PHYSICS 2B21: Mathematical Methods in Physics and Astronomy

Mid–Sessional Examination December 2003

College regulations do not allow us to return regular examination scripts to you in order that you can (hopefully) learn from your mistakes. The Christmas mock examinations are exceptions to this rule but you must realise that the examiner is in general not marking to give feedback to the candidate; rather is he explaining his reasoning to any second examiner checking the paper. Please note that this paper (worth 2%) has not been checked by a second examiner.

The percentage has been obtained by dividing the mark by 0.8. A percentage of 50 or over yield a continuous assessment mark of 2, one between 25 and 49% results in 1 extra continuous assessment mark. As a rough guide, the pass mark on this scale in the summer is likely to correspond to about 28%. This is to be compared to the class average of 34.9% achieved in this mock exam.

Before describing the most common errors on the different questions, let me mention some overall points.

- When the string is tied too tightly, the examiner has to tear the paper in order that it lie down flat on the table. This is very irritating.
- Some students do not always start a new question on a different page.
- You should not write in the columns of the answer books.
- When students answer more than four questions, I mark the first four and ignore completely the fifth.

Some comments on individual questions

1. By chance, I was requested to go over much of this material on the day before the test — but it didn't do much good!
   - ϕ is a scalar function. Many candidates wrote ϕ_x *etc.* and then got nowhere afterwards.
   - Many students seemed to neglect the necessity of writing basis vectors when writing out vectors in terms of components.
   - There was in fact much confusion between vectors and scalars which should have been clear from the 1B21 course.
2. If $\underline{D} = \underline{B}\,\underline{A}$ then $\underline{B} = \underline{D}\,\underline{A}^{-1}$. The order is important.
3. The normalisation was done a bit better this year. However, students who could not find the other two eigenvalues were completely lost after the first five marks.

4. When using d'Alembert, you must keep the $|x|$ factor in or otherwise it is meaningless. The differentiation of the cosh was not always very good.

5. Many students were not at ease with the standard bookwork at the beginning of the Fourier question. This is silly because it tends to come up very often. Candidates who tried this tended to use the integration by parts method. De Moivre is a MUCH simpler way and this should have been realised since I put it first.

6. There were VERY few attempts at this question and they were almost universally poor. Separation of variables is completely straightforward — much easier than the Fourier series integration for example.

Model solutions

1. (a) In Cartesian coordinates,

$$\underline{\nabla}\cdot(\phi\,\underline{S}) = \frac{\partial}{\partial x}(\phi\,S_x) + \frac{\partial}{\partial y}(\phi\,S_y) + \frac{\partial}{\partial z}(\phi\,S_z)$$

$$= \phi\,\frac{\partial S_x}{\partial x} + \phi\,\frac{\partial S_y}{\partial y} + \phi\,\frac{\partial S_z}{\partial z} + \frac{\partial \phi}{\partial x}\,S_x + \frac{\partial \phi}{\partial y}\,S_y + \frac{\partial \phi}{\partial z}\,S_z$$

$$= \phi\,(\underline{\nabla}\cdot\underline{S}) + (\underline{\nabla}\phi)\cdot\underline{S}\,.$$ **[3]**

Similarly

$$\underline{\nabla}\times(\phi\,\underline{S}) = \begin{vmatrix} \hat{\underline{e}}_x & \hat{\underline{e}}_y & \hat{\underline{e}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi\,S_x & \phi\,S_y & \phi\,S_z \end{vmatrix}$$

The x-component of this is

$$\frac{\partial}{\partial y}(\phi\,S_z) - \frac{\partial}{\partial z}(\phi\,S_y) = \phi\,\frac{\partial S_z}{\partial y} + \frac{\partial \phi}{\partial y}\,S_z - \phi\,\frac{\partial S_y}{\partial z} - \frac{\partial \phi}{\partial z}\,S_y$$

$$= \phi\left(\frac{\partial S_z}{\partial y} - \frac{\partial S_y}{\partial z}\right) + \left(\frac{\partial \phi}{\partial y}\,S_z - \frac{\partial \phi}{\partial z}\,S_y\right) = \phi\,(\underline{\nabla}\times\underline{S})_x + (\underline{\nabla}\phi\times\underline{S})_x\,.$$

The same is true for the other components, which proves that

$$\underline{\nabla}\times(\phi\,\underline{S}) = \underline{\nabla}\phi\times\underline{S} + \phi\,(\underline{\nabla}\times\underline{S})\,.$$ **[3]**

