Electromagnetism and Optics

An introductory course

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Contents

1 Introduction

These lecture notes are designed to accompany a lower-division college survey course covering electricity, magnetism, and optics. Students are expected to be familiar with calculus and elementary mechanics.

2 Vectors

2.1 Vector Algebra

In applied mathematics, physical quantities are (predominately) represented by two distinct classes of objects. Some quantities, denoted *scalars*, are represented by *real numbers*. Others, denoted *vectors*, are represented by directed line elements in space: *e.g.*, $\overrightarrow{\rm PQ}$ in see Fig. 2.1. Note that line elements (and, therefore, vectors) are movable, and do not carry intrinsic position information: *i.e.*, in Fig. 2.2, $\overrightarrow{\rm PS}$ and $\overrightarrow{\rm QR}$ are considered to be the *same* vector. In fact, vectors just possess a magnitude and a direction, whereas scalars possess a magnitude but no direction. By convention, vector quantities are denoted by bold-faced characters (*e.g.*, **a**) in typeset documents. Vector addition can be represented using a parallelogram: *e.g.*, $\overrightarrow{\rm PR} = \overrightarrow{\rm PQ} + \overrightarrow{\rm QR}$ in Fig. 2.2. $\overrightarrow{\rm PR}$ is said to be the *resultant* of $\overrightarrow{\rm PQ}$ and $\overrightarrow{\rm QR}$. Suppose that $\mathbf{a} \equiv \overrightarrow{\rm PQ} \equiv \overrightarrow{\rm SR}$, $\mathbf{b} \equiv \overrightarrow{\rm QR} \equiv \overrightarrow{\rm PS}$, and $\mathbf{c} \equiv \overrightarrow{\rm PR}$. It follows, from Fig. 2.2, that vector addition is *commutative*: *i.e.*, $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (since $\overrightarrow{\rm PR}$ is also the resultant of $\overrightarrow{\rm PS}$ and $\overrightarrow{\rm SR}$). It can also be shown that the *associative* law holds: *i.e.*, $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$.

There are two general approaches to vector analysis. The *geometric* approach is based on drawing line elements in space, and then making use of the theorems of Euclidian geometry. The *coordinate* approach assumes that space is defined by Cartesian coordinates, and uses these to characterize vectors. In Physics, we

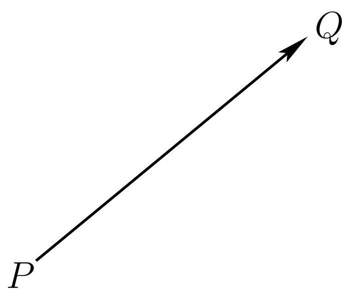


Figure 2.1: *A directed line element.*

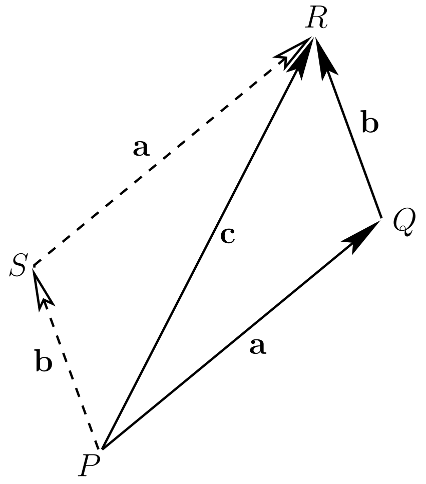


Figure 2.2: *Vector addition.*

generally adopt the second approach, because it is far more convenient.

In the coordinate approach, a vector is denoted as the row matrix of its components along each of the Cartesian axes (the x-, y-, and z-axes, say):

$$\mathbf{a} \equiv (a_x,\ a_y,\ a_z). \tag{2.1}$$

Here, a_x is the x-coordinate of the "head" of the vector minus the x-coordinate of its "tail," *etc*. If $\mathbf{a} \equiv (a_x, a_y, a_z)$ and $\mathbf{b} \equiv (b_x, b_y, b_z)$ then vector addition is defined

$$\mathbf{a} + \mathbf{b} \equiv (a_x + b_x,\ a_y + b_y,\ a_z + b_z). \tag{2.2}$$

If $\mathbf{a}$ is a vector and n is a scalar then the product of a scalar and a vector is defined

$$n\,\mathbf{a} \equiv (n\,a_x,\ n\,a_y,\ n\,a_z). \tag{2.3}$$

The vector $n\,\mathbf{a}$ is interpreted as a vector which points in the same direction as $\mathbf{a}$ (or in the opposite direction, if $n < 0$), and is $|n|$ times as long as $\mathbf{a}$. It is clear that vector algebra is *distributive* with respect to scalar multiplication: *i.e.*, $n\,(\mathbf{a} + \mathbf{b}) = n\,\mathbf{a} + n\,\mathbf{b}$.

Unit vectors can be defined in the x-, y-, and z-directions as $\mathbf{e}_x \equiv (1, 0, 0)$, $\mathbf{e}_y \equiv (0, 1, 0)$, and $\mathbf{e}_z \equiv (0, 0, 1)$. Any vector can be written in terms of these unit vectors: *i.e.*,

$$\mathbf{a} = a_x\,\mathbf{e}_x + a_y\,\mathbf{e}_y + a_z\,\mathbf{e}_z. \tag{2.4}$$

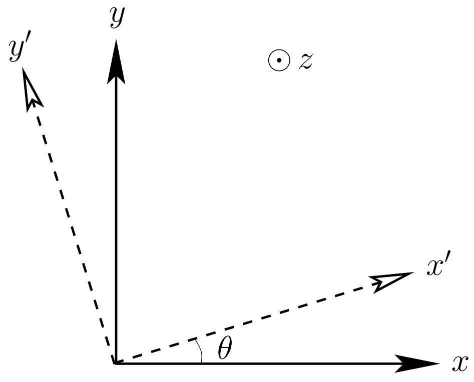


Figure 2.3: *Rotation of the basis about the z-axis.*

In mathematical terminology, three vectors used in this manner form a *basis* of the vector space. If the three vectors are mutually perpendicular then they are termed *orthogonal basis vectors*. However, any set of three non-coplanar vectors can be used as basis vectors.

Examples of vectors in Physics are displacements from an origin,

$$\mathbf{r} = (x,\, y,\, z), \tag{2.5}$$

and velocities,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \lim_{\delta t\to 0}\frac{\mathbf{r}(t+\delta t)-\mathbf{r}(t)}{\delta t}. \tag{2.6}$$

Suppose that we transform to a new orthogonal basis, the x'-, y'-, and z'-axes, which are related to the x-, y-, and z-axes via a rotation through an angle θ around the z-axis—see Fig. 2.3. In the new basis, the coordinates of the general displacement $\mathbf{r}$ from the origin are (x', y', z'). These coordinates are related to the previous coordinates via the transformation

$$x' = x\cos\theta + y\sin\theta, \tag{2.7}$$

$$y' = -x\sin\theta + y\cos\theta, \tag{2.8}$$

$$z' = z. \tag{2.9}$$

Now, we do not need to change our notation for the displacement in the new basis. It is still denoted $\mathbf{r}$. The reason for this is that the magnitude and direction of **r** are *independent* of the choice of basis vectors. The coordinates of **r** *do* depend on the choice of basis vectors. However, they must depend in a very specific manner [*i.e.*, Eqs. (2.7)–(2.9)] which preserves the magnitude and direction of **r**.

Since any vector can be represented as a displacement from an origin (this is just a special case of a directed line element), it follows that the components of a general vector **a** must transform in an similar manner to Eqs. (2.7)–(2.9). Thus,

$$\begin{aligned} a_{x'} &= a_x \cos\theta + a_y \sin\theta, && (2.10)\\ a_{y'} &= -a_x \sin\theta + a_y \cos\theta, && (2.11)\\ a_{z'} &= a_z, && (2.12) \end{aligned}$$

with analogous transformation rules for rotation about the y- and z-axes. In the coordinate approach, Eqs. (2.10)–(2.12) constitute the *definition* of a vector. The three quantities (a_x, a_y, a_z) are the components of a vector provided that they transform under rotation like Eqs. (2.10)–(2.12). Conversely, (a_x, a_y, a_z) *cannot* be the components of a vector if they do not transform like Eqs. (2.10)–(2.12). Scalar quantities are *invariant* under transformation. Thus, the individual components of a vector (a_x, say) are real numbers, but they are *not* scalars. Displacement vectors, and all vectors derived from displacements, automatically satisfy Eqs. (2.10)–(2.12). There are, however, other physical quantities which have both magnitude and direction, but which are not obviously related to displacements. We need to check carefully to see whether these quantities are vectors.

2.2 Vector Area

Suppose that we have a plane surface of scalar area S. We can define a vector area **S** whose magnitude is S, and whose direction is perpendicular to the plane, in the sense determined by a right-hand grip rule on the rim—see Fig. 2.4. This quantity clearly possesses both magnitude and direction. But is it a true vector? Well, we know that if the normal to the surface makes an angle α_x with the x-axis then the area seen looking along the x-direction is $S \cos\alpha_x$. Let this be the x-component of **S**. Similarly, if the normal makes an angle α_y with the y-axis then the area seen looking along the y-direction is $S \cos\alpha_y$. Let this be the y-component of **S**.

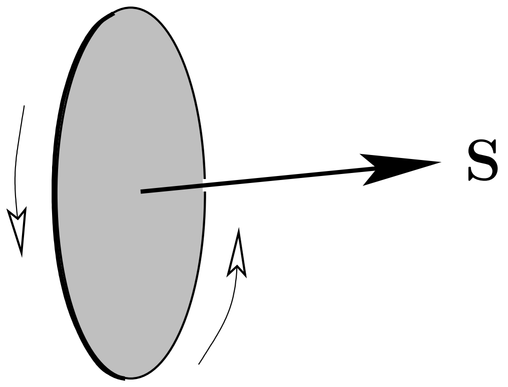


Figure 2.4: *A vector area.*

If we limit ourselves to a surface whose normal is perpendicular to the z-direction then $\alpha_x = \pi/2 - \alpha_y = \alpha$. It follows that $\mathbf{S} = S\,(\cos\alpha, \sin\alpha, 0)$. If we rotate the basis about the z-axis by θ degrees, which is equivalent to rotating the normal to the surface about the z-axis by $-\theta$ degrees, then

$$S_{x'} = S\,\cos\,(\alpha - \theta) = S\,\cos\alpha\,\cos\theta + S\,\sin\alpha\,\sin\theta = S_x\,\cos\theta + S_y\,\sin\theta, \quad (2.13)$$

which is the correct transformation rule for the x-component of a vector. The other components transform correctly as well. This proves that a vector area is a true vector.

According to the vector addition theorem, the projected area of two plane surfaces, joined together at a line, looking along the x-direction (say) is the x-component of the resultant of the vector areas of the two surfaces. Likewise, for many joined-up plane areas, the projected area in the x-direction, which is the same as the projected area of the *rim* in the x-direction, is the x-component of the resultant of all the vector areas: *i.e.*,

$$\mathbf{S} = \sum_i \mathbf{S}_i. \quad (2.14)$$

If we approach a limit, by letting the number of plane facets increase, and their areas reduce, then we obtain a continuous surface denoted by the resultant vector area

$$\mathbf{S} = \sum_i \delta\mathbf{S}_i. \quad (2.15)$$

It is clear that the projected area of the rim in the x-direction is just S_x. Note that the vector area of a given surface is completely determined by its rim. So, two different surfaces sharing the same rim both possess the *same* vector area.

In conclusion, a loop (not all in one plane) has a vector area **S** which is the resultant of the vector areas of any surface ending on the loop. The components of **S** are the projected areas of the loop in the directions of the basis vectors. As a corollary, a closed surface has $\mathbf{S} = \mathbf{0}$, since it does not possess a rim.

