THERMAL AND KINETIC PHYSICS (PHY 214)

EXERCISE 8 : WEEK 9

OUTLINE SOLUTIONS

QUESTION 1: (20 marks)

(a) We consider an infinitesimal change in temperature dT at constant total magnetization, $\mathcal{M}$. The thermodynamic identity tells us that for any change

$$dU = TdS + B_0 d\mathcal{M}$$

so for a change at constant $\mathcal{M}$ we have $dU = TdS$ and if we divide this equation by dT at constant $\mathcal{M}$ we get

$$\left.\frac{dU}{dT}\right|_{\mathcal{M}} = T\left.\frac{dS}{dT}\right|_{\mathcal{M}}$$

Identifying the partial derivatives gives

$$\left(\frac{\partial U}{\partial T}\right)_{\mathcal{M}} = T\left(\frac{\partial S}{\partial T}\right)_{\mathcal{M}} = C_{\mathcal{M}}$$ **[3mks]**

Alternatively consider a change $d\mathcal{M}$ at constant T so we may divide the thermodynamic identity by $d\mathcal{M}$, remembering that all changes are at constant T , to get

$$\left.\frac{dU}{d\mathcal{M}}\right|_{T} = T\left.\frac{dS}{d\mathcal{M}}\right|_{T} + B_0$$

However, if we regard U as a function of $\mathcal{M}$, then we see that

$$\left.\frac{dU}{d\mathcal{M}}\right|_{T} = \left(\frac{\partial U}{\partial \mathcal{M}}\right)_{T}$$

giving the required result

$$\left(\frac{\partial U}{\partial \mathcal{M}}\right)_{T} = T\left(\frac{\partial S}{\partial \mathcal{M}}\right)_{T} + B_0$$ **[2mks]**

(b)

i) Using the definition, $F = U - TS$ and considering a small change in the Helmholtz free energy we may write

$$dF = dU - TdS - SdT = TdS + B_0 d\mathcal{M} - TdS - SdT$$

$$dF = -SdT + B_0 d\mathcal{M}$$ **[3mks]**

ii) The above indicates that F has as its natural variables T and $\mathcal{M}$, so that

$$dF = \left(\frac{\partial F}{\partial T}\right)_{\mathcal{M}} dT + \left(\frac{\partial F}{\partial \mathcal{M}}\right)_T d\mathcal{M}$$ **[2mks]**

so that, identifying the two expressions for dF we have

$$-S = \left(\frac{\partial F}{\partial T}\right)_{\mathcal{M}} \qquad B_0 = \left(\frac{\partial F}{\partial \mathcal{M}}\right)_{\mathrm{T}}$$ **[2mks]**

Since the mixed second partial derivatives of a perfect differential obey

$$\left(\frac{\partial^2 F}{\partial \mathcal{M} \partial T}\right) = \left(\frac{\partial^2 F}{\partial T \partial \mathcal{M}}\right)$$

we deduce

$$\left(\frac{\partial^2 F}{\partial \mathcal{M} \partial T}\right) = \frac{\partial}{\partial \mathcal{M}} \frac{\partial F}{\partial T} = -\left(\frac{\partial S}{\partial \mathcal{M}}\right)_T = \frac{\partial}{\partial T} \frac{\partial F}{\partial M} = \left(\frac{\partial B_0}{\partial T}\right)_{\mathcal{M}}$$ **[2mks]**

$$-\left(\frac{\partial S}{\partial \mathcal{M}}\right)_T = \left(\frac{\partial B_0}{\partial T}\right)_{\mathcal{M}}$$

and substituting the above gives the required result

$$\left(\frac{\partial U}{\partial \mathcal{M}}\right)_T = -T\left(\frac{\partial B_0}{\partial \mathrm{T}}\right)_{\mathcal{M}} + B_0$$ **[1mk]**

(c) Using the equation of state we can evaluate the derivative of B_0 with respect to T,

$$B_0 = \frac{\mu_0 T}{CV}\mathcal{M} \rightarrow \left(\frac{\partial B_0}{\partial T}\right)_{\mathcal{M}} = \frac{\mu_0}{CV}\mathcal{M}$$ **[2mks]**

