University College London

Department of Physics and Astronomy

2B21 Mathematical Methods in Physics & Astronomy

Suggested Solutions for Problem Sheet M3 (2003–2004)

1. To order ε^2,

$$\ell n\left(\underline{I}+\varepsilon\,\underline{A}\right)\approx\varepsilon\,\underline{A}-\tfrac{1}{2}(\varepsilon\,\underline{A})^2\,. \qquad \textbf{[2]}$$

Taking the trace

$$\mathrm{tr}\left\{\ell n\left(\underline{I}+\varepsilon\,\underline{A}\right)\right\}\approx\varepsilon\,\mathrm{tr}(\underline{A})-\tfrac{1}{2}\varepsilon^2\mathrm{tr}(\underline{A}^2)\,. \qquad \textbf{[2]}$$

Note that the right hand side is now a scalar.

The final step

$$|\underline{I}+\varepsilon\,\underline{A}|=\exp\left[tr\left\{\ell n\left(\underline{I}+\varepsilon\,\underline{A}\right)\right\}\right]\approx 1+\varepsilon\,tr(\underline{A})-\tfrac{1}{2}\varepsilon^2 tr(\underline{A}^2)+\tfrac{1}{2}\left(\varepsilon\,tr(\underline{A})-\tfrac{1}{2}\varepsilon^2 tr(\underline{A}^2)\right)^2$$

$$=1+\varepsilon\,tr\,(\underline{A})+\tfrac{1}{2}\varepsilon^2\left[(tr\,\underline{A})^2-tr\,(\underline{A}^2)\right]+0(\varepsilon^3)\,. \qquad \textbf{[2]}$$

For the matrix

$$\underline{A}=\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\,,\quad \underline{A}^2=\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}=\begin{pmatrix}0 & 2i\\ 2i & 0\end{pmatrix}\,.$$

$$tr(\underline{A})=2\,,\quad tr\,(\underline{A}^2)=0\,.$$

Therefore

$$|\underline{I}+\varepsilon\,\underline{A}|\approx 1+2\varepsilon+\tfrac{1}{2}\varepsilon^2[4-0]=1+2\varepsilon+2\varepsilon^2\,. \qquad \textbf{[2]}$$

Working out the determinant explicitly,

$$|\underline{I}+\varepsilon\,\underline{A}|=\begin{vmatrix}1+\varepsilon & i\varepsilon\\ i\varepsilon & 1+\varepsilon\end{vmatrix}=(1+\varepsilon)^2+\varepsilon^2=1+2\varepsilon+2\varepsilon^2\,. \qquad \textbf{[2]}$$

The theorem can be used for any size matrix, even infinite!

2. The eigenvalues are given by

$$\begin{vmatrix} -\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 1 = 0\,,$$

which has solutions $\lambda_1 = 1+\sqrt{2}$ and $\lambda_2 = 1-\sqrt{2}$. **[2]**

$$\underline{A}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix} = \underline{I} + 2\underline{A}\,. \qquad \textbf{[1]}$$

Using this result,

$$\underline{A}^4 = \underline{A}^2\,\underline{A}^2 = (\underline{I}+2\underline{A})(\underline{I}+2\underline{A}) = \underline{I} + 4\underline{A} + 4\underline{A}^2$$
$$= \underline{I} + 4\underline{A} + 4\underline{I} + 8\underline{A} = 5\underline{I} + 12\underline{A}\,. \qquad \textbf{[2]}$$

Similarly,

$$\underline{A}^8 = \underline{A}^4\,\underline{A}^4 = (5\underline{I}+12\underline{A})(5\underline{I}+12\underline{A}) = 25\underline{I} + 120\underline{A} + 144\underline{A}^2$$
$$= 25\underline{I} + 120\underline{A} + 144\underline{I} + 288\underline{A} = 169\underline{I} + 408\underline{A}\,. \qquad \textbf{[1]}$$

With the definition, $t_2 = 6^{1/2} \approx 2.4495$, $t_4 = 34^{1/4} \approx 2.4147$, $t_8 = 954^{1/8} \approx 2.4142 \approx 1+\sqrt{2} = \lambda_1$. **[3]**

To see why this happens, we need to use two results. Firstly, if the eigenvalues of a matrix $\underline{A}$ are λ_1 and λ_2, then the eigenvalues of the matrix $\underline{A}^n$ are λ_1^n and λ_2^n. [The proof uses $\underline{A} = \underline{R}^{-1}\underline{\Lambda}\,\underline{R}$ multiplied together n times.]

Secondly, the trace of a matrix is equal to the sum of its eigenvalues. This means that

$$tr\{\underline{A}^n\} = \lambda_1^n + \lambda_2^n\,.$$

But $|\lambda_1| > |\lambda_2|$, so that the second term above is negligible as $n \to \infty$. Thus

$$\lim_{n\to\infty}(t_n) = (\lambda_1^n)^{1/n} = \lambda_1\,. \qquad \textbf{[3]}$$

3. The proof is very similar to that for Hermitian matrices. Let

$$\underline{A}\,\underline{x}_i = \lambda_i \underline{x}_i \,.$$

Taking the Hermitian conjugate,

$$\underline{x}_i^\dagger\,\underline{A}^\dagger = \lambda_i^*\,\underline{x}_i^\dagger = -\underline{x}_i^\dagger\,\underline{A} \,.$$ **[2]**

Multiplying the first equation on the left by $\underline{x}_i^\dagger$ and the second on the right by $\underline{x}_i$, we get

$$\begin{aligned} \underline{x}_i^\dagger\,\underline{A}\,\underline{x}_i &= \lambda_i\,\underline{x}_i^\dagger\underline{x}_i \,, \\ \underline{x}_i^\dagger\,\underline{A}\,\underline{x}_i &= -\lambda_i^*\,\underline{x}_i^\dagger\underline{x}_i \,. \end{aligned}$$ **[1]**

Adding the two

$$(\lambda_i - \lambda_i^*)\underline{x}_i^\dagger\,\underline{x}_i = 0 \,.$$ **[1]**

Unless the eigenvectors have zero norm, this means that the real part of the eigenvalue has to vanish. **[1]**

Proofs which rely on $i\underline{H}$ being Hermitian and then use the theorem about Hermitian matrices give only **3 marks**.

For the given matrix,

$$\begin{vmatrix} -\lambda & 1+i & i \\ -1+i & -\lambda & 1-i \\ i & -1-i & -\lambda \end{vmatrix} = 0$$

$$= -\lambda(\lambda^2+2) - (1+i)(\lambda(1-i) - i(1-i)) + i(2+\lambda i) = -\lambda^3 - 5\lambda + 4i \,.$$ **[2]**

We are given that $\lambda = i$ is one solution. To get the others we must factorise the expression as

$$-\lambda^3 - 5\lambda + 4i = -(\lambda - i)(\lambda^2 + \lambda i + 4) = 0 \,.$$

The other two solutions are therefore

$$\lambda = \left[-i \pm \sqrt{-1-16}\right]/2 = i\left[-1 \pm \sqrt{17}\right]/2 \,.$$ **[2]**

These also are pure imaginary roots, as required by the theorem. **[1]**