Now

$$\underline{P} = \underline{\nabla}\times\underline{S} = \begin{vmatrix} \hat{\underline{e}}_x & \hat{\underline{e}}_y & \hat{\underline{e}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ S_x & S_y & S_z \end{vmatrix} = \hat{\underline{e}}_x\left(\frac{\partial S_z}{\partial y} - \frac{\partial S_y}{\partial z}\right) - \hat{\underline{e}}_y\left(\frac{\partial S_z}{\partial x} - \frac{\partial S_x}{\partial z}\right) + \hat{\underline{e}}_z\left(\frac{\partial S_y}{\partial x} - \frac{\partial S_x}{\partial y}\right)$$

Taking just the x-component,

$$(\underline{C} \times (\underline{\nabla} \times \underline{S}))_x = C_y P_z - C_z P_y = C_y \left(\frac{\partial S_y}{\partial x} - \frac{\partial S_x}{\partial y} \right) + C_z \left(\frac{\partial S_z}{\partial x} - \frac{\partial S_x}{\partial z} \right) .$$

Consider now x-component of the right hand side of the identity:

$$(\underline{\nabla}\,(\underline{C} \cdot \underline{S}))_x = \frac{\partial}{\partial x}(C_x S_x + C_y S_y + C_z S_z) = C_x \frac{\partial S_x}{\partial x} + C_y \frac{\partial S_y}{\partial x} + C_z \frac{\partial S_z}{\partial x} \cdot$$

Also

$$((\underline{C} \cdot \underline{\nabla})\, \underline{S})_x = C_x \frac{\partial S_x}{\partial x} + C_y \frac{\partial S_x}{\partial y} + C_z \frac{\partial S_x}{\partial z} \cdot$$

Hence

$$\begin{aligned}(\mathrm{RHS})_x &= C_x \frac{\partial S_x}{\partial x} + C_y \frac{\partial S_y}{\partial x} + C_z \frac{\partial S_z}{\partial x} - C_x \frac{\partial S_x}{\partial x} - C_y \frac{\partial S_x}{\partial y} - C_z \frac{\partial S_x}{\partial z} \\ &= C_y \left(\frac{\partial S_y}{\partial x} - \frac{\partial S_x}{\partial y} \right) + C_z \left(\frac{\partial S_z}{\partial x} - \frac{\partial S_x}{\partial z} \right) = (\mathrm{LHS})_x .\end{aligned}$$

The identity is true for the x-component. There is nothing special about x compared to y or z so that it follows that

$$\underline{C} \times (\underline{\nabla} \times \underline{S}) = \underline{\nabla}\,(\underline{C} \cdot \underline{S}) - (\underline{C} \cdot \underline{\nabla})\, \underline{S} . \qquad \textbf{[4]}$$

(b) In spherical polar coordinates ($x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$), the line element is given by

$$d\underline{r} = dr\, \underline{\hat{e}}_r + r\, d\theta\, \underline{\hat{e}}_\theta + r \sin\theta\, d\phi\, \underline{\hat{e}}_\phi ,$$

so that

$$(d\underline{r})^2 = (dr)^2 + r^2\,(d\theta)^2 + r^2 \sin^2\theta\,(d\phi)^2 = h_r^2\,(dr)^2 + h_\theta^2\,(d\theta)^2 + h_\phi^2\,(d\phi)^2 .$$

Thus $h_r = 1$, $h_\theta = r$, and $h_\phi = r\sin\theta$. One can then accept for the marks

$$\underline{\nabla} f = \sum \frac{1}{h_i} \left(\frac{\partial f}{\partial x_i} \right) = \left(\frac{\partial f}{\partial r} \right) \underline{\hat{e}}_r + \frac{1}{r} \left(\frac{\partial f}{\partial \theta} \right) \underline{\hat{e}}_\theta + \frac{1}{r \sin\theta} \left(\frac{\partial f}{\partial \phi} \right) \underline{\hat{e}}_\phi \cdot$$

Alternatively, an argument on physical grounds is that the component of gradient along a particular basis vector $\hat{u}_i$ is the rate of change of f with distance when u_i changes by a small amount. The relation between $d\underline{r}$ and du_i can be read off from the original defining equation — it gives the same answer! **[4]**