2.3 The Scalar Product

A scalar quantity is invariant under all possible rotational transformations. The individual components of a vector are not scalars because they change under transformation. Can we form a scalar out of some combination of the components of one, or more, vectors? Suppose that we were to define the "ampersand" product,

$$\mathbf{a} \,\&\, \mathbf{b} = a_x\, b_y + a_y\, b_z + a_z\, b_x = \text{scalar number}, \tag{2.16}$$

for general vectors **a** and **b**. Is $\mathbf{a} \,\&\, \mathbf{b}$ invariant under transformation, as must be the case if it is a scalar number? Let us consider an example. Suppose that $\mathbf{a} = (1,\, 0,\, 0)$ and $\mathbf{b} = (0,\, 1,\, 0)$. It is easily seen that $\mathbf{a} \,\&\, \mathbf{b} = 1$. Let us now rotate the basis through 45° about the z-axis. In the new basis, $\mathbf{a} = (1/\sqrt{2},\, -1/\sqrt{2},\, 0)$ and $\mathbf{b} = (1/\sqrt{2},\, 1/\sqrt{2},\, 0)$, giving $\mathbf{a} \,\&\, \mathbf{b} = 1/2$. Clearly, $\mathbf{a} \,\&\, \mathbf{b}$ is *not* invariant under rotational transformation, so the above definition is a bad one.

Consider, now, the *dot product* or *scalar product*,

$$\mathbf{a} \cdot \mathbf{b} = a_x\, b_x + a_y\, b_y + a_z\, b_z = \text{scalar number}. \tag{2.17}$$

Let us rotate the basis though θ degrees about the z-axis. According to Eqs. (2.10)–(2.12), in the new basis $\mathbf{a} \cdot \mathbf{b}$ takes the form

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (a_x \cos\theta + a_y \sin\theta)\,(b_x \cos\theta + b_y \sin\theta) \\ &\quad + (-a_x \sin\theta + a_y \cos\theta)\,(-b_x \sin\theta + b_y \cos\theta) + a_z\, b_z \\ &= a_x\, b_x + a_y\, b_y + a_z\, b_z. \end{aligned} \tag{2.18}$$

Thus, $\mathbf{a}\cdot\mathbf{b}$ *is* invariant under rotation about the z-axis. It can easily be shown that it is also invariant under rotation about the x- and y-axes. Clearly, $\mathbf{a}\cdot\mathbf{b}$ is a true scalar, so the above definition is a good one. Incidentally, $\mathbf{a}\cdot\mathbf{b}$ is the *only* simple combination of the components of two vectors which transforms like a scalar. It is easily shown that the dot product is commutative and distributive:

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= \mathbf{b}\cdot\mathbf{a},\\ \mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) &= \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}.\end{aligned} \tag{2.19}$$

The associative property is meaningless for the dot product, because we cannot have $(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}$, since $\mathbf{a}\cdot\mathbf{b}$ is scalar.

We have shown that the dot product $\mathbf{a}\cdot\mathbf{b}$ is coordinate independent. But what is the physical significance of this? Consider the special case where $\mathbf{a}=\mathbf{b}$. Clearly,

$$\mathbf{a}\cdot\mathbf{b} = a_x^{\,2}+a_y^{\,2}+a_z^{\,2} = \text{Length }(OP)^2, \tag{2.20}$$

if $\mathbf{a}$ is the position vector of P relative to the origin O. So, the invariance of $\mathbf{a}\cdot\mathbf{a}$ is equivalent to the invariance of the length, or magnitude, of vector $\mathbf{a}$ under transformation. The length of vector $\mathbf{a}$ is usually denoted $|a|$ (“the modulus of a”) or sometimes just a, so

$$\mathbf{a}\cdot\mathbf{a} = |a|^2 = a^2. \tag{2.21}$$

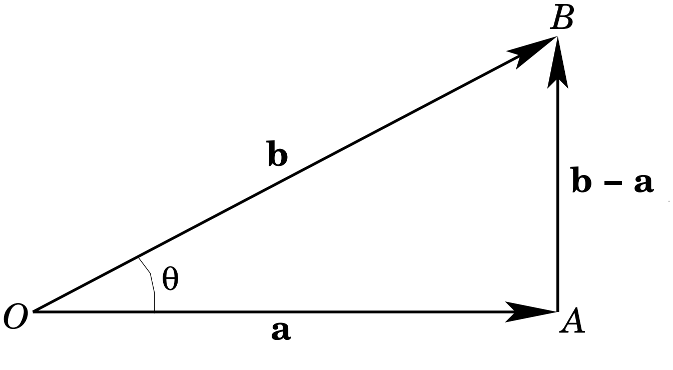


Figure 2.5: *A vector triangle.*

Let us now investigate the general case. The length squared of AB in Fig. 2.5 is

$$(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a}) = |a|^2+|b|^2-2\,\mathbf{a}\cdot\mathbf{b}. \tag{2.22}$$

However, according to the "cosine rule" of trigonometry,

$$(AB)^2 = (OA)^2 + (OB)^2 - 2\,(OA)\,(OB)\,\cos\theta, \tag{2.23}$$

where (AB) denotes the length of side AB. It follows that

$$\mathbf{a}\cdot\mathbf{b} = |a|\,|b|\,\cos\theta. \tag{2.24}$$

Clearly, the invariance of $\mathbf{a}\cdot\mathbf{b}$ under transformation is equivalent to the invariance of the angle subtended between the two vectors. Note that if $\mathbf{a}\cdot\mathbf{b} = 0$ then either $|a| = 0$, $|b| = 0$, or the vectors $\mathbf{a}$ and $\mathbf{b}$ are mutually perpendicular. The angle θ subtended between two vectors can easily be obtained from the dot product: *i.e.*,

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|a|\,|b|}. \tag{2.25}$$

Note that $a_x = a\,\cos\theta_x$, *etc.*, where θ_x is the angle subtended between vector $\mathbf{a}$ and the x-axis.

The work W performed by a constant force $\mathbf{F}$ which moves an object through a displacement $\mathbf{r}$ is the product of the magnitude of $\mathbf{F}$ times the displacement in the direction of $\mathbf{F}$. So, if the angle subtended between $\mathbf{F}$ and $\mathbf{r}$ is θ then

$$W = |F|\,(|r|\,\cos\theta) = \mathbf{F}\cdot\mathbf{r}. \tag{2.26}$$

2.4 The Vector Product

We have discovered how to construct a scalar from the components of two general vectors $\mathbf{a}$ and $\mathbf{b}$. Can we also construct a vector which is not just a linear combination of $\mathbf{a}$ and $\mathbf{b}$? Consider the following definition:

$$\mathbf{a}\,\mathrm{x}\,\mathbf{b} = (a_x\,b_x,\; a_y\,b_y,\; a_z\,b_z). \tag{2.27}$$

Is $\mathbf{a}\,\mathrm{x}\,\mathbf{b}$ a proper vector? Suppose that $\mathbf{a} = (1,\,0,\,0)$ and $\mathbf{b} = (0,\,1,\,0)$. Clearly, $\mathbf{a}\,\mathrm{x}\,\mathbf{b} = \mathbf{0}$. However, if we rotate the basis through 45° about the z-axis then $\mathbf{a} = (1/\sqrt{2},\,-1/\sqrt{2},\,0)$, $\mathbf{b} = (1/\sqrt{2},\,1/\sqrt{2},\,0)$, and $\mathbf{a}\,\mathrm{x}\,\mathbf{b} = (1/2,\,-1/2,\,0)$. Thus, $\mathbf{a}\,\mathrm{x}\,\mathbf{b}$ does not transform like a vector, because its magnitude depends on the choice of axes. So, above definition is a bad one.

Consider, now, the *cross product* or *vector product*,

$$\mathbf{a}\times\mathbf{b} = (a_y\,b_z - a_z\,b_y,\ a_z\,b_x - a_x\,b_z,\ a_x\,b_y - a_y\,b_x) = \mathbf{c}. \tag{2.28}$$

Does this rather unlikely combination transform like a vector? Let us try rotating the basis through θ degrees about the z-axis using Eqs. (2.10)–(2.12). In the new basis,

$$\begin{aligned} c_{x'} &= (-a_x\,\sin\theta + a_y\,\cos\theta)\,b_z - a_z\,(-b_x\,\sin\theta + b_y\,\cos\theta) \\ &= (a_y\,b_z - a_z\,b_y)\,\cos\theta + (a_z\,b_x - a_x\,b_z)\,\sin\theta \\ &= c_x\,\cos\theta + c_y\,\sin\theta. \end{aligned} \tag{2.29}$$

Thus, the x-component of $\mathbf{a}\times\mathbf{b}$ transforms correctly. It can easily be shown that the other components transform correctly as well, and that all components also transform correctly under rotation about the y- and z-axes. Thus, $\mathbf{a}\times\mathbf{b}$ is a proper vector. Incidentally, $\mathbf{a}\times\mathbf{b}$ is the *only* simple combination of the components of two vectors which transforms like a vector (which is non-coplanar with $\mathbf{a}$ and $\mathbf{b}$). The cross product is *anticommutative*,

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}, \tag{2.30}$$

distributive,

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}, \tag{2.31}$$

but is *not* associative:

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}. \tag{2.32}$$

Note that $\mathbf{a}\times\mathbf{b}$ can be written in the convenient, and easy to remember, determinant form

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}. \tag{2.33}$$

The cross product transforms like a vector, which means that it must have a well-defined direction and magnitude. We can show that $\mathbf{a}\times\mathbf{b}$ is *perpendicular* to both $\mathbf{a}$ and $\mathbf{b}$. Consider $\mathbf{a}\cdot\mathbf{a}\times\mathbf{b}$. If this is zero then the cross product must be

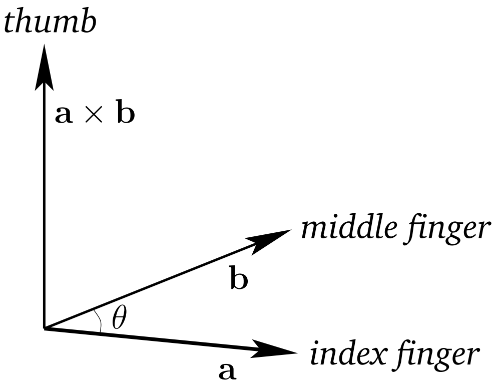


Figure 2.6: *The right-hand rule for cross products.*

perpendicular to **a**. Now

$$\begin{aligned}\mathbf{a}\cdot\mathbf{a}\times\mathbf{b} &= a_x\,(a_y\,b_z - a_z\,b_y) + a_y\,(a_z\,b_x - a_x\,b_z) + a_z\,(a_x\,b_y - a_y\,b_x)\\ &= 0. \qquad (2.34)\end{aligned}$$

Therefore, $\mathbf{a}\times\mathbf{b}$ is perpendicular to **a**. Likewise, it can be demonstrated that $\mathbf{a}\times\mathbf{b}$ is perpendicular to **b**. The vectors **a**, **b**, and $\mathbf{a}\times\mathbf{b}$ form a *right-handed* set, like the unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$. In fact, $\mathbf{e}_x \times \mathbf{e}_y = \mathbf{e}_z$. This defines a unique direction for $\mathbf{a}\times\mathbf{b}$, which is obtained from a right-hand rule—see Fig. 2.6.

Let us now evaluate the magnitude of $\mathbf{a}\times\mathbf{b}$. We have

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})^2 &= (a_y\,b_z - a_z\,b_y)^2 + (a_z\,b_x - a_x\,b_z)^2 + (a_x\,b_z - a_y\,b_x)^2\\ &= ({a_x}^2 + {a_y}^2 + {a_z}^2)\,({b_x}^2 + {b_y}^2 + {b_z}^2) - (a_x\,b_x + a_y\,b_y + a_z\,b_z)^2\\ &= |a|^2\,|b|^2 - (\mathbf{a}\cdot\mathbf{b})^2\\ &= |a|^2\,|b|^2 - |a|^2\,|b|^2\,\cos^2\theta = |a|^2\,|b|^2\,\sin^2\theta. \qquad (2.35)\end{aligned}$$

Thus,

$$|\mathbf{a}\times\mathbf{b}| = |a|\,|b|\,\sin\theta. \qquad (2.36)$$

Clearly, $\mathbf{a}\times\mathbf{a} = \mathbf{0}$ for any vector, since θ is always zero in this case. Also, if $\mathbf{a}\times\mathbf{b} = \mathbf{0}$ then either $|a| = 0$, $|b| = 0$, or **b** is parallel (or antiparallel) to **a**.

