the time component of the geodesic equation has the following form:

$$\frac{d^2t}{d\tau^2} + \frac{r_g}{r^2}\left(1-\frac{r_g}{r}\right)^{-1}\frac{dt}{d\tau}\frac{dr}{d\tau} = 0,$$

where τ is proper time ($ds = cd\tau$). Solve this equation for a particle which is falling radially towards a black hole. Show that

$$\frac{dt}{d\tau}\left(1-\frac{r_g}{r}\right) = 1.$$

Solution:

c) Show that when the particle in b) has zero velocity at infinity, then

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{c^2 r_g}{r}.$$

Find τ and t as functions of r and sketch them on the same graph. Explain why $t \to \infty$ when $r \to r_g$.

Solution:

Q2

Solution:

A rotating black hole is described by the Kerr metric.

$$ds^2 = (1-\frac{r_g r}{\rho^2})dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta)\sin^2\theta d\phi^2 +$$

$$\frac{2r_g r a}{\rho^2}\sin^2\theta d\phi dt,$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - r_g r + a^2$, and $a = \frac{J}{mc}$, where J is the angular momentum of the black hole.

Solution:

a) Explain why the location of the event horizon, r_{hor}, is given by $g_{rr} = \infty$ and show that

$$r_{hor} = \frac{r_g}{2} + \sqrt{(\frac{r_g}{2})^2 - a^2}.$$

Solution:

b) Compare this expression with the expression for the radius of the event horizon in the case of a non-rotating black hole.

Q3

a) What is the limit of stationarity and what is the ergosphere?
Solution:

b) Explain why the location of the sphere corresponding to the limit of stationarity, r_{ls}*, is determined by* $g_{tt} = 0$*, and show then that*

$$r_{ls} = \frac{r_g}{2} + \sqrt{(\frac{r_g}{2})^2 - a^2 \cos^2 \theta}.$$

Solution:

c) Sketch the rotating black hole as projected on i) the equatorial plane, $\theta = \pi/2$*, and ii) the perpendicular plane,* $\phi = 0$ *(indicate the event horizon, the limit of stationarity and the ergosphere).*

Q4

a) Explain qualitatively why it is possible to extract the energy of a rotating black hole despite the fact that no signal can escape outside from within the black hole horizon.
Solution:

b) Show that the circle $r = r_{hor}$ *and* $\theta = \pi/2$*, is the world line of a photon moving around the rotating black hole with angular velocity*

$$\Omega_{hor} = \frac{a}{r_g r_{hor}}.$$

(Hint: put $dr = 0$*,* $d\theta = 0$ *and* $d\phi = \Omega_{hor} dt$ *into* ds *and show that* $ds = 0$*.)*
Solution:

F. Solutions to course work 6

Q1.

a) A binary system consists of two neutron stars of the same mass M. The orbital period of the system is P. Using Newtonian mechanics, estimate to an order of magnitude the separation between the neutron stars, r, and the fractional relativistic corrections to the orbital motion.

Taking into account that

$$P = 2\pi/\omega,$$

where ω is the angular velocity which according Newtonian theory is related with the separation r as

$$\omega^2 r = \frac{GM}{r^2}, \text{ thus } \omega = \sqrt{\frac{GM}{r^3}}.$$

Hence

$$r = \left(\frac{GM}{\omega^2}\right)^{1/3} = \left(\frac{GMT^2}{4\pi^2}\right)^{1/3} = \left(\frac{T}{2\pi}\right)^{2/3} (GM)^{1/3}.$$

b) Evaluate the relativistic corrections if $P = 8$ min and $M = 1.5M_\odot$. Compare your estimate with relativistic effects in the solar system. It is known that the perihelion shift of Mercury is 43" per century. What analogous shift can you expect in the case of the binary system of neutron stars? (Hint: The relativistic shift per one orbital period is of order r_g/r, where r_g is gravitational radius of the neutron star.)

Per 1 year there are N revolutions and $N = \frac{1\ year}{P}$, hence the periastron shift per 1 year is

$$\Delta\varphi \sim \frac{r_g}{r} N \propto T^{-5/3}.$$

Comparing with the case of Mercury we have the ratio of the number of revolutions per year

$$N/N_{Merc} = P_{Merc}/P.$$

Hence

$$\frac{\Delta\varphi}{\Delta\varphi_{Merc}} \sim \left(\frac{M}{M_\odot}\right)\left(\frac{a}{r}\right)\left(\frac{P_{Merc}}{P}\right) = \left(\frac{M}{M_\odot}\right)\left(\frac{M}{M_\odot}\right)^{-1/3}\left(\frac{P}{P_{Merc}}\right)^{-2/3}\left(\frac{P_{Merc}}{P}\right) =$$

$$= \left(\frac{M}{M_\odot}\right)^{2/3}\left(\frac{P}{P_{Merc}}\right)^{-5/3}.$$

Taking to account that

$$P_{Merc} = (0.5)^{3/2} \text{ year, and } P = \frac{5.7}{365} \text{ year } \approx \frac{1}{64} \text{ year},$$

we obtain

$$\Delta\varphi = \left(\frac{43"}{100}\right) \times 10^{3\cdot 2/3} (0.5)^{3/2\cdot 5/3} (64)^{5/3} = \left(\frac{43}{60\times 60}\right)^o \times (0.5)^{5/2} \times 64^{5/3} =$$

$$\left(\frac{43\times 0.25\times 0.7\times 4^5}{3600}\right)^o \approx (0.7/3.5)^o = 0.2^o.$$

Q2.