For $f = x^2 + y^2$, in Cartesian coordinates

$$\underline{\nabla} f = 2(x\,\underline{\hat{e}}_x + y\,\underline{\hat{e}}_y)\ . \qquad \textbf{[1]}$$

In polar coordinates, $f = r^2 \sin^2\theta$, so that, using the above formula,

$$\underline{\nabla} f = 2r\sin^2\theta\,\underline{\hat{e}}_r + 2r\sin\theta\cos\theta\,\underline{\hat{e}}_\theta = 2r\sin\theta\,(\sin\theta\,\underline{\hat{e}}_r + \cos\theta\,\underline{\hat{e}}_\theta)\ . \qquad \textbf{[2]}$$

One can go directly to check the equality of these two forms, but some students might want to check the magnitudes first. In Cartesian coordinates

$$|\ \underline{\nabla} f\ |^2 = 4(x^2 + y^2) = 4r^2\sin^2\theta\ ,$$

whereas in polars it is

$$|\ \underline{\nabla} f\ |^2 = 4r^2(\sin^4\theta + \sin^2\theta\cos^2\theta) = 4r^2\sin^2\theta\ ,$$

and these two expressions are equal.

To show that the two vectors are identically equal, the student has either to work geometrically or use the given relation between the basis vectors:

$$\begin{aligned}
\underline{\hat{e}}_r &= \sin\theta\cos\phi\,\underline{\hat{e}}_x + \sin\theta\sin\phi\,\underline{\hat{e}}_y + \cos\theta\,\underline{\hat{e}}_z\ ,\\
\underline{\hat{e}}_\theta &= \cos\theta\cos\phi\,\underline{\hat{e}}_x + \cos\theta\sin\phi\,\underline{\hat{e}}_y - \sin\theta\,\underline{\hat{e}}_z\ ,\\
\underline{\hat{e}}_\phi &= -\sin\phi\,\underline{\hat{e}}_x + \cos\phi\,\underline{\hat{e}}_y\ ,
\end{aligned}$$

Now $2r\sin\theta\,(\sin\theta\,\underline{\hat{e}}_r + \cos\theta\,\underline{\hat{e}}_\theta) =$

$$\begin{aligned}
&2r\sin\theta\,[\sin\theta\,(\sin\theta\cos\phi\,\underline{\hat{e}}_x + \sin\theta\sin\phi\,\underline{\hat{e}}_y + \cos\theta\,\underline{\hat{e}}_z)\\
&\quad + \cos\theta\,(\cos\theta\cos\phi\,\underline{\hat{e}}_x + \cos\theta\sin\phi\,\underline{\hat{e}}_y - \sin\theta\,\underline{\hat{e}}_z)]\\
&\quad = 2r\sin\theta(\cos\phi\,\underline{\hat{e}}_x + \sin\phi\,\underline{\hat{e}}_y) = 2(x\,\underline{\hat{e}}_x + y\,\underline{\hat{e}}_y)\ . \qquad \textbf{[3]}
\end{aligned}$$

2. (a) The determinant of the matrix is

$$|\underline{A}| = \begin{vmatrix} 3 & 1 & -3 \\ 1 & 4 & 2 \\ -3 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 3 & 1 & -3 \\ 1 & 4 & 2 \\ 0 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -11 & -9 \\ 1 & 4 & 2 \\ 0 & 3 & 2 \end{vmatrix}$$

$$= -(-22 + 27) = -5\,. \qquad \textbf{[2]}$$

The matrix of minors and the cofactor matrix are

$$\underline{M} = \begin{pmatrix} 16 & 11 & 14 \\ 11 & 6 & 9 \\ 14 & 9 & 11 \end{pmatrix} \quad \text{and} \quad \underline{C} = \begin{pmatrix} 16 & -11 & 14 \\ -11 & 6 & -9 \\ 14 & -9 & 11 \end{pmatrix}\,. \qquad \textbf{[3]}$$

The adjoint matrix

$$\underline{A}^{\mathrm{adj}} = \underline{C}^T = \underline{C} \qquad \textbf{[1]}$$

because the matrix is symmetric.
Hence the inverse matrix

$$\underline{A}^{-1} = \frac{1}{|\underline{A}|}\,\underline{A}^{\mathrm{adj}} = \frac{1}{5}\begin{pmatrix} -16 & 11 & -14 \\ 11 & -6 & 9 \\ -14 & 9 & -11 \end{pmatrix}\,. \qquad \textbf{[1]}$$