Suppose that a force **F** is applied at position **r**—see Fig. 2.7. The moment, or torque, about the origin O is the product of the magnitude of the force and the

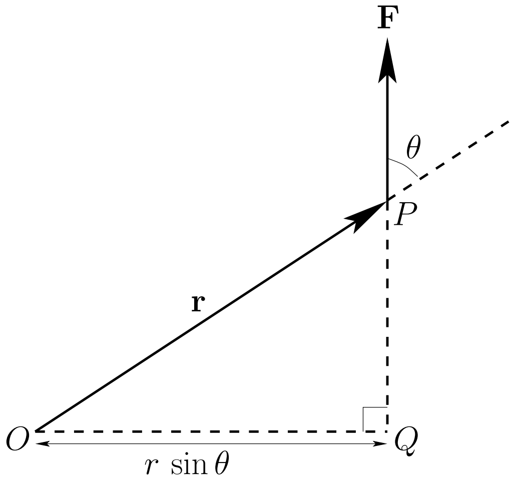


Figure 2.7: *A torque.*

length of the lever arm OQ. Thus, the magnitude of the moment is $|F|\,|r|\,\sin\theta$. The direction of the moment is conventionally the direction of the axis through O about which the force tries to rotate objects, in the sense determined by a right-hand grip rule. It follows that the vector moment is given by

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}. \tag{2.37}$$

2.5 Vector Calculus

Suppose that vector $\mathbf{a}$ varies with time, so that $\mathbf{a} = \mathbf{a}(t)$. The time derivative of the vector is defined

$$\frac{d\mathbf{a}}{dt} = \lim_{\delta t \to 0} \left[\frac{\mathbf{a}(t + \delta t) - \mathbf{a}(t)}{\delta t} \right]. \tag{2.38}$$

When written out in component form this becomes

$$\frac{d\mathbf{a}}{dt} = \left(\frac{da_x}{dt}, \frac{da_y}{dt}, \frac{da_z}{dt} \right). \tag{2.39}$$

Suppose that $\mathbf{a}$ is, in fact, the product of a scalar $\phi(t)$ and another vector $\mathbf{b}(t)$.

What now is the time derivative of **a**? We have

$$\frac{da_x}{dt} = \frac{d}{dt}(\phi\, b_x) = \frac{d\phi}{dt}\, b_x + \phi\, \frac{db_x}{dt}, \tag{2.40}$$

which implies that

$$\frac{d\mathbf{a}}{dt} = \frac{d\phi}{dt}\, \mathbf{b} + \phi\, \frac{d\mathbf{b}}{dt}. \tag{2.41}$$

It is easily demonstrated that

$$\frac{d}{dt}\,(\mathbf{a}\cdot\mathbf{b}) = \frac{d\mathbf{a}}{dt}\cdot\mathbf{b} + \mathbf{a}\cdot\frac{d\mathbf{b}}{dt}. \tag{2.42}$$

Likewise,

$$\frac{d}{dt}\,(\mathbf{a}\times\mathbf{b}) = \frac{d\mathbf{a}}{dt}\times\mathbf{b} + \mathbf{a}\times\frac{d\mathbf{b}}{dt}. \tag{2.43}$$

It can be seen that the laws of vector differentiation are fairly analogous to those in conventional calculus.

2.6 Line Integrals

A *vector field* is defined as a set of vectors associated with each point in space. For instance, the velocity $\mathbf{v}(\mathbf{r})$ in a moving liquid (*e.g.*, a whirlpool) constitutes a vector field. By analogy, a *scalar field* is a set of scalars associated with each point in space. An example of a scalar field is the temperature distribution $T(\mathbf{r})$ in a furnace.

Consider a general vector field $\mathbf{A}(\mathbf{r})$. Line integrals of the form

$$\int_P^Q \mathbf{A}\cdot d\mathbf{r} = \int_P^Q (A_x\, dx + A_y\, dy + A_z\, dz), \tag{2.44}$$

evaluated on some particular path taken between two fixed points P and Q, often arise in Physics. Here $d\mathbf{r} = (dx, dy, dz)$ is a path element. The path might be specified as $x = f(l)$, $y = g(l)$, and $z = h(l)$, where f, g, h are mathematical functions, and l is a parameter (such as path-length) which varies monotonically along the path. It follows that $\mathrm{d}\mathbf{r} = (\mathrm{d}f/\mathrm{d}l, \mathrm{d}g/\mathrm{d}l, \mathrm{d}h/\mathrm{d}l)\,\mathrm{d}l$. In particular, if $\mathbf{A}(\mathbf{r})$ is a force-field then the line integral is the work done by the force in going between points P and Q along the given path [*cf.*, Eq. (2.26)]. Finally, if the path is a closed loop (*i.e.*, if P and Q are the same point) then the integral is conventionally written

$$\oint \mathbf{A}\cdot \mathrm{d}\mathbf{r}. \tag{2.45}$$

As an example of a path integral, consider the work done in a repulsive, inverse-square, central field, $\mathbf{F} = -\mathbf{r}/|r^3|$. The element of work done is $\mathrm{d}W = \mathbf{F}\cdot\mathrm{d}\mathbf{r}$. Take $P = (\infty, 0, 0)$ and $Q = (a, 0, 0)$. Route 1 is along the x-axis, so

$$W = \int_{\infty}^{a}\left(-\frac{1}{x^2}\right)\,\mathrm{d}x = \left[\frac{1}{x}\right]_{\infty}^{a} = \frac{1}{a}. \tag{2.46}$$

The second route is, firstly, around a large circle ($r =$ constant) to the point (a, ∞, 0), and then parallel to the y-axis—see Fig. 2.8. In the first, part no work is done, since $\mathbf{F}$ is perpendicular to $\mathrm{d}\mathbf{r}$. In the second part,

$$W = \int_{\infty}^{0}\frac{-y\,\mathrm{d}y}{(a^2+y^2)^{3/2}} = \left[\frac{1}{(y^2+a^2)^{1/2}}\right]_{\infty}^{0} = \frac{1}{a}. \tag{2.47}$$

In this case, the integral is independent of the path taken between the beginning and end points. However, not all line integrals are path independent. Indeed, there are two different classes of line integral—those whose values only depend on the end points, and those whose values depend both on the end points and the path taken between these points.

2.7 Surface Integrals

Surface integrals often arise in Physics. For instance, the rate of flow of a liquid of velocity $\mathbf{v}$ through an infinitesimal surface of vector area $\mathrm{d}\mathbf{S}$ is $\mathbf{v}\cdot\mathrm{d}\mathbf{S}$ (*i.e.*, the product of the normal component of the velocity, $v\cos\theta$, and the magnitude of the area, $\mathrm{d}S$, where θ is the angle subtended between $\mathbf{v}$ and $\mathrm{d}\mathbf{S}$). The net rate of flow through a surface S made up of very many infinitesimal surfaces is

$$\int_S \mathbf{v}\cdot\mathrm{d}\mathbf{S} = \lim_{\mathrm{d}\mathbf{S}\rightarrow 0}\left[\sum v\,\cos\theta\,\mathrm{d}S\right], \tag{2.48}$$

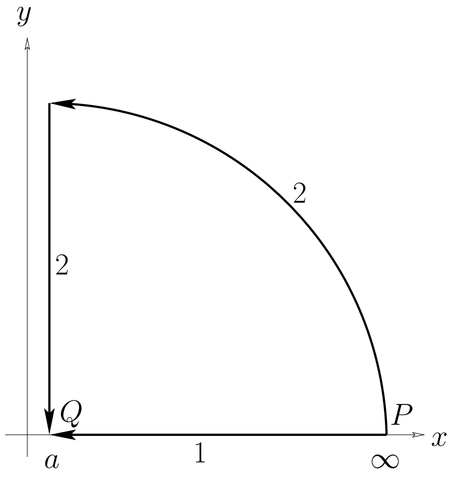


Figure 2.8: *An example line integral.*

where θ is the angle subtended between a surface element $d\mathbf{S}$ and the local flow velocity $\mathbf{v}(\mathbf{r})$. If the surface is closed, and the surface elements all point outward, then the integral is conventionally written

$$\oint_S \mathbf{v}\cdot d\mathbf{S}. \tag{2.49}$$

In this case, the integral is often termed the *flux* of the velocity field $\mathbf{v}$ out of the closed surface S.

2.8 Volume Integrals

A volume integral takes the form

$$\int_V F(x,y,z)\,dV, \tag{2.50}$$

where F is a three-dimensional mathematical function, V some volume in space, and $dV = dx\,dy\,dz$ an element of this volume. The volume element is sometimes written $d^3\mathbf{r}$.

As an example of a volume integral, let us evaluate the centre of gravity of a solid hemisphere of radius a (centered on the origin). The height of the centre of gravity is given by

$$\bar{z} = \int_V z\,dV \Big/ \int_V dV. \tag{2.51}$$

The bottom integral is simply the volume of the hemisphere, which is $2\pi\,a^3/3$. The top integral is most easily evaluated in spherical polar coordinates $(r,\,\theta,\,\phi)$, for which $z = r\,\cos\theta$ and $dV = r^2\,\sin\theta\,dr\,d\theta\,d\phi$. Thus,

$$\begin{aligned}\int_V z\,dV &= \int_0^a dr \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi\; r\,\cos\theta\; r^2\sin\theta \\ &= \int_0^a r^3\,dr \int_0^{\pi/2} \sin\theta\,\cos\theta\,d\theta \int_0^{2\pi} d\phi = \frac{\pi\,a^4}{4},\end{aligned} \tag{2.52}$$

giving

$$\bar{z} = \frac{\pi\,a^4}{4}\frac{3}{2\pi\,a^3} = \frac{3\,a}{8}. \tag{2.53}$$

3 Electricity

3.1 Historical Introduction

We usually associate electricity with the 20th Century, during which it revolutionized the lives of countless millions of ordinary people, in much the same manner as steam power revolutionized lives in the 19th Century. It is, therefore, somewhat surprising to learn that people have known about electricity for many thousands of years. In about 1000 BC, the ancient Greeks started to navigate the Black Sea, and opened up trade routes, via the river Dnieper, to the Baltic region. Amongst the many trade items that the Greeks obtained from the Baltic was a substance which they called "electron" (ἠλέκτρον), but which we nowadays call *amber*. Amber is fossilized pine resin, and was used by the Greeks, much as it is used today, as a gem stone. However, in about 600 BC, the ancient Greek philosopher Thales of Miletus discovered that amber possesses a rather peculiar property: *i.e.*, when it is rubbed with fur, it develops the ability to attract light objects, such as feathers. For many centuries, this strange phenomenon was thought to be a unique property of amber.

In Elizabethan times, the English physician William Gilbert coined the word "electric" (from the Greek word for amber) to describe the above mentioned effect. It was later found that many materials become electric when rubbed with certain other materials. In 1733, the French chemist du Fay discovered that there are, in fact, *two* different types of electricity. When amber is rubbed with fur, it acquires so-called "resinous electricity." On the other hand, when glass is rubbed with silk, it acquires so-called "vitreous electricity." Electricity repels electricity of the same kind, but attracts electricity of the opposite kind. At the time, it was thought that electricity was created by friction.

Scientists in the 18th Century eventually developed the concept of *electric charge* in order to account for a large body of observations made in countless electrical experiments. There are two types of charge: *positive* (which is the same as vitreous), and *negative* (which is the same as resinous). Like charges repel one another, whilst opposite charges attract. When two bodies are rubbed together, charge can be transfered from one to the other, but the total charge remains constant. Thus, when amber is rubbed with fur, there is transfer of charge such that the amber acquires a negative charge, and the fur an equal positive charge. Likewise, when glass is rubbed with silk, the glass acquires a positive charge, and the silk an equal negative charge. The idea that electrical charge is a conserved quantity is attributed to the American scientist Benjamin Franklin (who is also to blame for the unfortunate sign convention in electricity). The *law of charge conservation* can be written:

> In any closed system, the total electric charge remains constant.

Of course, when summing charge, positive charges are represented as positive numbers, and negative charges as negative numbers.

In the 20th Century, scientists discovered that the atoms out of which ordinary matter is composed consist of two components: a relatively massive, positively charged nucleus, surrounded by a cloud of relatively light, negatively charged particles called *electrons*. Electrons and atomic nuclii carry fixed electrical charges, and are essentially indestructible (provided that we neglect nuclear reactions). Under normal circumstances, only the electrons are mobile. Thus, when amber is rubbed with fur, electrons are transferred from the fur to the amber, giving the amber an excess of electrons, and, hence, a negative charge, and the fur a deficit of electrons, and, hence, a positive charge. Substances normally contain neither an excess nor a deficit of electrons, and are, therefore, electrically neutral.

