University College London

Department of Physics and Astronomy

2B21 Mathematical Methods in Physics & Astronomy

Suggested Solutions for Problem Sheet M8 (2003–2004)

1. Using integration by parts

$$I = \int \sin nx \sinh x \, dx = \int \sin nx \, d(\cosh x) = \sin nx \cosh x - n \int \cos nx \cosh x \, dx$$

$$= \sin nx \cosh x - n \int \cos nx \, d(\sinh x) = \sin nx \cosh x - n \cos nx \sinh x - n^2 I \, .$$

Hence

$$I = \frac{1}{1+n^2} \left[\cosh x \sin nx - n \cos nx \sinh x\right] \, .$$ **[4]**

Alternative methods include writing the sin and sinh functions in terms of exponentials or using $\sinh(x) = -i\sin(ix)$, but both are significantly longer.

The sinh function

$$f(x) = \sinh x$$

in the interval $-\pi \leq x \leq \pi$, is clearly <u>odd</u>. **[1]**

All the coefficients of the cosine terms in

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

must therefore vanish since the cosine function is even, *i.e.* $a_n = 0$. **[1]**

Turning to the sine coefficients,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sinh x \, \sin nx \, dx \, .$$

$$= \frac{1}{\pi} \Big[\cosh x \sin nx - n \cos nx \sinh x\Big]_{-\pi}^{+\pi} = \frac{2}{\pi} \sinh \pi (-1)^{n+1} \frac{n}{n^2+1} \, .$$ **[3]**

The Fourier series is therefore

$$f(x) = \frac{2}{\pi} \sinh \pi \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1} \sin nx \, .$$ **[1]**

Parseval's theorem states that the average value of f^2 is given by

$$< \, f^2(x) \, > = \frac{1}{2\pi} \int_{-\pi}^{+\pi} [f(x)]^2 \, dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \, .$$ **[2]**

$$< f^2(x) > = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \sinh^2 x \, dx = \frac{1}{4\pi} \int_{-\pi}^{+\pi} (\cosh 2x - 1)$$

$$= \frac{1}{8\pi} \left[\sinh 2x - 2x \right]_{-\pi}^{+\pi} = \frac{1}{4\pi} \left[\sinh 2\pi - 2\pi \right]$$ **[2]**

Parseval's identity then gives

$$\frac{1}{4\pi} \left[\sinh 2\pi - 2\pi \right] = \frac{1}{2} \left(\frac{2 \sinh \pi}{\pi} \right)^2 \sum_{n=1}^{\infty} \frac{n^2}{(n^2+1)^2} \, ,$$

from which we get

$$\sum_{n=1}^{\infty} \frac{n^2}{(n^2+1)^2} = \frac{\pi}{8} \frac{(\sinh 2\pi - 2\pi)}{\sinh^2 \pi} .$$ **[2]**

2. The required Fourier transform is the integral

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^{+1} e^{-ax} \, e^{i\omega x} \, dx = \frac{1}{\sqrt{2\pi}} \frac{1}{(a - i\omega)} \cdot$$ **[2]**

Parseval's theorem (for Fourier transforms) states that

$$\int_{-\infty}^{+\infty} |f(x)|^2 \, dx = \int_{-\infty}^{+\infty} |g(\omega)|^2 \, d\omega \, .$$ **[1]**

$$\text{Now LHS} = \int_0^{\infty} e^{-2ax} \, dx = \frac{1}{2a} \cdot$$ **[1]**

$$\text{RHS} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{a^2 + \omega^2} \left(\frac{d\omega}{d=} \right) \frac{1}{2\pi a} \left[\tan^{-1} \left(\frac{x}{a} \right) \right]_{-\infty}^{\infty} = \frac{1}{2a} \cdot$$ **[3]**

Parseval's theorem survives!