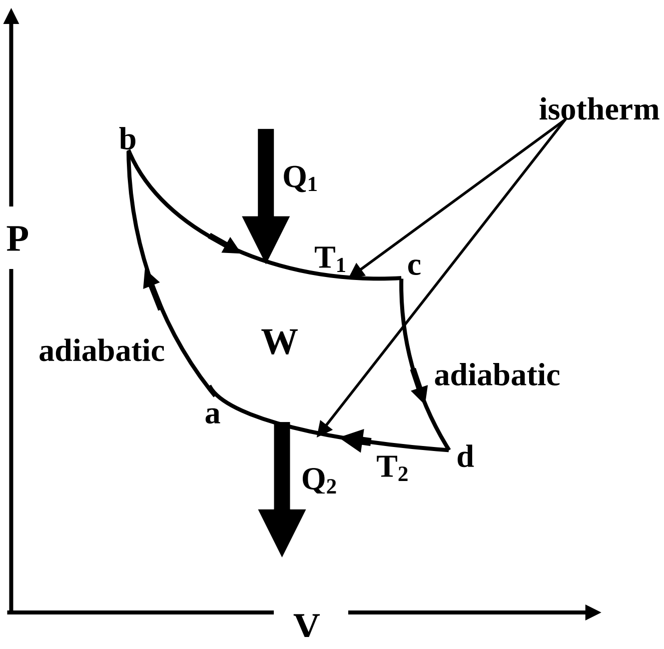
and substituting into ii) above gives

$$\left(\frac{\partial U}{\partial \mathcal{M}}\right)_T = -T\left(\frac{\mu_0}{CV}\mathcal{M}\right) + B_0 = -B_0 + B_0 = 0$$ **[2mks]**

Since this derivative is identically zero, U cannot depend on $\mathcal{M}$ and must be a function of T alone. **[1mk]**

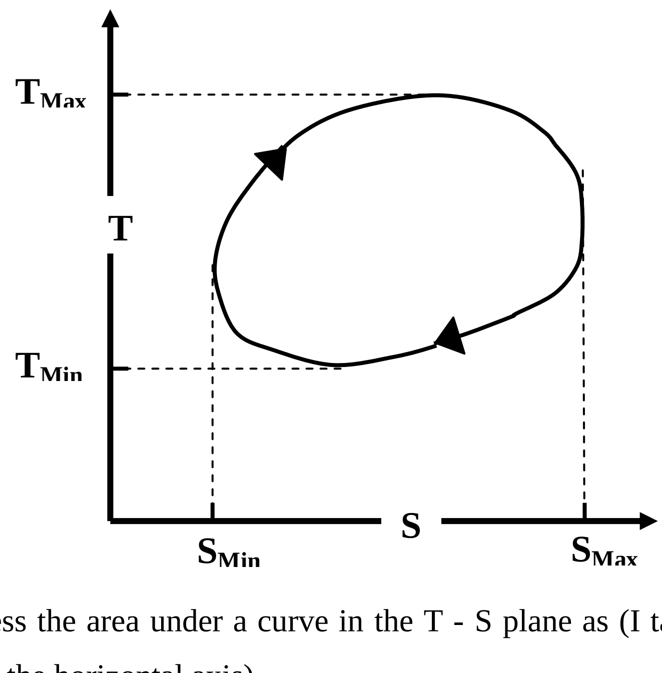
QUESTION 2: (20 marks)

(a) The Carnot cycle consists of two isothermal paths on which heat is exchanged and two adiabatic paths. For a fluid as the working substance, the corresponding P - V diagram is given in the lecture notes.



[3mks]

(b) For the general reversible engine cycle shown below



we can express the area under a curve in the T - S plane as (I take T on the vertical axis and S on the horizontal axis)

$$Area = \int_{S_i}^{S_f} TdS = \int_i^f đQ = Heat\ \ absorbed\ \ in\ \ going\ \ \ from\ \ i\ \ to\ \ f$$

We can split the engine cycle into an upper path starting at $S_{\min}$ and ending at $S_{\max}$ and the return (lower) path starting at $S_{\max}$ and finishing at $S_{\min}$. On the upper path we have

$$A_{UP} = \int_{min}^{max} đQ = \Delta Q_{S_{min} \to S_{max}}$$ **[4mks]**

Since the entropy of the substance is increasing, it is absorbing heat so that the area computed just above is the total heat Q_1 supplied to the engine,

$$A_{UP} = Q_1$$

On the return path, the positive area A_{LOW} is also a heat flow given by

$$A_{LOW} = \left| \int_{max}^{min} đQ \right| = \left| -Q_2 \right|$$ **[4mks]**