Now

$$\underline{B} = \underline{D}\,\underline{A}^{-1} = \frac{1}{5}\begin{pmatrix} -2 & 12 & 11 \\ 14 & 17 & -3 \\ -5 & 16 & 19 \end{pmatrix}\begin{pmatrix} -16 & 11 & -14 \\ 11 & -6 & 9 \\ -14 & 9 & -11 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 5 & -2 \\ -2 & 4 & 1 \end{pmatrix}. \qquad \textbf{[3]}$$

Students who do the multiplication the wrong way around should find that

$$\underline{A}^{-1}\,\underline{D} = \frac{1}{5}\begin{pmatrix} 256 & -229 & -475 \\ -151 & 174 & 310 \\ 209 & -191 & -390 \end{pmatrix}\,,$$

but this gives a maximum of **two** marks.

(b) The three matrices have many properties in common. Choose one from Hermitean, zero trace, unitary, or having determinants equal to -1. **[1]**

$$\underline{\sigma}_1^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

$$\underline{\sigma}_2^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

$$\underline{\sigma}_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$ **[2]**

$$\underline{\sigma}_1 \, \underline{\sigma}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\underline{\sigma}_3 .$$ **[1]**

$$\underline{\sigma}_2 \, \underline{\sigma}_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\underline{\sigma}_3 .$$ **[1]**

$$Z = \exp\left[i\alpha \, \underline{\sigma}_2\right] = \sum_{n=0}^{\infty} \frac{1}{n!} \; (i\alpha)^n \; (\underline{\sigma}_2)^n \; .$$

Now if n is odd,

$$(\underline{\sigma}_2)^n = \underline{\sigma}_2 \; ,$$

whereas if n is even,

$$(\underline{\sigma}_2)^n = \underline{I} \; .$$ **[2]**

Hence

$$Z = \sum_{n=even} \frac{1}{n!} \; (i\alpha)^n \; \underline{I} + \sum_{n=odd} \frac{1}{n!} \; (i\alpha)^n \; \underline{\sigma}_2$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{1}{(2m)!} \alpha^{2m} \, \underline{I} + i \sum_{m=0}^{\infty} (-1)^m \frac{1}{(2m+1)!} \alpha^{2m+1} \, \underline{\sigma}_2 = \underline{I} \cos\alpha + i\underline{\sigma}_2 \sin\alpha .$$ **[3]**

3. For

$$\underline{A} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -2 & -1 \\ 2 & -1 & 1 \end{pmatrix}$$

$$|\underline{A}| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & -2 & -1 \\ 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{vmatrix} = 0\,.$$

(Here we have subtracted $2R_1$ from R_3 and R_1 from R_2.) Hence one of the eigenvalues is $\lambda_1 = 0$. **[2]**

To verify that $\underline{v}_1$ is the associated eigenvector,

$$(\underline{A} - \lambda_1 \underline{I})\,\underline{v}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 2 \\ 1 & -2 & -1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1+1-2 \\ 1-2+1 \\ 2-1-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

as required. It is a normalised eigenvector by inspection. **[3]**

The general eigenvalue equation is

$$|\underline{A} - \lambda \underline{I}| = 0 = \begin{vmatrix} 1-\lambda & 1 & 2 \\ 1 & -2-\lambda & -1 \\ 2 & -1 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)[(\lambda+2)(\lambda-1) - 1] - [(1-\lambda)+2] + 2[-1 + 2(2+\lambda)]$$

$$= (1-\lambda)(\lambda^2 + \lambda - 3) - (3-\lambda) + 2(3+2\lambda) = -\lambda^3 + 9\lambda\,.$$

Hence, the other two eigenvalues are $\lambda_2 = +3$ and $\lambda_3 = -3$. **[3]**

For the second eigenvalue,

$$(\underline{A} - \lambda_2 \underline{I})\,\underline{v}_2 = \begin{pmatrix} -2 & 1 & 2 \\ 1 & -5 & -1 \\ 2 & -1 & -2 \end{pmatrix} \begin{pmatrix} v_{12} \\ v_{22} \\ v_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

so that $v_{22} = 0$ and $v_{12} = v_{32}$. The normalised eigenvector is therefore **[2]**

$$\underline{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

[1]

