Lecture 3 Jeans instability and star formation. Spherically symmetric accretion and stellar winds

We believe that at least some star formation takes place in interstellar gas clouds. Three pieces of observational evidence are that (1) associations of young, bright massive stars are found in nebulae; (2) nebulosity is seen in young, open clusters; and (3) infrared observations reveal young stellar objects (YSOs) obscured by gas.

3.1 Jeans instability

The linearized equations for small perturbations are equations (2.19). We shall consider the simplest possible system, which is a homogeneous cloud, infinite in all directions, so p_0 and ρ_0 are independent of position, as too is ψ_0 by virtue of the hydrostatic equation. Thus equation (1.19a) becomes

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla p' - \rho_0 \nabla \psi'. \qquad (3.1)$$

Taking the divergence of this equation, and using (2.19c) to eliminate $\nabla^2 \psi'$ gives

$$\rho_0 \frac{\partial (\nabla \cdot \mathbf{u})}{\partial t} = -\nabla^2 p' - 4\pi G \rho_0 \rho'. \qquad (3.2)$$

In a uniform medium, $q' = \delta q$ for any quantity q. Suppose that the gas is ideal and isothermal, so $\delta p / p = \delta \rho / \rho$. Hence

$$p' = a^2 \rho', \qquad (3.3)$$

where $a = \sqrt{p_0 / \rho_0}$ is the isothermal sound speed. Also in a uniform medium (19.b) becomes

$$\frac{\partial \rho'}{\partial t} = -\rho_0 \nabla \cdot \mathbf{u}. \qquad (3.4)$$

Hence, using equations (3.3) and (3.4) to eliminate p' and $\nabla \cdot \mathbf{u}$ from (3.2), we obtain

$$\frac{\partial^2 \rho'}{\partial t^2} = a^2 \nabla^2 \rho' + 4\pi G \rho_0 \rho'. \qquad (3.5)$$

This is a linear PDE for ρ', with coefficients that are independent of position and time. Hence we seek solutions $\rho' \propto \exp(i\omega t + i\mathbf{k}\cdot\mathbf{r})$. In this case $\partial\rho'/\partial t = i\omega\rho'$ and $\nabla\rho' = i\mathbf{k}\rho'$. Hence, equation (3.5) implies

$$-\omega^2 \rho' = -k^2 a^2 \rho' + 4\pi G \rho_0 \rho', \qquad (3.6)$$

where $k = |\mathbf{k}|$; and so for a nontrivial solution ($\rho' \neq 0$) we obtain the dispersion relation

$$\omega^2 = k^2 a^2 - 4\pi G \rho_0 . \qquad (3.7)$$

If k and ω are real, this represents an oscillation. However, if the right-hand side of (3.7) is negative, as it will be for sufficiently small k, then ω^2 will be negative and so the cloud will be unstable because there will be a solution ρ' where $|\exp(i\omega t)|$ grows exponentially with time. Thus the cloud is unstable to fluctuations of wavenumber k if

$$k^2 a^2 < 4\pi G \rho_0 ,$$

i.e. if

$$k^{-1} > \left(\frac{a^2}{4\pi G \rho_0}\right)^{1/2} \equiv (2\pi)^{-1} \lambda_J .$$

Now a real cloud is of finite size, so one cannot have arbitrarily large wavelengths λ, *i.e.* arbitrarily small wavenumbers. If the cloud is roughly sperical with radius R, one must have $\lambda < 2R$. So such a cloud is unstable to density perturbations if

$$R > R_J \equiv \frac{1}{2}\lambda_J = \frac{1}{2}\left(\frac{\pi a^2}{G\rho_0}\right)^{1/2}. \qquad (3.8)$$

If the density grows unstably, it is because mass is falling in to some region from surrounding regions. So put another way, the cloud collapses if its mass exceeds the critical Jeans mass

$$M_J = \frac{4}{3}\pi R_J^3 \rho_0 = \frac{\pi}{6}\rho_0 \lambda_J^3. \qquad (3.9)$$

Note the crucial role of the perturbation to the gravitational potential in this instability. If this had been neglected in equation (3.1), equation (3.7) would have reduced to $\omega^2 = k^2 a^2$, the dispersion relation for isothermal sound waves (cf equation (2.32)), which would give stable oscillatory solutions always.