The quadrupole formula for the metric perturbation associated with gravitational waves is given by

$$h_{\alpha\beta} = -\frac{2G}{3c^4 R}\frac{d^2 D_{\alpha\beta}}{dt^2}(t - R/c),$$

where R is the distance to the source of the gravitational waves and

$$D_{\alpha\beta} = \int (3x_\alpha x_\beta - r^2\delta_{\alpha\beta})dM$$

is the quadrupole tensor of the source. Consider a mass m moving along circular orbit around the black hole of mass M, assuming that $m \ll M$.

a) Show that all the amplitudes $h_{\alpha\beta}$ of gravitational wave, emitted by such system, are periodic functions of time with $\omega = 2\omega_0$, where $\omega_0 = 2\pi/T$, and T is the orbital period.

$$x_1 = r\cos\omega_0 t,$$

$$x_2 = r\sin\omega_0 t,$$

$$D_{11} = mr_c^2(3\cos^2\omega_0 t - 1) = \frac{1}{2}mr^2(1 + 3\cos 2\omega_0 t),$$

$$D_{22} = mr_c^2(3\sin^2\omega_0 t - 1) = \frac{1}{2}mr^2(1 - 3\cos 2\omega_0 t),$$

$$D_{12} = \frac{3}{2}mr_c^2\sin 2\omega_0 t,$$

then

$$h_{11} = -\frac{2Gmr^2}{3c^4R}\frac{3}{2}(2\omega_0)^2\cos 2\omega_0 t = \frac{4\omega_0^2 Gmr^2}{c^4R}\cos 2\omega_0,$$

$$h_{22} = \frac{2Gmr^2}{3c^4R}\frac{3}{2}(2\omega_0)^2\cos 2\omega_0 t = -\frac{4\omega_0^2 Gmr^2}{c^4R}\sin 2\omega_0,$$

$$h_{12} = \frac{2Gmr^2}{3c^4R}\frac{3}{2}(2\omega_0)^2\sin 2\omega_0 t = \frac{4\omega_0^2 Gmr^2}{c^4R}\sin 2\omega_0,$$

it is clear, that

$$\omega = 2\omega_0.$$

b) Show that, to an order of magnitude (omitting the indices α and β)

$$h \approx \frac{r_g}{R}\left(\frac{R_g\omega}{c}\right)^{2/3},$$

where r_g is the gravitational radius of the mass m and R_g is the gravitational radius of the black hole.

From

$$r\omega_0^2 = \frac{GM}{r^2},$$

we have

$$\frac{1}{r^3} = \frac{\omega_0^2}{GM},$$

and finally

$$r_c^{-1} = (4GM)^{-1/3}\omega^{2/3}.$$

Thus

$$h \approx \frac{4\omega_0^2 Gmr^2}{c^4 R} = \frac{r_g R_g}{rR} \approx \frac{r_g}{R}\left(\frac{R_g\omega}{c}\right)^{2/3}.$$

Q3.

The future LISA mission will be able to detect gravitational waves with $h > 10^{-23}$, if $10^{-4}Hz < \omega < 3 \cdot 10^{-3}Hz$. From what distance will it be possible to detect gravitational radiation from the binary system, containing the black hole of mass $m = 3M_\odot$, moving along a circular orbit with radius $r = 10^4 R_g$ around the massive black hole of mass $M = 10^3 M_\odot$?

$$\omega_0^2 = \frac{GM}{r^3} = \frac{+}{c^2} 2\frac{2GM}{c^2 r^3} = c^2 \frac{R_g}{2r^3},$$

hence,

$$\omega_0 = c\sqrt{\frac{R_g}{2r^3}} = c\sqrt{\frac{R_g}{2 \cdot 10^{12} R_g^3}} = \frac{10^{-6}c}{\sqrt{2}R_g} = \frac{10^{-4}Hz}{\sqrt{2}},$$

thus

$$\omega = 2\omega_0 = \sqrt{2}10^{-4}Hz \geq 10^{-4}Hz,$$

which means that the radiation is within LISA frequency range.

$$h = \frac{3 \cdot 10^5}{3 \cdot 10^{18}} \left(\frac{3 \cdot 10^5 \cdot 10^{-4}}{3 \cdot 10^{10}}\right)^{2/3} (\frac{m}{M.})(\frac{R}{1pc})^{-1}(\frac{M}{M.})^{2/3}(\frac{\omega}{10^{-4}Hz})^{2/3}$$

$$\approx 10^{-19}(\frac{m}{M.})(\frac{R}{1\ pc})^{-1}(\frac{M}{M.})^{2/3}(\frac{\omega}{10^{-4}Hz})^{2/3}.$$

Then

$$h = \frac{3 \cdot 10^5 cm}{R}(\frac{3 \cdot 10^5 \cdot 10^3 \cdot 1.4 \cdot 10^{-4}s^{-1}cm}{3 \cdot 10^{10}})^{2/3} > 10^{-23},$$

if

$$R < 3 \cdot 10^{23} \cdot 10^5 cm \cdot 10^{-4} \approx 1Mpc.$$