Similarly for the third eigenvalue,

$$(\underline{A} - \lambda_3 \underline{I})\, \underline{v}_3 = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 1 & -1 \\ 2 & -1 & 4 \end{pmatrix} \begin{pmatrix} v_{13} \\ v_{23} \\ v_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} ,$$

so that

$$v_{13} + v_{23} - v_{33} = 0, \; ,$$
$$2v_{13} - v_{23} + 4v_{33} = 0.$$

This has solution $v_{13} = -v_{33}$ and $v_{23} = 2v_{33}$ and so

$$\underline{v}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} .$$

[2]

Now

$$\begin{aligned} \underline{v}_1 \cdot \underline{v}_2 &= \frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}} = 0 \,, \\ \underline{v}_2 \cdot \underline{v}_3 &= -\frac{1}{\sqrt{12}} + 0 + \frac{1}{\sqrt{12}} = 0 \,, \\ \underline{v}_3 \cdot \underline{v}_1 &= -\frac{1}{\sqrt{18}} + \frac{2}{\sqrt{18}} - \frac{1}{\sqrt{18}} = 0 \,. \end{aligned}$$

The vectors are therefore all mutually orthogonal. The normalisation factors are not necessary to get these marks. **[3]**

Taking the cross product,

$$\underline{v}_1 \times \underline{v}_2 = \frac{1}{\sqrt{6}} \begin{vmatrix} \underline{\hat{e}}_1 & \underline{\hat{e}}_2 & \underline{\hat{e}}_3 \\ 1 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix} = \frac{1}{\sqrt{6}} (\underline{\hat{e}}_1 - 2\underline{\hat{e}}_2 - \underline{\hat{e}}_3) = -\underline{v}_3 \,.$$

[2]

This last two results follow because:
a) they are orthogonal vectors due to the matrix $\underline{A}$ being real and symmetric (thus Hermitean), **[1]**
b) the eigenvectors have length one — they were normalised. Otherwise there would have been some overall scale in the cross-product relation. **[1]**

4. Look for a solution of the second order differential equation

$$(1-x^2)\,\frac{d^2y}{dx^2} - x\,\frac{dy}{dx} - y = 0\,,$$

in the form

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n\, x^{n+k}\,, \\ y' &= \sum_{n=0}^{\infty} a_n (n+k)\, x^{n+k-1}\,, \\ y'' &= \sum_{n=0}^{\infty} a_n (n+k)(n+k-1)\, x^{n+k-2}\,. \end{aligned}$$ **[1]**

Inserting these into the equation, we obtain

$$\sum_{n=0}^{\infty} a_n(n+k)(n+k-1)\,x^{n+k-2} - \sum_{n=0}^{\infty} a_n(n+k)(n+k-1)\,x^{n+k}$$

$$-\sum_{n=0}^{\infty} a_n(n+k)\,x^{n+k} - \sum_{n=0}^{\infty} a_n\,x^{n+k} = 0\,.$$ **[1]**

Grouping like powers together, this simplifies to

$$\sum_{n=0}^{\infty} a_n\,(n+k)(n+k-1)\,x^{n+k-2} = \sum_{n=0}^{\infty} a_n\,\Big[(n+k)^2+1\Big]\,x^{n+k}\,.$$ **[2]**

If this is to be true for a range of values of x, it must be true power by power in x. The lowest power comes from $n=0$ in the first term. Since there is no x^{k-2} power in the second term, we demand that

$$a_0\,k(k-1) = 0\,.$$

However, by assumption, $a_0 \neq 0$ so that $k=0$ or $k=1$. **[2]**

To get the recurrence relation, change the dummy index so that we have the same powers of x everywhere by putting $n \to n+2$ in the first term:

$$\sum_{n=-1}^{\infty} a_{n+2}\,(n+k+1)(n+k+2)\,x^{n+k} = \sum_{n=0}^{\infty} a_n\,\Big[(n+k)^2+1\Big]\,x^{n+k}\,.$$ **[2]**

This gives us immediately the recurrence relation

$$\frac{a_{n+2}}{a_n} = \frac{(n+k)^2+1}{(n+k+2)(n+k+1)}\,,$$ **[2]**

with $k = 0$ or $k = 1$.