For a typical HI region, $n_H \approx 10\,\mathrm{cm}^{-3}$ (the number of hydrogen atoms per cubic centimetre), so the density is about $2\times 10^{-23}\,\mathrm{g\,cm}^{-3}$. The temperature is about 100K. Hence, using $a^2 = \Re T/\mu$ with mean molecular weight $\mu \approx 1$, one gets that $\lambda_J \approx 10^{20}\,\mathrm{cm}$ (about 100 light years) and $M_J \approx 3\times 10^{37}\,\mathrm{g} \approx 10^4$ solar masses. But this is very much greater than the mass of the most massive known stars. Hence it cannot simply be that a cloud exceeding its Jeans mass collapses under its own self-gravity to form a single star [3.1].

If we suppose that the collapse continues isothermally (*i.e.* a^2 remains constant) then $\lambda_J \propto \rho^{-1/2}$ and similarly $M_J \propto \rho^{-1/2}$. Hence, as the cloud collapses and the density grows, the Jeans mass decreases. One can therefore envisage that subcondensations form in the cloud as smaller and smaller masses start to collapse in upon themselves. This picture is known as fragmentation. However, fragmentation must not continue indefinitely, as we wish eventually to form stars with observed stellar masses. Now in the later stages of collapse, the cloud presumably becomes too opaque for radiation to smooth out temperature fluctuations, so that isothermal collapse is no longer a good assumption. If we assume that we enter the opposite regime, of adiabatic collapse, then

$$\frac{1}{p}\frac{Dp}{Dt} = \frac{\Gamma_1}{\rho}\frac{D\rho}{Dt}$$

so $p \propto \rho^{\Gamma_1}$. Instead of the isothermal sound speed we use the adiabatic sound speed c, with $c^2 = \Gamma_1 p/\rho \propto \rho^{\Gamma_1 - 1}$. From (3.8) and (3.9) we then have $\lambda_J \propto \rho^{(\Gamma_1 - 2)/2}$ and $M_J \propto \rho^{(3\Gamma_1/2 - 2)}$. E.g., for $\Gamma_1 = 5/3, \quad M_J \propto \rho^{1/2}$. Hence the Jeans mass is no longer decreasing with increasing cloud density, and fragmentation halts.

The problem with the fragmentation picture as propounded above is that dispersion relation (3.7) implies that ω^2 is most negative for the smallest values of k. Thus, the cloud collapses fastest at the largest scales. Although the cloud becomes unstable on smaller scales as the density increases, these smaller-scale perturbations would get overwhelmed by the faster overall collapse of the cloud.

Exercise 3.1 You should be at least a little concerned as to whether the infinite homogeneous cloud obeys the equilibrium equations (2.15). Think about this.

Obviously, although the model of a homogenous cloud has led us to a useful criterion (the Jeans mass) for the collapse of a cloud under its own self-gravity, the model is too simplistic to explain in detail the formation of real stars. The Galaxy, and virtually everything within it, rotates. Unless it loses angular momentum by some mechanism, a rotating cloud will rotate faster as it collapses. Indeed, the centrifugal force will eventually balance self-gravity so that the cloud can no longer collapse perpendicularly to the rotation axis. (It can still collapse along the rotation axis.) Thus the cloud tends to flatten into a disk. This could lead ultimately to the formation of planetary systems such as our own solar system.

3.2 Bernoulli's theorem

The problem of how material falls radially onto a central object is sometimes called the Bondi problem, after [3.3] (see also [3.2]).

As a preliminary, we prove a standard result in fluid dynamics, Bernoulli's theorem for steady inviscid flow. A standard identity from vector calculus gives

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla\left(\frac{1}{2}u^2\right) - \mathbf{u} \times (\nabla \times \mathbf{u}), \qquad (3.10)$$

where $u = |\mathbf{u}|$. Neglecting viscosity and setting time derivatives to zero, and using equation (3.10), the equation of motion becomes

$$\nabla\left(\frac{1}{2}u^2\right) - \mathbf{u}\times(\nabla\times\mathbf{u}) = -\frac{1}{\rho}\nabla p - \nabla\psi. \qquad (3.11)$$