The SI unit of electric charge is the *coulomb* (C). The charge on an electron is -1.602×10^{-19} C.

3.2 Conductors and Insulators

Suppose that we were to electrically charge two isolated metal spheres: one with a positive charge, and the other with an equal negative charge. We could then perform a number of simple experiments. For instance, we could connect the spheres together using a length of string. In this case, we would find that the charges residing on the two spheres were unaffected. Next, we could connect the spheres using a copper wire. In this case, we would find that there was no charge remaining on either sphere. Further investigation would reveal that charge must have *flowed* through the wire, from one sphere to the other, such that the positive charge on the first sphere completely canceled the negative charge on the second, leaving zero charge on either sphere. Substances can be classified into two main groups, depending on whether they allow the free flow of electric charge. *Conductors* allow charge to pass freely through them, whereas *insulators* do not. Obviously, string is an insulator, and copper is a conductor. As a general rule, substances which are good conductors of heat are also good conductors of electricity. Thus, all metals are conductors, whereas air, (pure) water, plastics, glasses, and ceramics are insulators. Incidentally, the distinction between conductors and insulators was first made by the English scientist Stephen Gray in 1729.

Metals are good conductors (both of heat and electricity) because at least one electron per atom is *free*: *i.e.*, it is not tied to any particular atom, but is, instead, able to move freely throughout the metal. In good insulators, such as glass, all of the electrons are tightly bound to atoms (which are fixed), and so there are no free electrons.

3.3 Electrometers and Electroscopes

Electric charge is measured using a device called an *electrometer*, which consists of a metal knob connected via a conducting shaft to a flat, vertical metal plate. A very light gold leaf, hinged at the top, is attached to the plate. Both the plate and the gold leaf are enclosed in a glass vessel to protect the delicate leaf from air currents. When charge is deposited on the knob, some fraction is conducted to the plate and the gold leaf, which consequently repel one another, causing the leaf to pull away from the plate. The angular deflection of the gold leaf with respect to the plate is proportional to the charge deposited on the knob. An electrometer can be calibrated in such a manner that the angular deflection of the gold leaf can be used to calculate the absolute magnitude of the charge deposited on the knob.

An *electroscope* is a somewhat cruder charge measuring device than an electrometer, and consists of two gold leaves, hinged at the top, in place of the metal plate and the single leaf. When the knob is charged, the two leaves also become charged and repel one another, which causes them to move apart. The mutual deflection of the leaves can be used as a rough measure of the amount of electric charge deposited on the knob.

3.4 Induced Electric Charge

We have seen how an electroscope can be used to measure the absolute magnitude of an electric charge. But, how can we determine the sign of the charge? In fact, this is fairly straightforward. Suppose that an electroscope carries a charge of unknown sign. Consider what happens when we bring a negatively charged amber rod, produced by rubbing the rod with fur, close to the knob of the electroscope. The excess electrons in the rod repel the free electrons in the knob and shaft of the electroscope. The repelled electrons move as far away from the rod as possible, ending up in the gold leaves. Thus, the charge on the leaves becomes more negative. If the original charge on the electroscope is negative then the magnitude of the charge on the leaves increases in the presence of the rod, and the leaves consequently move further apart. On the other hand, if the original charge on the electroscope is positive then the magnitude of the charge on the leaves decreases in the presence of the rod, and the leaves consequently move closer together. The general rule is that the deflection of the leaves increases when a charge of the same sign is brought close to the knob of the electroscope, and *vice versa*. The sign of the charge on an electroscope can easily be determined in this manner.

Suppose that we bring a negatively charged rod close to the knob of an uncharged electroscope. The excess electrons in the rod repel the free electrons in the knob and shaft of the electroscope so that they collect in the gold leaves, which, therefore, move apart. It follows that whenever a charged object is brought close to the knob of an uncharged electroscope, the electroscope registers a charge. Thus, an uncharged electroscope can be used to *detect* electric charge residing on nearby objects, without disturbing that charge.

Suppose that we bring a negatively charged rod close to the knob of an uncharged electroscope which is attached, via a conducting wire, to a large uncharged conductor. The excess electrons in the rod repel the free electrons in the knob and shaft of the electroscope. The repelled electrons move as far away from the rod as possible, which means that they flow down the wire into the external conductor. Suppose that we disconnect the wire and then remove the charged rod. By disconnecting the wire we have stranded the electrons which were repelled down the wire on the external conductor. Thus, the electroscope, which was initially uncharged, acquires a deficit of electrons. In other words, the electroscope becomes positively charged. Clearly, by bringing a charged object close to an uncharged electroscope, transiently connecting the electroscope to a large uncharged conductor, and then removing the object, we can *induce* a charge of the opposite sign on the electroscope without affecting the charge on the object. This process is called charging by *induction*.

But where are we going to find a large uncharged conductor? Well, it turns out that we standing on one. The ground (*i.e.*, the Earth) is certainly large, and it turns out that it is also a reasonably good electrical conductor. Thus, we can inductively charge an electroscope by transiently connecting it to the ground (*i.e.*, “grounding” or “earthing” it) whilst it is in the presence of a charged object. The most effective way of earthing an object is to connect it to a conducting wire which is attached, at the other end, to a metal stake driven into the ground. A somewhat less effective way of grounding an object is simply to touch it. It turns out that we are sufficiently good electrical conductors that charge can flow though us to the ground.

Charges can also be induced on good insulators, although to nothing like the same extent that they can be induced on good conductors. Suppose that a negatively charged amber rod is brought close to a small piece of paper (which is an insulator). The excess electrons on the rod repel the electrons in the atoms which make up the paper, but attract the positively charged nuclei. Since paper is an insulator, the repelled electrons are not free to move through the paper.

Instead, the atoms in the paper *polarize*: *i.e.*, they distort in such a manner that their nuclei move slightly towards, and their electrons slightly away from, the rod. The electrostatic force of attraction between the excess electrons in the rod and the atomic nuclei in the paper is slightly greater than the repulsion between the electrons in the rod and those in the paper, since the electrons in the paper are, on average, slightly further away from the rod than the nuclei (and the force of electrostatic attraction falls off with increasing distance). Thus, there is a net *attractive* force between the rod and the paper. In fact, if the piece of paper is sufficiently light then it can actually be picked up using the rod. In summary, whenever a charged object is brought close to an insulator, the atoms in the insulator polarize, resulting in a net attractive force between the object and the insulator. This effect is used commercially to remove soot particles from the exhaust plumes of coal-burning power stations.

3.5 Coulomb's Law

The first precise measurement of the force between two electric charges was performed by the French scientist Charles-Augustin de Coulomb in 1788. Coulomb concluded that:

> The electrical force between two charges at rest is directly proportional to the product of the charges, and inversely proportional to the square of the distance between the charges

This law of force is nowadays known as *Coulomb's law*. Incidentally, an electrical force exerted between two stationary charges is known as an *electrostatic* force. In algebraic form, Coulomb's law is written

$$f = \frac{q\,q'}{4\pi\epsilon_0\,r^2}, \tag{3.1}$$

where f is the magnitude of the force, q and q' are the magnitudes of the two charges (with the appropriate signs), and r is the distance between the two charges. The force is repulsive if $f > 0$, and attractive if $f < 0$. The universal constant

$$\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{N^{-1}\,m^{-2}\,C^2} \tag{3.2}$$

is called the *permittivity of free space* or the *permittivity of the vacuum*. We can also write Coulomb's law in the form

$$f = k_e\,\frac{q\,q'}{r^2}, \tag{3.3}$$

where the constant of proportionality $k_e = 1/4\pi\epsilon_0$ takes the value

$$k_e = 8.988 \times 10^9\ \mathrm{N\,m^2\,C^{-2}}. \tag{3.4}$$

Coulomb's law has an analogous form to Newton's law of gravitation,

$$f = -G\,\frac{m\,m'}{r^2}, \tag{3.5}$$

with electric charge playing the role of mass. One major difference between the two laws is the sign of the force. The electrostatic force between two like charges is repulsive (*i.e.*, $f > 0$), whereas that between two unlike charges is attractive (*i.e.*, $f < 0$). On the other hand, the gravitational force between two masses is always attractive (since there is no such thing as a negative mass). Another major difference is the relative magnitude of the two forces. For instance, the electrostatic repulsion between two electrons is approximately 10^{42} times larger than the corresponding gravitational attraction.

The electrostatic force $\mathbf{f}_{ab}$ exerted by a charge q_a on a second charge q_b, located a distance r from the first charge, has the magnitude

$$f = \frac{q_a\,q_b}{4\pi\epsilon_0\,r^2}, \tag{3.6}$$

and is directed *radially away* from the first charge if $f > 0$, and *radially towards* it if $f < 0$. The force $\mathbf{f}_{ba}$ exerted by the second charge on the first is equal and opposite to $\mathbf{f}_{ab}$, so that

$$\mathbf{f}_{ba} = -\mathbf{f}_{ab}, \tag{3.7}$$

in accordance with Newton's third law of motion.

Suppose that we have three point charges, q_a, q_b, and q_c. It turns out that electrostatic forces are *superposable*. That is, the force $\mathbf{f}_{ba}$ exerted by q_b on q_a is completely unaffected by the presence of q_c. Likewise, the force $\mathbf{f}_{ca}$ exerted by q_c on q_a is unaffected by the presence of q_b. Thus, the net force $\mathbf{f}_a$ acting on q_a is the *resultant* of these two forces: *i.e.*,

$$\mathbf{f}_a = \mathbf{f}_{ba} + \mathbf{f}_{ca}. \tag{3.8}$$

This rule can be generalized in a straightforward manner to the case where there are more than three point charges.

3.6 Electric Fields

According to Coulomb's law, a charge q exerts a force on a second charge q', and *vice versa*, even in a vacuum. But, how is this force transmitted through empty space? In order to answer this question, physicists in the 19th Century developed the concept of an *electric field*. The idea is as follows. The charge q generates an electric field $\mathbf{E}(\mathbf{r})$ which fills space. The electrostatic force exerted on the second charge q' is actually produced locally by the electric field $\mathbf{E}$ at the position of this charge, in accordance with Coulomb's law. Likewise, the charge q' generates its own electric field $\mathbf{E}'(\mathbf{r})$ which also fills space. The equal and opposite reaction force exerted on q is produced locally by the electric field $\mathbf{E}'$ at the position of this charge, again, in accordance with Coulomb's law. Of course, an electric field cannot exert a force on the charge which generates it, in just the same way as we cannot pick ourselves up with our own shoelaces. Incidentally, electric fields have a real physical existence, and are not just theoretical constructs invented by physicists to get around the problem of the transmission of electrostatic forces through vacuums. We can say this with certainty because, as we shall see later, there is an *energy* associated with an electric field filling space. Indeed, it is actually possible to convert this energy into heat or work, and *vice versa*.

The electric field $\mathbf{E}(\mathbf{r})$ generated by a set of fixed electric charges is a vector field which is defined as follows. If $\mathbf{f}(\mathbf{r})$ is the electrostatic force experienced by some small positive test charge q' located at a certain point $\mathbf{r}$ in space, then the electric field at this point is simply the force divided by the magnitude of the test charge. In other words,

$$\mathbf{E} = \frac{\mathbf{f}}{q'}. \tag{3.9}$$

Electric field has dimensions of force per unit charge, and units of newtons per coulomb ($\mathrm{N\,C^{-1}}$). Incidentally, the reason that we specify a small, rather than a large, test charge is to avoid disturbing any of the fixed charges which generate the electric field.

Let us use the above rule to reconstruct the electric field generated by a point charge q. According to Coulomb's law, the electrostatic force exerted by a point charge q on a positive test charge q', located a distance r from it, has the magnitude

$$f = \frac{q\,q'}{4\pi\epsilon_0\,r^2}, \tag{3.10}$$

and is directed radially away from the former charge if $q > 0$, and radially towards it if $q < 0$. Thus, the electric field a distance r away from a charge q has the magnitude

$$E = \frac{q}{4\pi\epsilon_0\,r^2}, \tag{3.11}$$

and is directed radially away from the charge if $q > 0$, and radially towards the charge if $q < 0$. Note that the field is independent of the magnitude of the test charge.