The series converges if, when $n \to \infty$,

$$\left|\frac{a_{n+2}x^{n+2}}{a_n x^n}\right| = \left|\frac{a_{n+2}}{a_n}\right| |x^2| \ < \ 1\,. \qquad \textbf{[1]}$$

This means that we need for convergence

$$\left|\frac{(n+k)^2+1}{(n+k+2)(n+k+1)}\right| |x^2| \to |x^2| < 1\,.$$

Thus the series will converge for all $-1 < x < +1$. **[2]**

Trying

$$y = \cosh\left(\arcsin x\right)\ ,$$

$$y' = \sinh\left(\arcsin x\right)\ \frac{1}{\sqrt{1-x^2}}\ , \qquad \textbf{[1]}$$

$$y'' = \cosh\left(\arcsin x\right)\ \frac{1}{1-x^2} + \sinh\left(\arcsin x\right)\ \frac{x}{(1-x^2)^{3/2}}\ . \qquad \textbf{[2]}$$

Inserting this into the differential equation gives

$$\cosh\left(\arcsin x\right) + \sinh\left(\arcsin x\right)\ \frac{x}{\sqrt{1-x^2}} - x\sinh\left(\arcsin x\right)\ \frac{1}{\sqrt{1-x^2}} - \cosh\left(\arcsin x\right) = 0,$$

as expected. **[2]**

Now the inverse sine function only gives real solutions for $|x| < 1$ and so we should expect something strange to happen at these limits. In fact the series diverges there. **[2]**

Students could argue instead that the equation has a regular singularity at $x = \pm 1$. Though true, this is not what was asked for and would only get one mark.

5. We are given that

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \, .$$

Integrating both sides with respect to x from $-\pi$ to π gives

$$\int_{-\pi}^{\pi} f(x)\, dx = a_0\,\pi + \sum_{n=1}^{\infty} a_n \, \frac{1}{n} \sin nx \Big|_{-\pi}^{\pi} - \sum_{n=1}^{\infty} a_n \, \frac{1}{n} \cos nx \Big|_{-\pi}^{\pi} = a_0\,\pi \, . \qquad \textbf{[1]}$$

Multiplying both sides by $\cos mx$ and integrating from $-\pi$ to π gives (for $m > 0$)

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx \, dx \, . \qquad \textbf{[1]}$$

We have here used the fact that the integrals of the terms involving a_0 and b_n will vanish. But the cosine functions are orthogonal over this interval:

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \pi \, \delta_{m\,n} \, . \qquad \textbf{[1]}$$

The Kronecker-delta just picks out the term in the sum where $m = n$, which means that

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, \cos nx \, dx \, , \qquad \textbf{[1]}$$

a formula that is also valid when $n = 0$.

In exactly the same way, multiplying both sides by $\sin mx$ and integrating yields

$$\int_{-\pi}^{\pi} f(x) \sin mx \, dx = \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin mx \, dx \, , \qquad \textbf{[1]}$$

and thus

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, \sin nx \, dx \, . \qquad \textbf{[1]}$$

$$\int_{-\pi}^{\pi} e^{\lambda x} e^{inx} \, dx = \int_{-\pi}^{\pi} e^{\lambda x} \left[\cos nx + i \sin nx\right] \, dx = \frac{1}{\lambda + in} \, e^{(\lambda + in)x} \Big|_{-\pi}^{\pi}$$

$$= (-1)^n \, \frac{2}{\lambda + in} \sinh \lambda\pi \, . \qquad \textbf{[2]}$$

Comparing real and imaginary parts of this equation, we see that

$$\begin{aligned} \int_{-\pi}^{\pi} e^{\lambda x} \cos nx \, dx &= 2(-1)^n \sinh \lambda\pi \, \frac{\lambda}{\lambda^2 + n^2} \\ \int_{-\pi}^{\pi} e^{\lambda x} \sin nx \, dx &= -2(-1)^n \sinh \lambda\pi \, \frac{n}{\lambda^2 + n^2} \, , \end{aligned} \qquad \textbf{[2]}$$

Using **INSTEAD** integration by parts, things get rather messy and the following shows the alternative distribution of the previous four marks.