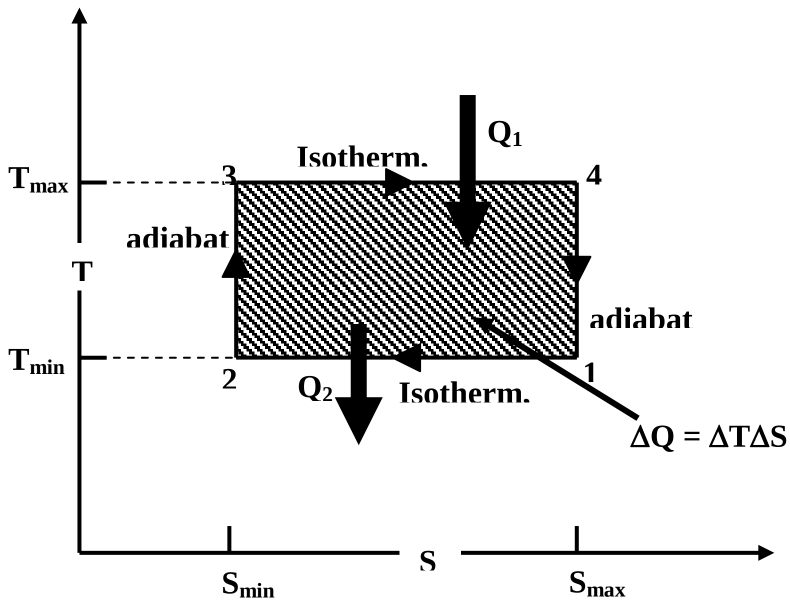
where Q_2 is the waste heat coming out of the engine,

$$A_{LOW} = Q_2$$

The engine efficiency is given then as

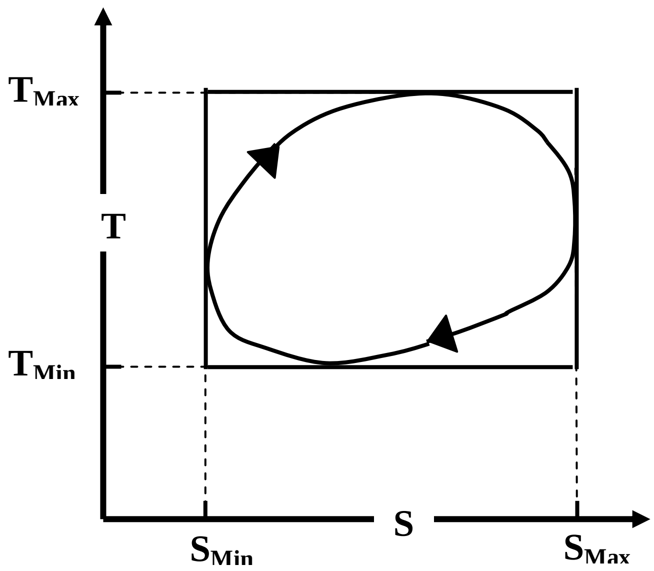
$$\eta_E = 1 - \frac{Q_2}{Q_1} = 1 - \frac{A_{LOW}}{A_{UP}}$$ **[1mk]**

(c) In the T - S plane, an isotherm is a horizontal straight line while an adiabat is a vertical straight line (constant entropy). Thus the Carnot cycle is just a rectangle.



[3mks]

(d) To compare the general engine shown in (b) with the Carnot engine shown in (c), enclose the general cycle in the smallest rectangle that will just hold it, i.e., a rectangle with horizontal sides at $T_{\max}$ and $T_{\min}$ and with vertical sides at $S_{\max}$ and $S_{\min}$



By the geometry of this diagram it is obvious that the following inequalities hold:

$$A_{LOW}^{Engine} > A_{LOW}^{Carnot} \qquad A_{UP}^{Carnot} > A_{UP}^{Engine}$$ **[2mks]**

It follows that

$$\frac{A_{LOW}^{Engine}}{A_{UP}^{Engine}} > \frac{A_{LOW}^{Carnot}}{A_{UP}^{Carnot}}$$ **[2mks]**

And hence that

$$1 - \frac{A_{LOW}^{Engine}}{A_{UP}^{Engine}} < 1 - \frac{A_{LOW}^{Carnot}}{A_{UP}^{Carnot}}$$ **[2mks]**

or

$$\eta_{General\ \ Engine} < \eta_{Carnot\ \ Engine}$$