A corollary of the above definition of an electric field is that a stationary charge q located in an electric field $\mathbf{E}$ experiences an electrostatic force

$$\mathbf{f} = q\,\mathbf{E}, \tag{3.12}$$

where $\mathbf{E}$ is the electric field at the location of the charge (excluding the field produced by the charge itself).

Since electrostatic forces are superposable, it follows that electric fields are also superposable. For example, if we have three stationary point charges, q_a, q_b, and q_c, located at three different points in space, then the net electric field which fills space is simply the vector sum of the fields produced by each point charge taken in isolation.

3.7 Worked Examples

Example 3.1: Electrostatic force between three colinear point charges

Question: A particle of charge $q_1 = +6.0\,\mu\text{C}$ is located on the x-axis at coordinate $x_1 = 5.1\,\text{cm}$. A second particle of charge $q_2 = -5.0\,\mu\text{C}$ is placed on the x-axis at $x_2 = -3.4\,\text{cm}$. What is the magnitude and direction of the total electrostatic force acting on a third particle of charge $q_3 = +2.0\,\mu\text{C}$ placed at the origin ($x = 0$)?

Solution: The force f acting between charges 1 and 3 is given by

$$f = k_e\,\frac{q_1\,q_3}{{x_1}^2} = (8.988\times 10^9)\,\frac{(6\times 10^{-6})\,(2\times 10^{-6})}{(5.1\times 10^{-2})^2} = +41.68\,\text{N}.$$

Since $f > 0$, the force is repulsive. This means that the force f_{13} exerted by charge 1 on charge 3 is directed along the $-x$-axis (*i.e.*, from charge 1 towards charge 3), and is of magnitude $|f|$. Thus, $f_{13} = -41.69\,\text{N}$. Here, we adopt the convention that forces directed along the $+x$-axis are positive, and *vice versa*. The force f' acting between charges 2 and 3 is given by

$$f' = k_e\,\frac{q_2\,q_3}{|x_2|^2} = (8.988\times 10^9)\,\frac{(-5\times 10^{-6})\,(2\times 10^{-6})}{(3.4\times 10^{-2})^2} = -77.75\,\text{N}.$$

Since $f' < 0$, the force is attractive. This means that the force f_{23} exerted by charge 2 on charge 3 is directed along the $-x$-axis (*i.e.*, from charge 3 towards charge 2), and is of magnitude $|f'|$. Thus, $f_{23} = -77.75\,\text{N}$.

The resultant force f_3 acting on charge 3 is the algebraic sum of the forces exerted by charges 1 and 2 separately (the sum is algebraic because all the forces act along the x-axis). It follows that

$$f_3 = f_{13} + f_{23} = -41.69 - 77.75 = -119.22\,\text{N}.$$

Thus, the magnitude of the total force acting on charge 3 is $119.22\,\text{N}$, and the force is directed along the $-x$-axis (since $f_3 < 0$).

Example 3.2: Electrostatic force between three non-colinear point charges

Question: Suppose that three point charges, q_a, q_b, and q_c, are arranged at the vertices of a right-angled triangle, as shown in the diagram. What is the magnitude and direction of the electrostatic force acting on the third charge if $q_a = -6.0\,\mu\text{C}$, $q_b = +4.0\,\mu\text{C}$, $q_c = +2.0\,\mu\text{C}$, $a = 4.0\,\text{m}$, and $b = 3.0\,\text{m}$?

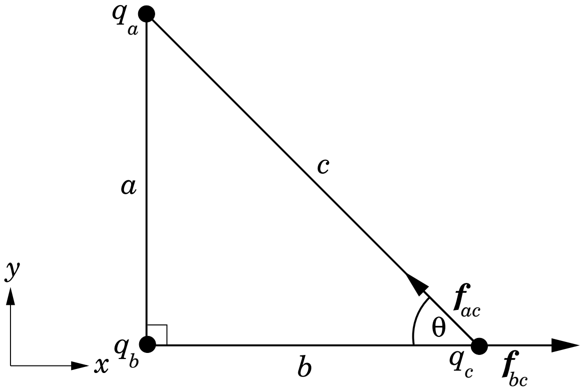
Solution: The magnitude f_{ac} of the force $\mathbf{f}_{ac}$ exerted by charge q_a on charge q_c is given by

$$f_{ac} = k_e\,\frac{|q_a|\,q_c}{c^2} = (8.988\times 10^9)\,\frac{(6\times 10^{-6})\,(2\times 10^{-6})}{(4^2+3^2)} = 4.31\times 10^{-3}\,\text{N},$$

where use has been made of the Pythagorean theorem. The force is attractive (since charges q_a and q_c are of opposite sign). Hence, the force is directed from charge q_c towards charge q_a, as shown in the diagram. The magnitude f_{bc} of the force $\mathbf{f}_{bc}$ exerted by charge q_b on charge q_c is given by

$$f_{bc} = k_e\,\frac{q_b q_c}{b^2} = (8.988\times 10^9)\,\frac{(4\times 10^{-6})\,(2\times 10^{-6})}{(3^2)} = 7.99\times 10^{-3}\,\text{N}.$$

The force is repulsive (since charges q_b and q_c are of the same sign). Hence, the force is directed from charge q_b towards charge q_c, as shown in the diagram. Now, the net force acting on charge q_c is the sum of $\mathbf{f}_{ac}$ and $\mathbf{f}_{bc}$. Unfortunately, since $\mathbf{f}_{ab}$ and $\mathbf{f}_{bc}$ are vectors pointing in *different* directions, they *cannot* be added together algebraically. Fortunately, however, their components along the x- and y-axes *can* be added algebraically. Now, it is clear, from the diagram, that $\mathbf{f}_{bc}$ is



directed along the $+x$-axis. If follows that

$$\begin{aligned} f_{bc\,x} &= f_{bc} = 7.99\times 10^{-3}\,\mathrm{N}, \\ f_{bc\,y} &= 0. \end{aligned}$$

It is also clear, from the diagram, that $\mathbf{f}_{ac}$ subtends an angle

$$\theta = \tan^{-1}(a/b) = \tan^{-1}(4/3) = 53.1^\circ$$

with the $-x$-axis, and an angle $90^\circ - \theta$ with the $+y$-axis. It follows from the conventional laws of vector projection that

$$\begin{aligned} f_{ac\,x} &= -f_{ac}\cos\theta = -(4.31\times 10^{-3})\,(0.6) = -2.59\times 10^{-3}\,\mathrm{N}, \\ f_{ac\,y} &= f_{ac}\cos(90^\circ-\theta) = f_{ac}\sin\theta = (4.31\times 10^{-3})\,(0.8) = 3.45\times 10^{-3}\,\mathrm{N}. \end{aligned}$$

The x- and y-components of the resultant force $\mathbf{f}_c$ acting on charge q_c are given by

$$\begin{aligned} f_{c\,x} &= f_{ac\,x} + f_{bc\,x} = -2.59\times 10^{-3} + 7.99\times 10^{-3} = 5.40\times 10^{-3}\,\mathrm{N}, \\ f_{c\,y} &= f_{ac\,y} + f_{bc\,y} = 3.45\times 10^{-3}\,\mathrm{N}. \end{aligned}$$

Thus, from the Pythagorean theorem, the magnitude of the resultant force is

$$f_c = \sqrt{(f_{c\,x})^2 + (f_{c\,y})^2} = 6.4\times 10^{-3}\,\mathrm{N}.$$

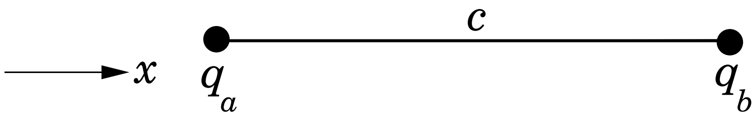
Furthermore, the resultant force subtends an angle

$$\phi = \tan^{-1}(f_{c\,y}/f_{c\,x}) = 32.6^\circ$$

with the $+x$-axis, and an angle $90^\circ - \phi = 57.4^\circ$ with the $+y$-axis.

Example 3.3: Electric field generated by two point charges

Question: Two point charges, q_a and q_b, are separated by a distance c. What is the electric field at a point halfway between the charges? What force would be exerted on a third charge q_c placed at this point? Take $q_a = 50\,\mu\mathrm{C}$, $q_b = 100\,\mu\mathrm{C}$,



$q_c = 20\,\mu\text{C}$, and $c = 1.00$ m.

Solution: Suppose that the line from q_a to q_b runs along the x-axis. It is clear, from Coulomb's law, that the electrostatic force exerted on any charge placed on this line is parallel to the x-axis. Thus, the electric field at any point along this line must also be aligned along the x-axis. Let the x-coordinates of charges q_a and q_b be $-c/2$ and $+c/2$, respectively. It follows that the origin ($x = 0$) lies halfway between the two charges. The electric field E_a generated by charge q_a at the origin is given by

$$E_a = k_e\,\frac{q_a}{(c/2)^2} = (8.988\times 10^9)\,\frac{(50\times 10^{-6})}{(0.5)^2} = 1.80\times 10^6\,\text{N}\,\text{C}^{-1}.$$

The field is positive because it is directed along the $+x$-axis (*i.e.*, from charge q_a towards the origin). The electric field E_b generated by charge q_b at the origin is given by

$$E_b = -k_e\,\frac{q_b}{(c/2)^2} = -(8.988\times 10^9)\,\frac{(100\times 10^{-6})}{(0.5)^2} = -3.60\times 10^6\,\text{N}\,\text{C}^{-1}.$$

The field is negative because it is directed along the $-x$-axis (*i.e.*, from charge q_b towards the origin). The resultant field E at the origin is the algebraic sum of E_a and E_b (since all fields are directed along the x-axis). Thus,

$$E = E_a + E_b = -1.8\times 10^6\,\text{N}\,\text{C}^{-1}.$$

Since E is negative, the resultant field is directed along the $-x$-axis.

The force f acting on a charge q_c placed at the origin is simply

$$f = q_c\,E = (20\times 10^{-6})\,(-1.8\times 10^6) = -36\,\text{N}.$$

Since $f < 0$, the force is directed along the $-x$-axis.

4 Gauss' Law

4.1 Electric Field-Lines

An electric field can be represented diagrammatically as a set of lines with arrows on, called *electric field-lines*, which fill space. Electric field-lines are drawn according to the following rules:

> The direction of the electric field is everywhere tangent to the field-lines, in the sense of the arrows on the lines. The magnitude of the field is proportional to the number of field-lines per unit area passing through a small surface normal to the lines.

Thus, field-lines determine the magnitude, as well as the direction, of the electric field. In particular, the field is strong at points where the field-lines are closely spaced, and weak at points where they are far apart.

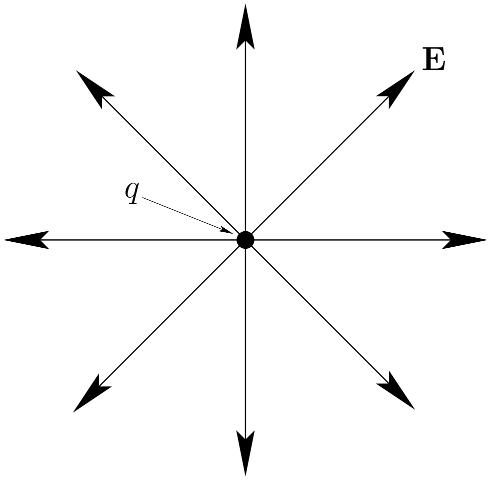


Figure 4.1: *The electric field-lines of a positive point charge.*

The electric field-lines associated with a *positive* point charge are a set of unbroken, evenly spaced (in solid angle) straight-lines which radiate from the charge—see Fig. 4.1. Thus, the tangent to the field-lines is always directed radially away from the charge, giving the correct direction for the electric field. The number of electric field-lines per unit area normal to the lines falls off like $1/r^2$, where r is the radial distance from the charge, since the total number of lines is fixed, whereas the area normal to the lines increases like r^2. Thus, the electric field-strength falls off like $1/r^2$, in accordance with Coulomb's law.

By analogy, the electric field-lines associated with a *negative* point charge are a set of unbroken, evenly spaced (in solid angle) straight lines which converge on the charge.

As a general rule, electric field-lines generated by fixed charges begin on positive charges, end on negative charges, and are unbroken and never cross in the vacuum regions between charges.

4.2 Gauss' Law

One of the most useful results in electrostatics is named after the celebrated German mathematician Karl Friedrich Gauss (1777–1855).