$$I = \int_{-\pi}^{\pi} e^{\lambda x} \cos nx \, dx = \frac{1}{\lambda} \int_{-\pi}^{\pi} \cos nx \, d\left(e^{\lambda x}\right)$$

$$= \frac{1}{\lambda} \cos nx \, e^{\lambda x}\Big|_{-\pi}^{\pi} + \frac{n}{\lambda} \int_{-\pi}^{\pi} \sin nx \, e^{\lambda x} \, dx = (-1)^n \, \frac{2}{\lambda} \sinh \lambda\pi + \frac{n}{\lambda^2} \int_{-\pi}^{\pi} \sin nx \, d\left(e^{\lambda x}\right)$$

$$= (-1)^n \, \frac{2}{\lambda} \sinh \lambda\pi + \frac{n}{\lambda^2} \sin nx \, e^{-\lambda x}\Big|_{-\pi}^{\pi} - \frac{n^2}{\lambda^2} \int_{-\pi}^{\pi} \cos nx \, e^{-\lambda x} \, dx$$

$$= (-1)^n \, \frac{2}{\lambda} \sinh \lambda\pi - \frac{n^2}{\lambda^2} \, I \, . \quad \mathbf{(2)}$$

Hence

$$I = 2(-1)^n \sinh \lambda\pi \, \frac{\lambda}{\lambda^2 + n^2} \, . \quad \mathbf{(1)}$$

The integral involving $\sin nx$ can be done in an analogous manner but one can also recognise that this integral occurs during the above integration by parts. Defining

$$J = \int_{-\pi}^{\pi} e^{\lambda x} \sin nx \, dx \, ,$$

we see that

$$I = (-1)^n \, \frac{2}{\lambda} \sinh \lambda\pi + \frac{n}{\lambda} \, J \, ,$$

from which it follows that

$$J = -2(-1)^n \sinh \lambda\pi \, \frac{n}{\lambda^2 + n^2} \, . \quad \mathbf{(1)}$$

Use of these integrals in the Fourier series gives immediately

$$\begin{aligned} e^{\lambda x} \quad &= \quad \frac{1}{\lambda\pi} \sinh \lambda\pi + 2 \sinh \lambda\pi \, \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\lambda}{\lambda^2 + n^2} \, (-1)^n \cos nx \\ &\quad -2 \sinh \lambda\pi \, \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{n}{\lambda^2 + n^2} \, (-1)^n \sin nx \, . \end{aligned} \qquad \mathbf{[2]}$$

Parseval's theorem states that the average value of f^2 is given by

$$< \; f^2(x) \; > = \frac{1}{2\pi} \int_{-\pi}^{+\pi} [f(x)]^2 \, dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \, . \qquad \mathbf{[2]}$$

$$< \; f^2(x) \; > = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{2\lambda x} \, dx = \frac{1}{4\lambda\pi} \; e^{2\lambda x}\Big|_{-\pi}^{\pi} = \frac{1}{2\lambda\pi} \sinh 2\lambda\pi \, . \qquad \mathbf{[2]}$$

The right hand side of Parseval's identity in this case is

$$\text{RHS} = \left(\frac{\sinh \lambda\pi}{\pi}\right)^2 \left[\frac{1}{\lambda^2} + 2\sum_{n=1}^{\infty}\left(\frac{\lambda^2}{(\lambda^2+n^2)^2} + \frac{n^2}{(\lambda^2+n^2)^2}\right)\right] \qquad \textbf{[2]}$$

Thus

$$\frac{1}{\lambda\pi}\sinh\lambda\pi\cosh\lambda\pi = \sinh^2\lambda\pi\frac{1}{\pi^2}\left[\frac{1}{\lambda^2} + 2\sum_{n=1}^{\infty}\frac{1}{\lambda^2+n^2}\right]$$

and so

$$\sum_{n=1}^{\infty}\frac{\lambda^2}{\lambda^2+n^2} = \tfrac{1}{2}\left(\pi\lambda\coth\lambda\pi - 1\right)\ . \qquad \textbf{[2]}$$

6. (a) Look for a solution of the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial Z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 Z}{\partial \theta^2}=\frac{1}{v^2}\frac{\partial^2 Z}{\partial t^2},$$

in the separated form

$$Z(r,\theta,t)=R(r)\times\Theta(\theta)\times T(t).$$

Hence

$$\Theta\,T\,\frac{d}{dr}\left(r\,\frac{dR}{dr}\right)+R\,T\,\frac{1}{r^2}\,\frac{d^2\Theta}{d\theta^2}=R\,\Theta\,\frac{1}{v^2}\,\frac{d^2T}{dt^2}.$$

