EXERCISE 5.1

Consider a thin spherical shell with inner radius r and outer radius $r + dr$, with volume $V = 4\pi r^2 dr$. Vector flux to the outside of V across its boundary S is

$$\int_S \mathbf{u} \cdot \hat{\mathbf{n}}\, dS = 4\pi(r + dr)^2 u(r + dr) - 4\pi r^2 u(r) =$$

$$= 8\pi r u(r) dr + 4\pi r^2 \frac{du}{dr} dr = 4\pi \frac{d}{dr}\left(r^2 u\right) dr$$

with dr small: here we use $u(r + dr) = u(r) + (du / dr)dr$ and linearize in dr. According to the divergence theorem, this vector flux is

$$\int_V \nabla \cdot \mathbf{u}\, dV = (\nabla \cdot \mathbf{u})V = (\nabla \cdot \mathbf{u}) 4\pi r^2 dr$$

since $\nabla \cdot \mathbf{u}$ can be considered uniform in V (dr is small). Comparing the two expressions, we get

$$\nabla \cdot \mathbf{u} = \frac{1}{r^2} \frac{d}{dr}\left(r^2 u\right).$$

From equations (5.3b,c), we have

$$r^2 \frac{d}{dr}(\psi_1) = -4\pi G r^2 \rho_0 U + \text{const}.$$

For this equation to be valid at $r = 0$, the integration constant needs to be zero.

Now use the equation (5.3d) to express ρ_1 through p_1 and U: the result is

$$\rho_1 = \frac{1}{c^2} p_1 - \left(\frac{d\rho_0}{dr} - \frac{1}{c^2} \frac{dp_0}{dr}\right) U = \frac{1}{c^2} p_1 + \frac{\rho_0}{g_0} N^2 U.$$

Finally, substitute this expression into (5.3a,b) to get (5.4b,a).