Suppose that the flow is baratropic - so the pressure is a known function of density, $p = p(\rho)$. (This is a common simplification in astrophysical fluid dynamics - by assuming a given relation between pressure and density, we can often avoid needing to give specific consideration to the energy equation). Define the enthalpy

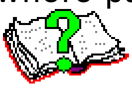
$$h = \int \frac{dp}{\rho}, \qquad (3.12)$$

so $\nabla h = \rho^{-1}\nabla p$. Then equation (3.11) becomes

$$\mathbf{u}\times(\nabla\times\mathbf{u}) = \nabla\left(\frac{1}{2}u^2 + h + \psi\right), \qquad (3.13)$$

from which it follows by taking the dot product with $\mathbf{u}$ that

$$\mathbf{u}\cdot\nabla\left(\frac{1}{2}u^2 + h + \psi\right) = 0. \qquad (3.14)$$

This result shows that $\frac{1}{2}u^2 + h + \psi$ is constant along a streamline, *i.e.* a line everywhere parallel to $\mathbf{u}$. For a simple physical derivation of the Bernoulli's theorem, press .

An everyday application of Bernoulli's theorem (3.14) is to consider flow from a kitchen tap. In this case ρ is essentially uniform, so $h = p/\rho$. Bernoulli's theorem says that

$$\frac{1}{2}u^2 + \frac{p}{\rho} - gz \qquad (3.15)$$

is constant along streamlines, z being measured downwards. In particular, along a surface streamline the pressure p is everywhere equal to the atmospheric pressure (constant). Therefore as the flow falls from the tap, (3.15) implies that u increases. Now suppose that the stream from the tap has horizontal cross-sectional area

$A(z)$. The direction of u is essentially vertical, and the flow is incompressible. Thus mass conservation implies that $\rho u A$, the mass flow per unit time through a horizontal plane, is independent of z. Thus, as the water falls, u increases and A decreases.

Note, however, that for sufficiently small A, the surface tension cannot be ignored. Also the flow is not stable: a Kelvin-Helmholtz instability sets up an oscillatory disturbance on the surface.

3.3 The de Laval nozzle

We consider now how the picture of incompressible-type flow from a tap will be modified in a situation where the compressibility of the fluid is important. We still take the flow to be steady, baratropic and one-dimensional. As an example, consider the flow from a jet engine. The spatial variation of the cross-section A is given (by the walls of the combustion chamber), and we can neglect gravity. Bernoulli and mass conservation imply

$$\left.\begin{aligned} \tfrac{1}{2}u^2 + h &= \text{constant} \\ \rho u A &= \text{constant} \end{aligned}\right\} \qquad (3.16)$$

The spatial variation of A induces variations in the other quantities. The first of equations (3.16) implies that

$$u\,du + \frac{c^2}{\rho}d\rho = 0, \qquad (3.17)$$

where $c^2 = dp/d\rho$ (c is thus the sound speed). This equation, relating changes in density and in velocity, can be rewritten

$$\frac{d\rho}{\rho} = -\mathcal{M}^2\frac{du}{u}, \qquad (3.18)$$

where $\mathcal{M} \equiv u/c$ is the Mach number.

Note that if $\mathcal{M} \ll 1$, fractional changes in density are negligible compared with fractional changes in u. Thus we can generally neglect compressibility if $\mathcal{M} \ll 1$. On the other hand, supersonic flight past obstacles involves substantial compressions and expansions. Also, equation (3.18) and the second of equations (3.16) together give

$$\left(1-\mathcal{M}^2\right)\frac{du}{u} = -\frac{dA}{A}. \qquad (3.19)$$

Exercise 3.2 Derive equation (3.19).

We now consider equation (3.19) for three cases. If $\mathcal{M} < 1$ (subsonic flow), an increase in u corresponds to a decrease in A. This was the situation for the running tap.

If $\mathcal{M} > 1$ (supersonic flow), an increase in u requires an increase in the area of the nozzle! The explanation for this is that the density decreases faster than the velocity increases (equation 3.18); thus mass conservation requires an increase in A.

For $\mathcal{M} = 1$, the sonic transition between subsonic and supersonic flow, for a smooth transition (du finite) equation (3.19) implies that dA must be zero at the transition point. This is important for jet design. The nozzle needs to converge (A decreasing) to provide the necessary acceleration from subsonic speeds, but should smoothly stop converging and start to diverge where the flow gets to supersonic speeds. In astrophysical situations, the same acceleration can be achieved by external body forces, such as gravity.