After dividing through by $R\,\Theta\,T$,

$$\frac{1}{R}\,\frac{1}{r}\,\frac{d}{dr}\left(r\,\frac{dR}{dr}\right)+\frac{1}{\Theta}\,\frac{1}{r^2}\,\frac{d^2\Theta}{d\theta^2}=\frac{1}{T}\,\frac{1}{v^2}\,\frac{d^2T}{dt^2},$$ **[1]**

we see that the RHS depends purely upon time, whereas the LHS is independent of t. Therefore they are both equal to a constant, which we put equal to $-k^2$. Then

$$\frac{d^2T}{dt^2}+k^2v^2\,T=0,$$ **[1]**

$$\frac{r}{R}\,\frac{d}{dr}\left(r\,\frac{dR}{dr}\right)+k^2r^2+\frac{1}{\Theta}\,\frac{d^2\Theta}{d\theta^2}=0.$$

Now the third term above depends purely upon θ while the first two are functions purely of r. Hence they must both be constant, say equal to $-n^2$. Then the r and θ dependence factorises into two equations

$$r\,\frac{d}{dr}\left(r\,\frac{dR}{dr}\right)+(k^2r^2-n^2)\,R=0,$$ **[1]**

$$\frac{d^2\Theta}{d\theta^2}+n^2\Theta=0.$$ **[1]**

The θ equation has solutions

$$\Theta=a_n\cos n\theta+b_n\sin n\theta.$$ **[1]**

The drumhead is clearly in the same position at θ and $\theta+2\pi$, which means that n must be an *integer*. **[1]**

The solution for $T(t)$ which vanishes at $t=0$ is

$$T=\sin(kvt).$$ **[1]**

Since Bessel's equation has not been covered in the course, students will (in general) not have seen the equation for $R(r)$. Letting $\rho = kr$, this becomes

$$\rho \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + (\rho^2 - n^2)\, R = 0\,,$$

which has solutions of the form $R_n(\rho) = R_n(kr)$. Most students will probably miss the point about the argument being kr. **[1]**

The full solution is therefore

$$Z = R_n(kr)\sin(kvt)\left[a_n \cos n\theta + b_n \sin n\theta\right], \qquad \textbf{[1]}$$

where n is an integer.

We have not yet imposed the boundary condition on the circumference. This requires that $R_n(ka) = 0$. This will make k a discrete variable, though this point is not needed for the mark. **[1]**

(b) To obtain the first integral, multiply

$$(2n+1)x\,P_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)\,,$$

by $P_{n+1}(x)$ and integrate to get

$$(2n+1)\int_{-1}^{+1} P_{n+1}(x)\,x\,P_n(x)\,dx =$$

$$(n+1)\int_{-1}^{+1} P_{n+1}(x)\,P_{n+1}(x)\,dx + n\int_{-1}^{+1} P_{n+1}(x)\,P_{n-1}(x)\,dx = 2\frac{n+1}{2n+3} + 0. \qquad \textbf{[2]}$$

Hence

$$\int_{-1}^{+1} P_{n+1}(x)\,x\,P_n(x) = \frac{2(n+1)}{(2n+1)(2n+3)}\,. \qquad \textbf{[1]}$$

For the second, multiplying

$$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$$

by $P_n(x)$ and integrating gives

$$(2n+1)\int_{-1}^{+1}\left[P_n(x)\right]^2\,dx = \int_{-1}^{+1} P_n(x)\,P'_{n+1}(x)\,dx - \int_{-1}^{+1} P_n(x)\,P'_{n-1}(x)\,dx\,. \qquad \textbf{[1]}$$

However, the last integral vanishes because $P'_{n-1}(x)$ is a polynomial of degree $n-2$. **[1]**

Hence

$$\int_{-1}^{+1} P'_{n+1}(x)\,P_n(x)\,dx = 2\,. \qquad \textbf{[1]}$$

In the case of $n = 1$, $P_{n+1}(x) = \frac{1}{2}(3x^2 - 1)$ and $P_n(x) = x$.

$$\int_{-1}^{+1} P_{n+1}(x)\, x\, P_n(x)\, dx = \int_{-1}^{+1} \tfrac{1}{2} x^2 (3x^2 - 1)\, dx$$
$$= \left(\frac{3}{5} - \frac{1}{3}\right) = \frac{4}{15}\,,$$

which agrees with the formula. **[2]**

For the second identity, $P'_{n+1}(x) = 3x$, and

$$\int_{-1}^{+1} P'_{n+1}(x)\, P_n(x)\, dx = 3 \int_{-1}^{+1} x^2\, dx = 2\,.$$ **[2]**