Suppose that a positive point charge q generates an electric field $\mathbf{E}$. Consider a spherical surface of radius R, centred on the charge. The normal to this surface is everywhere parallel to the direction of the electric field $\mathbf{E}$, since the field always points radially away from the charge. The area of the surface is $4\pi R^2$. Finally, the strength of the electric field at radius R is $E(R) = q/(4\pi\epsilon_0 R^2)$. Hence, if we multiply the electric field-strength by the area of the surface, we obtain

$$E(R)\, 4\pi R^2 = \frac{q}{4\pi\epsilon_0 R^2}\, 4\pi R^2 = \frac{q}{\epsilon_0}. \tag{4.1}$$

Note that the final result is independent of the radius of the sphere. Thus, the same result would be obtained for any sphere centred on the charge. This is the essence of *Gauss' law*.

You may be wondering why it took a famous German mathematician to prove such a trivial-seeming law. Well, Gauss proved that this law also applies to *any* closed surface, and *any* distribution of electric charges. Thus, if we multiply each outward element of a general closed surface S by the component of the electric field normal to that element, and then sum over the entire surface, the result is the total charge enclosed by the surface, divided by ϵ_0. In other words,

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \frac{Q}{\epsilon_0}, \tag{4.2}$$

where S is a closed surface, and Q is the charge enclosed by it. The integral is termed the *electric flux*, Φ_E, through the surface, and is proportional to the number of electric field-lines which pierce this surface. We adopt the convention that the flux is positive if the electric field points outward through the surface, and negative if the field points inward. Thus, Gauss' law can be written:

> The electric flux through any closed surface is equal to the total charge enclosed by the surface, divided by ϵ_0.

Gauss' law is especially useful for evaluating the electric fields produced by charge distributions which possess some sort of symmetry. Let us examine three examples of such distributions.

4.3 Electric Field of a Spherical Conducting Shell

Suppose that a thin, spherical, conducting shell carries a negative charge $-Q$. We expect the excess electrons to mutually repel one another, and, thereby, become uniformly distributed over the surface of the shell. The electric field-lines produced outside such a charge distribution point towards the surface of the conductor, and end on the excess electrons. Moreover, the field-lines are *normal* to the surface of the conductor. This must be the case, otherwise the electric field would have a component parallel to the conducting surface. Since the excess electrons are free to move through the conductor, any parallel component of the field would cause a redistribution of the charges on the shell. This process will only cease when the parallel component has been reduced to zero over the whole surface of the shell. It follows that:

> The electric field immediately above the surface of a conductor is directed normal to that surface.

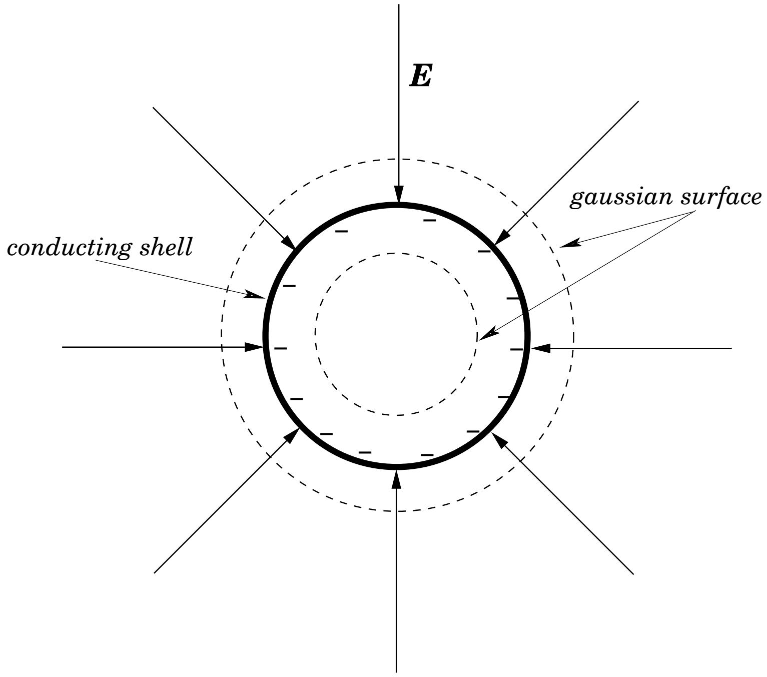


Figure 4.2: *The electric field generated by a negatively charged spherical conducting shell.*

Let us consider an imaginary surface, usually referred to as a *gaussian surface*, which is a sphere of radius R lying just above the surface of the conductor. Since the electric field-lines are everywhere normal to this surface, Gauss' law tells us that

$$\Phi_E = E\,A = \frac{-Q}{\epsilon_0}, \tag{4.3}$$

where Φ_E is the electric flux through the gaussian surface, $A = 4\pi\,R^2$ the area of this surface, and E the electric field-strength just above the surface of the conductor. Note that, by symmetry, E is uniform over the surface of the conductor. It follows that

$$E = \frac{-Q}{\epsilon_0\,A} = -\frac{Q}{4\pi\epsilon_0\,R^2}. \tag{4.4}$$

But, this is the same result as would be obtained from Coulomb's law for a point charge of magnitude $-Q$ located at the centre of the conducting shell. Now, a simple extension of the above argument leads to the conclusion that Eq. (4.4) holds *everywhere* outside the shell (with R representing the radial distance from the center of the shell). Hence, we conclude the electric field outside a charged, spherical, conducting shell is the same as that generated when all the charge is concentrated at the centre of the shell.

Let us repeat the above calculation using a spherical gaussian surface which lies just inside the conducting shell. Now, the gaussian surface encloses no charge, since all of the charge lies on the shell, so it follows from Gauss' law, and symmetry, that the electric field inside the shell is zero. In fact, the electric field inside *any* closed hollow conductor is *zero* (assuming that the region enclosed by the conductor contains no charges).

4.4 Electric Field of a Uniformly Charged Wire

Consider a long straight wire which carries the uniform charge per unit length λ. We expect the electric field generated by such a charge distribution to possess cylindrical symmetry. We also expect the field to point radially (in a cylindrical sense) away from the wire (assuming that the wire is positively charged).

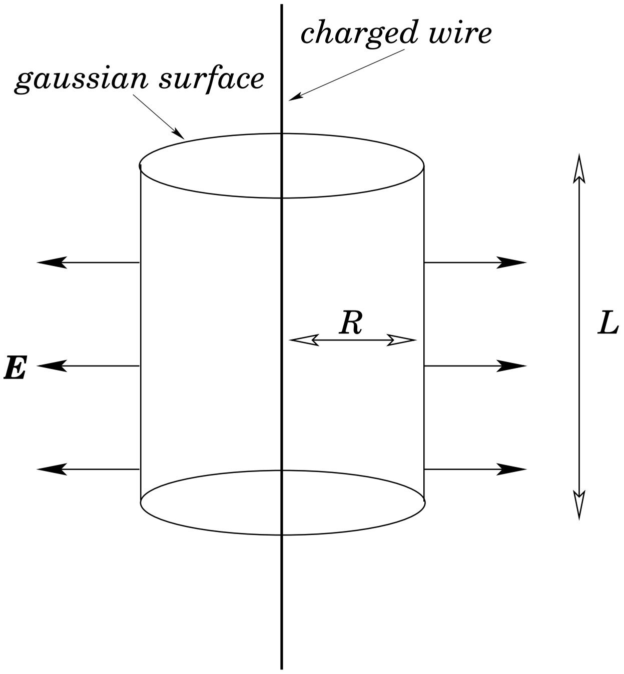


Figure 4.3: *The electric field generated by a uniformly charged wire.*

Let us draw a cylindrical gaussian surface, co-axial with the wire, of radius R and length L—see Fig. 4.3. The above symmetry arguments imply that the electric field generated by the wire is everywhere perpendicular to the curved surface of the cylinder. Thus, according to Gauss' law,

$$E(R)\,2\pi\,R\,L = \frac{\lambda\,L}{\epsilon_0}, \tag{4.5}$$

where $E(R)$ is the electric field-strength a perpendicular distance R from the wire. Here, the left-hand side represents the electric flux through the gaussian surface. Note that there is no contribution from the two flat ends of the cylinder, since the field is parallel to the surface there. The right-hand side represents the total charge enclosed by the cylinder, divided by ϵ_0. It follows that

$$E(R) = \frac{\lambda}{2\pi\epsilon_0\,R}. \tag{4.6}$$

The field points radially (in a cylindrical sense) away from the wire if $\lambda > 0$, and radially towards the wire if $\lambda < 0$.

4.5 Electric Field of a Uniformly Charged Plane

Consider an infinite plane which carries the uniform charge per unit area σ. Suppose that the plane coincides with the y–z plane (*i.e.*, the plane which satisfies $x = 0$). By symmetry, we expect the electric field on either side of the plane to be a function of x only, to be directed normal to the plane, and to point away from/towards the plane depending on whether σ is positive/negative.

Let us draw a cylindrical gaussian surface, whose axis is normal to the plane, and which is cut in half by the plane—see Fig. 4.4. Let the cylinder run from $x = -a$ to $x = +a$, and let its cross-sectional area be A. According to Gauss' law,

$$2\,E(a)\,A = \frac{\sigma\,A}{\epsilon_0}, \tag{4.7}$$

where $E(a) = -E(-a)$ is the electric field strength at $x = +a$. Here, the left-hand side represents the electric flux out of the surface. Note that the only contributions to this flux come from the flat surfaces at the two ends of the cylinder. The

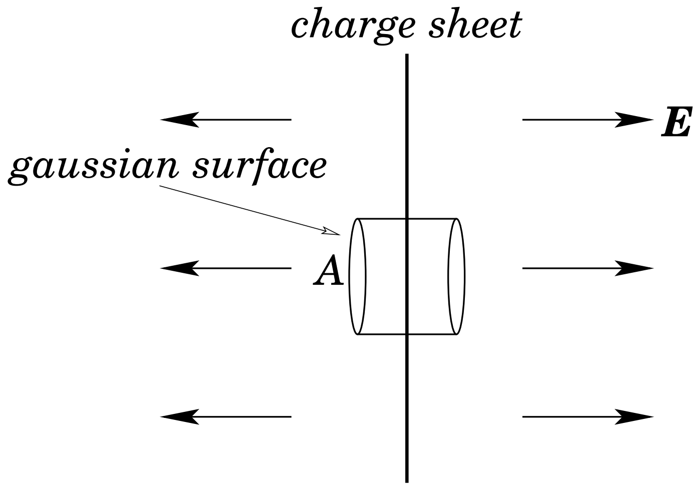


Figure 4.4: *The electric field generated by a uniformly charged plane.*

right-hand side represents the charge enclosed by the cylindrical surface, divided by ϵ_0. It follows that

$$E = \frac{\sigma}{2\,\epsilon_0}. \tag{4.8}$$

Note that the electric field is uniform (*i.e.*, it does not depend on x), normal to the charged plane, and oppositely directed on either side of the plane. The electric field always points away from a positively charged plane, and *vice versa*.

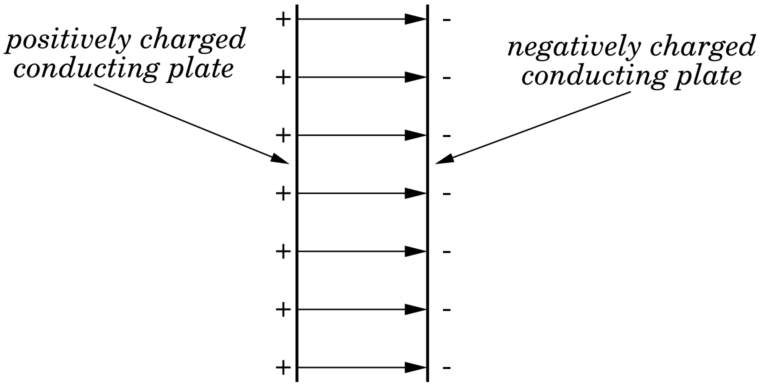

positively charged conducting plate
negatively charged conducting plate
+
-

Figure 4.5: *The electric field generated by two oppositely charged parallel planes.*

Consider the electric field produced by two parallel planes which carry equal and opposite uniform charge densities $\pm\sigma$. We can calculate this field by superposing the electric fields produced by each plane taken in isolation. It is easily seen, from the above discussion, that in the region between the planes the field is uniform, normal to the planes, directed from the positively to the negatively charged plane, and of magnitude

$$E = \frac{\sigma}{\epsilon_0} \tag{4.9}$$

—see Fig. 4.5. Outside this region, the electric field cancels to zero. The above result is only valid for two charged planes of infinite extent. However, the result is approximately valid for two charged planes of finite extent, provided that the spacing between the planes is small compared to their typical dimensions.

