Spacetime and Gravity: Assignment 5 Solutions

November 30, 2009

In what follows, unless otherwise stated, we will use units such that the speed of light, $c = 1$.

1.

We are given a spherical shell of mass M and Radius R. Outside the shell we have a spherically symmetric mass distribution, so the metric will take the Schwarzschild form:

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \quad (1)$$

Inside there is no mass enclosed and therefore by Gauss's law there is no gravitational field strength and thus the metric is flat

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \quad (2)$$

However, this does not mean that the gravitational potential inside the shell is zero. Recall from electromagnetism where the field strength is given by the gradient of the potential $E = -\nabla V$. Thus if the field strength is zero it implies that the potential is constant, but not necessarily also zero. In our case the potential inside the shell is not equal to zero. By continuity of the metric (i.e. to avoid singularities in the metric) one has to match the metric at the surface of the shell with that in the interior, this will tell us the constant graviational potential that we have inside the shell. Since we are only interested in the time dilation effects we focus on the time component of the metric. So, at the boundary of the shell:

$$ds^2 = -\left(1 - \frac{2GM}{Rc^2}\right) dt^2 \quad (3)$$

and so this will be the constant time component of the metric we feel inside the shell. Outside, the laptop is placed quot;far awayquot; from the shell, meaning far enough that it won't feel the gravitational effects it causes. The laptop is in flat space. So, we sit inside the shell with a constant Schwarzschild-like time component of the metric, whilst the laptop is in flat space. Now, in a Schwarzschild environment, the time dilation effects are given by:

$$t' = \sqrt{\left(1 - \frac{2GM}{rc^2}\right)} t \quad (4)$$

where t' is the time measured by an observer in the field caused by the gravitational object (i.e. us) and t is the time measured by an observer outside the field (the laptop).

Rearranging this expression for the mass of the shell M and substituting for its radius $r = R$ we obtain:

$$M = \frac{Rc^2}{2G}\left(1 - \frac{t'^2}{t^2}\right) \tag{5}$$

which yields an answer

$$M = 6.75 \times 10^{27} Kg \tag{6}$$

The problem seems evident, the mass needed is extremely large. To achieve such a time dilation one needs to surround oneself with a shell of mass approximately equal (one thousandth) to that of the sun! Also, even though your in the shell and the laptop does the equivalent of 100 years of calculations, the whole world outside the shell will have moved forward by 100 years! So maybe when you come out the money you wanted to make is not useful any more. A better way would be to sit inside the shell on a savings account and let interest make you rich, if the bank doesn't collapse in the 100 years you are in there.
2.

Space in the vicinity of the neutron star will have a Schwarzschild metric. Space far away from it will however not feel its graviational effects and thus will be flat.

Close to the star

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right) dt^2 \tag{7}$$

but far from it

$$ds^2 = dt'^2 \tag{8}$$

where the minus sign of the flat metric is not shown by choice of signature. To obtain the redshift caused by the star we match the two solutions:

$$\begin{aligned} dt'^2 &= \left(1 - \frac{2GM}{rc^2}\right) dt^2 && (9) \\ \Rightarrow \frac{dt'}{dt} &= \sqrt{\frac{1}{1 - \frac{2GM}{r}}} && (10) \end{aligned}$$

Now we can approximate the last expression via an expansion of the square root to obtain:

$$\frac{dt'}{dt} = 1 + \frac{GM}{r} \tag{11}$$

Finally we are in a position to obtain the form of the graviational redshift. We define the redshift Red:

$$Red = \frac{dt'}{dt} - 1 = \frac{\lambda_o - \lambda_e}{\lambda_e} = \frac{GM}{r} \tag{12}$$

where λ_o is the observed wavelength very far away and λ_e is the emitted wavelength at the surface of the netruon star. Rearranging

$$\lambda_o = \lambda_e \left(1 + \frac{GM}{c^2 r}\right) \tag{13}$$

Substituting for the mass and radius of the star we obtain

$$\lambda_o = 344nm \tag{14}$$

This is called red-shift (as opposed to blueshift) and is of important use in every-day astrophysics and cosmology. We want to now calculate the effective speed of light of a radially emitted photon at the surface of the star. For a photon $ds = 0$ and therefore using the Schwarzschild solution (with constant angular variations) we obtain (using the Newtonian approximation):

$$c^2 dt^2 \left(1 - \frac{2GM}{c^2 r}\right) = dr^2 \left(1 + \frac{2GM}{c^2 r}\right) \tag{15}$$

and therefore

$$\begin{aligned} \frac{dr}{dt} &= \sqrt{\frac{\left(1 - \frac{2GM}{c^2 r}\right)}{\left(1 + \frac{2GM}{c^2 r}\right)}} c & (16) \\ &= 2.21 \times 10^8 ms^{-1} & (17) \end{aligned}$$

Note that the effective speed appears lower than what it actually is.

3.

We are given the flat Minkowski metric

$$ds^2 = -dx_0^2 + \sum_{i=1}^{4} dx^i dx^i \tag{18}$$

with the constraint that defines de Sitter space

$$-x_0^2 + \sum_i x^i x^i = 1 \tag{19}$$

Now we make a change of coordinates to

$$x_0 = \sinh(t) \qquad x_i = \cosh(t) y_i \tag{20}$$

The constraint in these coordinates is

$$-\sinh^2(t) + \cosh^2(t) y_i y^i = 1 \tag{21}$$

which suggests

$$y_i y^i = 1 \tag{22}$$

and the summation over the index i is used. We substitute these coordinates back into the original metric to obtain the de Sitter metric:

$$
\begin{aligned}
-dx_0^2 + \sum_{i=1}^{4} dx^i dx^i & \quad - \cosh^2(t) dt^2 + \sinh^2(t) dt^2 y_i y^i && (23) \\
& \quad +2 \sinh(t) \cosh(t) dt y_i dy_i + \cosh^2(t) dy_i dy^i && (24) \\
& = \quad -dt^2 + \cosh^2(t) dy_i dy^i && (25)
\end{aligned}
$$

where in the last equation we used $y_i dy^i = 0$ which comes from

$$
\begin{aligned}
y_i y^i &= 1 && (26) \\
\Rightarrow 2 dy_i y^i &= 0 && (27) \\
\Rightarrow y_i dy^i = 0 & && (28)
\end{aligned}
$$

1 Summary of important concepts

1.Time feels the effect of gravity! So even though someone moving at high speed in a plane will have its clock ticking slower because of special relativistic effects it will also have it (to a minor effect) clicking faster because he feels less of the effect of Earth's gravity.
2.The light we see coming from other planets/stars is not strictly the same as the one they emit. The light is red/blue-shifted according to the relative movement of source and observer.