UNIVERSITY COLLEGE LONDON
DEPARTMENT OF PHYSICS AND ASTRONOMY

2B21 MATHEMATICAL METHODS IN PHYSICS AND ASTRONOMY

Problem Sheet M4 (2003–2004)

Solutions to be handed in on Tuesday 4 November 2003

1. A coupled dynamical system with two degrees of freedom, x_1 and x_2, is described by the equations

$$\begin{aligned} 2\,\frac{d^2x_1}{dt^2} &= -5x_1 + 3x_2\,, \\ 2\,\frac{d^2x_2}{dt^2} &= 3x_1 - 5x_2\,. \end{aligned}$$

   Use matrix techniques to diagonalise these equations and find the normal frequencies and modes. **[10 marks]**

   At time $t = 0$, the coordinates and velocities are given by

$$x_1 = 0,\ x_2 = 2a \ \text{ and } \ \frac{dx_1}{dt} = \frac{dx_2}{dt} = 0\,.$$

   Find x_1 and x_2 at later times. **[4 marks]**

2. Demonstrate that the eigenvalues λ of the Hermitian matrix

$$\underline{A} = \begin{pmatrix} 1 & i & 3i \\ -i & 1 & -3 \\ -3i & -3 & -3 \end{pmatrix}$$

   satisfy the characteristic equation

$$\lambda^3 + \lambda^2 - 24\lambda + 36 = 0\,.$$ **[3 marks]**

   Prove that one eigenvalue is $\lambda_1 = 2$ and find the other two solutions. **[2 marks]**

   Verify in this case that:
   (i) the trace of the matrix is equal to the sum of the eigenvalues, and **[1 mark]**
   (ii) the determinant is equal to the product of the eigenvalues. **[2 marks]**

   Find the three (complex) eigenvectors $\underline{x}_i$, normalised to have unit length, $\underline{x}_i^\dagger\,\underline{x}_i = 1$, where the † denotes Hermitian conjugation. **[9 marks]**

   Prove that the eigenvectors are orthogonal,

$$\underline{x}_i^\dagger\,\underline{x}_j = 0 \ \text{ for } \ i \neq j\,.$$