4.6 Charged Conductors

Suppose that we put a negative charge on an arbitrarily shaped, solid, conducting object. Where does the excess negative charge end up after the charges have attained their equilibrium positions?

Let us construct a gaussian surface which lies just inside the surface of the conductor. Application of Gauss' law yields

$$\oint \mathbf{E}\cdot d\mathbf{S} = \frac{Q}{\epsilon_0}, \tag{4.10}$$

where Q is the enclosed charge. But, the electric field-strength inside a conductor must be *zero*, since the charges are free to move through the conductor, and will, thus, continue to move until no field remains. Hence, the left-hand side of the above equation is zero, and, therefore, the right-hand side must also be zero. This can only be the case if there are no charges enclosed by the gaussian surface. In other words, there can be no excess charge in the interior of the conductor. Instead, all of the excess charge must be distributed over the surface of the conductor. It follows that:

> Any excess charge on a solid conductor resides entirely on the outer surface of the conductor.

4.7 Worked Examples

Example 4.1: Electric field of a uniformly charged sphere

Question: An insulating sphere of radius a carries a total charge Q which is *uniformly* distributed over the volume of the sphere. Use Gauss' law to find the electric field distribution both inside and outside the sphere.

Solution: By symmetry, we expect the electric field generated by a spherically symmetric charge distribution to point radially towards, or away from, the center of the distribution, and to depend only on the radial distance r from this point. Consider a gaussian surface which is a sphere of radius r, centred on the centre of the charge distribution. Gauss' law gives

$$A(r)\,E_r(r) = \frac{q(r)}{\epsilon_0},$$

where $A(r) = 4\pi\,r^2$ is the area of the surface, $E_r(r)$ the radial electric field-strength at radius r, and $q(r)$ the total charge enclosed by the surface. It is easily seen that

$$q(r) = \left\{ \begin{array}{lcl} Q & & r \geq a \\ Q\,(r/a)^3 & & r < a \end{array} \right. .$$

Thus,

$$E_r(r) = \left\{ \begin{array}{lcl} \frac{Q}{4\pi\epsilon_0\,r^2} & & r \geq a \\ \frac{Q\,r}{4\pi\epsilon_0\,a^3} & & r < a \end{array} \right. .$$

Clearly, the electric field-strength is proportional to r inside the sphere, but falls off like $1/r^2$ outside the sphere.

5 Electric Potential

5.1 Electric Potential Energy

Consider a charge q placed in a uniform electric field $\mathbf{E}$ (*e.g.*, the field between two oppositely charged, parallel conducting plates). Suppose that we *very slowly* displace the charge by a vector displacement $\mathbf{r}$ in a straight-line. How much work must we perform in order to achieve this? Well, the force $\mathbf{F}$ we must exert on the charge is equal and opposite to the electrostatic force $q\,\mathbf{E}$ experienced by the charge (*i.e.*, we must overcome the electrostatic force on the charge before we are free to move it around). The amount of work W we would perform in displacing the charge is simply the product of the force $\mathbf{F} = -q\,\mathbf{E}$ we exert, and the displacement of the charge *in the direction of this force*. Suppose that the displacement vector subtends an angle θ with the electric field $\mathbf{E}$. It follows that

$$W = \mathbf{F}\cdot\mathbf{r} = -q\,\mathbf{E}\cdot\mathbf{r} = -q\,E\,r\,\cos\theta. \tag{5.1}$$

Thus, if we move a positive charge in the direction of the electric field then we do negative work (*i.e.*, we gain energy). Likewise, if we move a positive charge in the opposite direction to the electric field then we do positive work (*i.e.*, we lose energy).

Consider a set of point charges, distributed in space, which are rigidly clamped in position so that they cannot move. We already know how to calculate the electric field $\mathbf{E}$ generated by such a charge distribution (see Sect. 3). In general, this electric field is going to be non-uniform. Suppose that we place a charge q in the field, at point A, say, and then slowly move it along some curved path to a different point B. How much work must we perform in order to achieve this? Let us split up the charge's path from point A to point B into a series of N straight-line segments, where the ith segment is of length Δr_i and subtends an angle θ_i with the local electric field E_i. If we make N sufficiently large then we can adequately represent any curved path between A and B, and we can also ensure that E_i is approximately uniform along the ith path segment. By a simple generalization of Eq. (5.1), the work W we must perform in moving the charge from point A to point B is

$$W = -q \sum_{i=1}^{N} E_i \, \Delta r_i \, \cos\theta_i. \tag{5.2}$$

Finally, taking the limit in which N goes to infinity, the right-hand side of the above expression becomes a line integral:

$$W = -q \int_A^B \mathbf{E} \cdot d\mathbf{r}. \tag{5.3}$$

Let us now consider the special case where point B is identical with point A. In other words, the case in which we move the charge around a *closed loop* in the electric field. How much work must we perform in order to achieve this? It is, in fact, possible to prove, using rather high-powered mathematics, that the net work performed when a charge is moved around a closed loop in an electric field generated by fixed charges is *zero*. However, we do not need to be mathematical geniuses to appreciate that this is a sensible result. Suppose, for the sake of argument, that the net work performed when we take a charge around some closed loop in an electric field is non-zero. In other words, we lose energy every time we take the charge around the loop in one direction, but gain energy every time we take the charge around the loop in the opposite direction. This follows from Eq. (5.2), because when we switch the direction of circulation around the loop the electric field E_i on the ith path segment is unaffected, but, since the charge is moving along the segment in the opposite direction, $\theta_i \to 180^\circ + \theta_i$, and, hence, $\cos\theta_i \to -\cos\theta_i$. Let us choose to move the charge around the loop in the direction in which we gain energy. So, we move the charge once around the loop, and we gain a certain amount of energy in the process. Where does this energy come from? Let us consider the possibilities. Maybe the electric field of the movable charge does negative work on the fixed charges, so that the latter charges lose energy in order to compensate for the energy which we gain? But, the fixed charges cannot move, and so it is impossible to do work on them. Maybe the electric field loses energy in order to compensate for the energy which we gain? (Recall, from the previous section, that there is an energy associated with an electric field which fills space). But, all of the charges (*i.e.*, the fixed charges and the movable charge) are in the same position before and after we take the movable charge around the loop, and so the electric field is the same before and after (since, by Coulomb's law, the electric field only depends on the positions and magnitudes of the charges), and, hence, the energy of the field must be the same before and after. Thus, we have a situation in which we take a charge around a closed loop in an electric field, and gain energy in the process, but nothing loses energy. In other words, the energy appears out of "thin air," which clearly violates the first law of thermodynamics. The only way in which we can avoid this absurd conclusion is if we adopt the following rule:

> The work done in taking a charge around a closed loop in an electric field generated by fixed charges is zero.

One corollary of the above rule is that the work done in moving a charge between two points A and B in such an electric field is *independent* of the path taken between these points. This is easily proved. Consider two different paths, 1 and 2, between points A and B. Let the work done in taking the charge from A to B along path 1 be W_1, and the work done in taking the charge from A to B along path 2 be W_2. Let us take the charge from A to B along path 1, and then from B to A along path 2. The net work done in taking the charge around this closed loop is $W_1 - W_2$. Since we know this work must be zero, it immediately follows that $W_1 = W_2$. Thus, we have a new rule:

> The work done in taking a charge between two points in an electric field generated by fixed charges is independent of the path taken between the points.

A force which has the special property that the work done in overcoming it in order to move a body between two points in space is independent of the path taken between these points is called a *conservative force*. The electrostatic force between stationary charges is clearly a conservative force. Another example of a conservative force is the force of gravity (the work done in lifting a mass only depends on the difference in height between the beginning and end points, and not on the path taken between these points). Friction is an obvious example of a non-conservative force.

Suppose that we move a charge q very slowly from point A to point B in an electric field generated by fixed charges. The work W which we must perform in order to achieve this can be calculated using Eq. (5.3). Since we lose the energy W as the charge moves from A to B, something must gain this energy. Let us, for the moment, suppose that this something is the charge. Thus, the charge *gains* the energy W when we move it from point A to point B. What is the nature of this energy gain? It certainly is not a gain in kinetic energy, since we are moving the particle *slowly*: *i.e.*, such that it always possesses negligible kinetic energy. In fact, if we think carefully, we can see that the gain in energy of the charge depends only on its *position*. For a fixed starting point A, the work W done in taking the charge from point A to point B depends only on the position of point B, and not, for instance, on the route taken between A and B. We usually call energy a body possess by virtue of its position *potential energy*: *e.g.*, a mass has a certain *gravitational potential energy* which depends on its height above the ground. Thus, we can say that when a charge q is taken from point A to point B in an electric field generated by fixed charges its *electric potential energy* P increases by an amount W:

$$P_B - P_A = W. \tag{5.4}$$

Here, P_A denotes the electric potential energy of the charge at point A, *etc.* This definition uniquely defines the *difference* in the potential energy between points A and B (since W is independent of the path taken between these points), but the absolute value of the potential energy at point A remains arbitrary.

We have seen that when a charged particle is taken from point A to point B in an electric field its electric potential energy increases by the amount specified in Eq. (5.4). But, how does the particle store this energy? In fact, the particle does not store the energy at all. Instead, the energy is stored in the electric field surrounding the particle. It is possible to calculate this increase in the energy of the field directly (once we know the formula which links the energy density of an electric field to the magnitude of the field), but it is a very tedious calculation. It is far easier to calculate the work W done in taking the charge from point A to point B, via Eq. (5.3), and then use the conservation of energy to conclude that the energy of the electric field must have increased by an amount W. The fact that we conventionally ascribe this energy increase to the particle, rather than the field, via the concept of electric potential energy, does not matter for all practical purposes. For instance, we call the money which we have in the bank "ours," despite the fact that the bank has possession of it, because we know that the bank will return the money to us any time we ask them. Likewise, when we move a charged particle in an electric field from point A to point B then the energy of the field increases by an amount W (the work which we perform in moving the particle from A to B), but we can safely associate this energy increase with the particle because we know that if the particle is moved back to point A then the field will give all of the energy back to the particle *without loss*. Incidentally, we can be sure that the field returns the energy to the particle without loss because if there were any loss then this would imply that non-zero work is done in taking a charged particle around a closed loop in an electric field generated by fixed charges. We call a force-field which stores energy without loss a *conservative field*. Thus, an electric field, or rather an *electrostatic field* (*i.e.*, an electric field generated by stationary charges), is conservative. It should be clear, from the above discussion, that the concept of potential energy is only meaningful if the field which generates the force in question is conservative.

A gravitational field is another example of a conservative field. It turns out that when we lift a body through a certain height the increase in gravitational potential energy of the body is actually stored in the surrounding gravitational field (*i.e.*, in the distortions of space-time around the body). It is possible to determine the increase in energy of the gravitational field directly, but it is a very difficult calculation involving General Relativity. On the other hand, it is very easy to calculate the work done in lifting the body. Thus, it is convenient to calculate the increase in the energy of the field from the work done, and then to ascribe this energy increase to the body, via the concept of gravitational potential energy.

In conclusion, we can evaluate the increase in electric potential energy of a charge when it is taken between two different points in an electrostatic field from the work done in moving the charge between these two points. The energy is actually stored in the electric field surrounding the charge, but we can safely ascribe this energy to the charge, because we know that the field stores the energy without loss, and will return the energy to the charge whenever it is required to do so by the laws of Physics.