3.4 The Bondi problem

We consider the steady, spherically symmetric accretion of gas onto a gravitating point mass M. We assume a baratropic flow, so $p = p(\rho)$. Also we neglect the self-gravity of the infalling gas, which is a good approximation if its total mass is much less than that of the central point mass.

The velocity is wholly in the inward radial direction. Since the flow is steady, integrating the continuity equation over the region between concentric spherical surfaces and using the divergence theorem gives the mass conservation equation

$$4\pi r^2 \rho u = \text{constant} = -\dot{M} \qquad (3.20)$$

where $\dot{M}$ is a positive constant. Bernoulli's theorem yields

$$\frac{1}{2}u^2 + h - \frac{GM}{r} = 0 \qquad (3.21)$$

where

$$h = \int_{\rho_\infty}^{\rho} \frac{dp}{\rho}, \qquad (3.22)$$

ρ_∞ being the density at infinity. Note that Bernoulli's theorem applies to a given radial streamline, and following a streamline out to infinity shows that the constant on the right-hand side of (3.21) is zero: for at infinity $u = 0$, and $h = 0$ there by equation (3.22). Since every point in space is on some radial streamline, and the constant is zero on each one of them, equation (3.21) holds not just on a single streamline but everywhere in space.

In the particular case of isothermal flow, $p = c_\infty^2 \rho$ where c_∞ is a constant. Evaluating equation (3.22) gives

$$h = c_\infty^2 \ln(\rho / \rho_\infty). \qquad (3.23)$$

A characteristic length is

$$r_B = \frac{GM}{c_\infty^2}. \qquad (3.24)$$

This is called the Bondi radius.

We may define a dimensionless radial variable, speed and density by

$$x = \frac{r}{r_B}, \qquad v = \frac{u}{c_\infty}, \qquad a = \frac{\rho}{\rho_\infty} \qquad (3.25)$$

and a dimensionless accretion rate λ by measuring $\dot{M}$ in units of a mass flux $\rho_\infty c_\infty$ across an area $4\pi r_B^2$:

$$\lambda = \frac{\dot{M}}{4\pi\rho_\infty (GM)^2 / c_\infty^3}. \qquad (3.26)$$

The governing equations (3.20), (3.21) can then be written in dimensionless form as

$$x^2 a v = \lambda \qquad (3.27)$$

and

$$\frac{1}{2}v^2 + H(a) - \frac{1}{x} = 0 \qquad (3.28)$$

where $H(a) = \ln a$ for isothermal flow. Equations (3.27) and (3.28) imply that changes in the different dimensionless variables are related by

$$2\frac{dx}{x} + \frac{da}{a} + \frac{dv}{v} = 0, \qquad (3.29)$$

$$v\,dv + \frac{da}{a} + \frac{dx}{x^2} = 0; \qquad (3.30)$$

and eliminating da between these gives a relation between dx and dv:

$$\left(v - \frac{1}{v}\right)dv = \left(\frac{2}{x} - \frac{1}{x^2}\right)dx. \qquad (3.31)$$

Exercise 3.3 Derive equations (3.27) - (3.31) for yourself.

The sonic transition ($v = 1$) occurs when $x = 1/2$. At this point, equation (3.28) implies that $a = \exp(3/2)$ and (3.27) gives $\lambda = (1/4)\exp(3/2) \equiv \lambda_c$. This implies that

$$\dot{M} = \lambda_c 4\pi\rho_\infty \frac{(GM)^2}{c_\infty^3} \qquad (3.32)$$

and so this is the rate at which mass will be accreted (steadily) onto a point mass, assuming spherical symmetry and isothermal flow.

3.5 The Parker solar-wind solution

Parker's model for a thermally-driven solar wind is closely related mathematically to the Bondi problem. It is discussed in detail in [3.4]. There are some differences since the material is now being accelerated from rest at the central object to a large velocity far away, and the conditions at infinity are no longer specified *a priori*. You are encouraged to look at Q. 3f of Problem Set 2 of [3.2]. For a review on the subject of the solar wind, see [3.5].

LITERATURE

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