UNIVERSITY COLLEGE LONDON
DEPARTMENT OF PHYSICS AND ASTRONOMY

2B21 MATHEMATICAL METHODS IN PHYSICS AND ASTRONOMY

Problem Sheet M7 (2003–2004)

Solutions to be handed in on Tuesday 25 November 2003

1. The Legendre polynomials $P_n(x)$ satisfy the orthonormality relation

$$\int_{-1}^{+1} P_n(x)\, P_m(x)\, dx = \delta_{nm} \; ,$$

provided that n and m are non-negative integers.

Given that $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = \frac{1}{2}(3x^2 - 1)$, show by explicit integration that the above relation is satisfied for $n, m \leq 2$. **[7 marks]**

Assuming that for $n = 3$

$$P_3(x) = a\,[x^3 + bx^2 + cx + d] \; ,$$

use the orthogonality relation to find the coefficients b, c and d. **[4 marks]**

Use the normalisation integral above, OR OTHERWISE, determine the value of a. **[3 marks]**

2. The Laguerre polynomials $L_n(x)$ arise in the solution of the Schrödinger equation for the hydrogen atom. They may be defined by the generating function

$$g(x,t) = \frac{\exp\left(-xt/(1-t)\right)}{1-t} = \sum_{n=0}^{\infty} L_n(x)\, t^n \; .$$

- By expanding the left hand side in powers of t, show that $L_0(x) = 1$, $L_1(x) = 1 - x$, and $L_2(x) = \frac{1}{2}(x^2 - 4x + 2)$. **[3 marks]**

- By differentiating the generating function equation with respect to x and comparing powers of t, show that the polynomials satisfy the recurrence relation **[4 marks]**

$$L_n(x) = L_n'(x) - L_{n+1}'(x) \; .$$

- By multiplying the expansion of $g(x,t)$ by that for $g(x,u)$ and integrating, show that **[6 marks]**

$$\int_0^{\infty} e^{-x} L_n(x)\, L_m(x) = \delta_{nm} \; .$$