5.2 Electric Potential

Consider a charge q placed in an electric field generated by fixed charges. Let us chose some arbitrary reference point A in the field. At this point, the electric potential energy of the charge is defined to be zero. This *uniquely* specifies the electric potential energy of the charge at every other point in the field. For instance, the electric potential energy P_B at some point B is simply the work W done in moving the charge from A to B along any path. Now, W can be calculated using Eq. (5.3). It is clear, from this equation, that P_B depends both on the particular charge q which we place in the field, and the magnitude and direction of the electric field along the chosen route between points A and B. However, it is also clear that P_B is *directly proportional* to the magnitude of the charge q. Thus, if the electric potential energy of a charge q at point B is P_B then the electric potential energy of a charge $2\,q$ at the same point is $2\,P_B$. We can exploit this fact to define a quantity known as the *electric potential*. The difference in electric potential between two points B and A in an electric field is simply the work done in moving some charge between the two points divided by the magnitude of the charge. Thus,

$$V_B - V_A = \frac{W}{q}, \tag{5.5}$$

where V_A denotes the electric potential at point A, *etc.* This definition uniquely defines the difference in electric potential between points A and B, but the absolute value of the potential at point A remains arbitrary. We can therefore, without loss of generality, set the potential at point A equal to zero. It follows that the potential energy of a charge q at some point B is simply the product of the magnitude of the charge and the electric potential V_B at that point:

$$P_B = q\,V_B. \tag{5.6}$$

It is clear, from a comparison of Eqs. (5.3) and (5.4), that the electric potential at point B (relative to point A) is solely a property of the electric field, and is, therefore, the same for any charge placed at that point. We shall see exactly how the electric potential is related to the electric field later on.

The dimensions of electric potential are work (or energy) per unit charge. The units of electric potential are, therefore, joules per coulomb ($\mathrm{J\,C^{-1}}$). A joule per coulomb is usually referred to as a volt (V): *i.e.*,

$$1\,\mathrm{J\,C^{-1}} \equiv 1\,\mathrm{V}. \tag{5.7}$$

Thus, the alternative (and more conventional) units of electric potential are volts. The difference in electric potential between two points in an electric field is usually referred to as the *potential difference*, or even the difference in "voltage," between the two points.

A battery is a convenient tool for generating a difference in electric potential between two points in space. For instance, a twelve volt (12 V) battery generates an electric field, usually via some chemical process, which is such that the potential difference $V_+ - V_-$ between its positive and negative terminals is twelve volts. This means that in order to move a positive charge of 1 coulomb from the negative to the positive terminal of the battery we must do 12 joules of work against the electric field. (This is true irrespective of the route taken between the two terminals). This implies that the electric field must be directed predominately from the positive to the negative terminal.

More generally, in order to move a charge q through a potential difference ΔV we must do work $W = q\,\Delta V$, and the electric potential energy of the charge increases by an amount $\Delta P = q\,\Delta V$ in the process. Thus, if we move an electron, for which $q = -1.6 \times 10^{-19}\,\mathrm{C}$, through a potential difference of minus 1 volt then we must do 1.6×10^{-19} joules of work. This amount of work (or energy) is called an *electronvolt* (eV): *i.e.*,

$$1\,\mathrm{eV} \equiv 1.6 \times 10^{-19}\,\mathrm{J}. \tag{5.8}$$

The electronvolt is a convenient measure of energy in atomic physics. For instance, the energy required to break up a hydrogen atom into a free electron and a free proton is $13.6\,\mathrm{eV}$.

5.3 Electric Potential and Electric Field

We have seen that the difference in electric potential between two arbitrary points in space is a function of the electric field which permeates space, but is independent of the test charge used to measure this difference. Let us investigate the relationship between electric potential and the electric field.

Consider a charge q which is slowly moved an infinitesimal distance dx along the x-axis. Suppose that the difference in electric potential between the final and initial positions of the charge is dV. By definition, the change dP in the charge's electric potential energy is given by

$$dP = q\,dV \tag{5.9}$$

From Eq. (5.1), the work W which we perform in moving the charge is

$$W = -q\,E\,dx\,\cos\theta, \tag{5.10}$$

where E is the local electric field-strength, and θ is the angle subtended between the direction of the field and the x-axis. By definition, $E\,\cos\theta = E_x$, where E_x is the x-component of the local electric field. Energy conservation demands that $\Delta P = W$ (*i.e.*, the increase in the charge's energy matches the work done on the charge), or

$$q\,dV = -q\,E_x\,dx, \tag{5.11}$$

which reduces to

$$E_x = -\frac{dV}{dx}. \tag{5.12}$$

We call the quantity dV/dx the *gradient* of the electric potential in the x-direction. It basically measures how fast the potential V varies as the coordinate x is changed (but the coordinates y and z are held constant). Thus, the above formula is saying that the x-component of the electric field at a given point in space is equal to *minus* the local gradient of the electric potential in the x-direction.

According to Eq. (5.12), electric field strength has dimensions of potential difference over length. It follows that the units of electric field are volts per meter ($\mathrm{V\,m^{-1}}$). Of course, these new units are entirely equivalent to newtons per coulomb: *i.e.*,

$$1\,\mathrm{V\,m^{-1}} \equiv 1\,\mathrm{N\,C^{-1}}. \tag{5.13}$$

Consider the special case of a uniform x-directed electric field E_x generated by two uniformly charged parallel planes normal to the x-axis. It is clear, from

Eq. (5.12), that if E_x is to be constant between the plates then V must vary *linearly* with x in this region. In fact, it is easily shown that

$$V(x) = V_0 - E_x\, x, \tag{5.14}$$

where V_0 is an arbitrary constant. According to Eq. (5.14), the electric potential V *decreases* continuously as we move along the direction of the electric field. Since a positive charge is accelerated in this direction, we conclude that positive charges are accelerated *down* gradients in the electric potential, in much the same manner as masses fall down gradients of gravitational potential (which is, of course, proportional to height). Likewise, negative charges are accelerated *up* gradients in the electric potential.

According to Eq. (5.12), the x-component of the electric field is equal to minus the gradient of the electric potential in the x-direction. Since there is nothing special about the x-direction, analogous rules must exist for the y- and z-components of the field. These three rules can be combined to give

$$\mathbf{E} = -\left(\frac{dV}{dx}, \frac{dV}{dy}, \frac{dV}{dz}\right). \tag{5.15}$$

Here, the x derivative is taken at constant y and z, *etc.* The above expression shows how the electric field $\mathbf{E}(\mathbf{r})$, which is a vector field, is related to the electric potential $V(\mathbf{r})$, which is a scalar field.

We have seen that electric fields are superposable. That is, the electric field generated by a set of charges distributed in space is simply the *vector sum* of the electric fields generated by each charge taken separately. Well, if electric fields are superposable, it follows from Eq. (5.15) that electric potentials must also be superposable. Thus, the electric potential generated by a set of charges distributed in space is just the *scalar sum* of the potentials generated by each charge taken in isolation. Clearly, it is far easier to determine the potential generated by a set of charges than it is to determine the electric field, since we can sum the potentials generated by the individual charges algebraically, and do not have to worry about their directions (since they have no directions).

Equation (5.15) looks rather forbidding. Fortunately, however, it is possible to rewrite this equation in a more appealing form. Consider two neighboring points

A and B. Suppose that $d\mathbf{r} = (dx, dy, dz)$ is the vector displacement of point B relative to point A. Let dV be the difference in electric potential between these two points. Suppose that we travel from A to B by first moving a distance dx along the x-axis, then moving dy along the y-axis, and finally moving dz along the z-axis. The net increase in the electric potential dV as we move from A to B is simply the sum of the increases d_xV as we move along the x-axis, d_yV as we move along the y-axis, and d_zV as we move along the z-axis:

$$dV = d_xV + d_yV + d_zV. \tag{5.16}$$

But, according to Eq. (5.15), $d_xV = -E_x\,dx$, *etc.* So, we obtain

$$dV = -E_x\,dx - E_y\,dy - E_z\,dy, \tag{5.17}$$

which is equivalent to

$$dV = -\mathbf{E}\cdot d\mathbf{r} = -E\,dr\,\cos\theta, \tag{5.18}$$

where θ is the angle subtended between the vector $d\mathbf{r}$ and the local electric field $\mathbf{E}$. Note that dV attains its most negative value when $\theta = 0$. In other words, the direction of the electric field at point A corresponds to the direction in which the electric potential V decreases most rapidly. A positive charge placed at point A is accelerated in this direction. Likewise, a negative charge placed at A is accelerated in the direction in which the potential increases most rapidly (*i.e.*, $\theta = 180^\circ$). Suppose that we move from point A to a neighboring point B in a direction perpendicular to that of the local electric field (*i.e.*, $\theta = 90^\circ$). In this case, it follows from Eq. (5.18) that the points A and B lie at the same electric potential (*i.e.*, $dV = 0$). The locus of all the points in the vicinity of point A which lie at the same potential as A is a plane perpendicular to the direction of the local electric field. More generally, the surfaces of constant electric potential, the so-called *equipotential surfaces*, exist as a set of non-interlocking surfaces which are everywhere perpendicular to the direction of the electric field. Figure 5.1 shows the equipotential surfaces (dashed lines) and electric field-lines (solid lines) generated by a positive point charge. In this case, the equipotential surfaces are spheres centred on the charge.

In Sect. 4.3, we found that the electric field immediately above the surface of a conductor is directed perpendicular to that surface. Thus, it is clear that

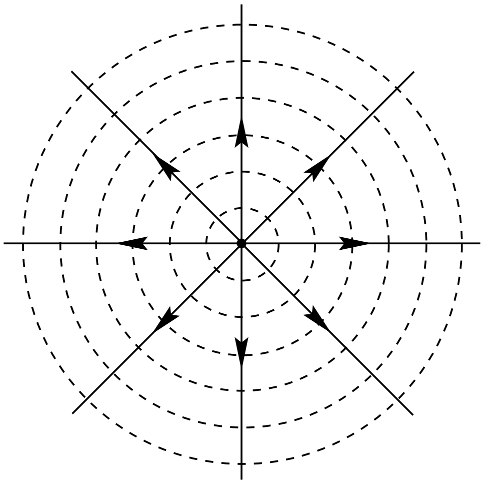


Figure 5.1: *The equipotential surfaces (dashed lines) and the electric field-lines (solid lines) of a positive point charge.*

the surface of a conductor must correspond to an equipotential surface. In fact, since there is no electric field inside a conductor (and, hence, no gradient in the electric potential), it follows that the whole conductor (*i.e.*, both the surface and the interior) lies at the same electric potential.

5.4 Electric Potential of a Point Charge

Let us calculate the electric potential $V(\mathbf{r})$ generated by a point charge q located at the origin. It is fairly obvious, by symmetry, and also by looking at Fig. 5.1, that V is a function of r only, where r is the radial distance from the origin. Thus, without loss of generality, we can restrict our investigation to the potential $V(x)$ generated along the positive x-axis. The x-component of the electric field generated along this axis takes the form

$$E_x(x) = \frac{q}{4\pi\epsilon_0\, x^2}. \tag{5.19}$$

Both the y- and z-components of the field are zero. According to Eq. (5.12), $E_x(x)$ and $V(x)$ are related via

$$E_x(x) = -\frac{dV(x)}{dx}. \tag{5.20}$$

Thus, by integration,

$$V(x) = \frac{q}{4\pi\epsilon_0\, x} + V_0, \tag{5.21}$$

where V_0 is an arbitrary constant. Finally, making use of the fact that $V = V(r)$, we obtain

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0\, r}. \tag{5.22}$$

Here, we have adopted the common convention that the potential at infinity is zero. A potential defined according to this convention is called an *absolute potential*.

Suppose that we have N point charges distributed in space. Let the ith charge q_i be located at position vector $\mathbf{r}_i$. Since electric potential is superposable, and is also a scalar quantity, the absolute potential at position vector $\mathbf{r}$ is simply the algebraic sum of the potentials generated by each charge taken in isolation:

$$V(\mathbf{r}) = \sum_{i=1}^{N} \frac{q_i}{4\pi\epsilon_0\, |\mathbf{r} - \mathbf{r}_i|}. \tag{5.23}$$

The work W we would perform in taking a charge q from infinity and slowly moving it to point $\mathbf{r}$ is the same as the increase in electric potential energy of the charge during its journey [see Eq. (5.4)]. This, by definition, is equal to the product of the charge q and the increase in the electric potential. This, finally, is the same as q times the absolute potential at point $\mathbf{r}$: *i.e.*,

$$W = q\, V(\mathbf{r}). \tag{5.24}$$

5.5 Worked Examples

Example 5.1: Charge in a uniform electric field

Question: A charge of $q = +1.20\,\mu\text{C}$ is placed in a uniform x-directed electric field of magnitude $E_x = 1.40 \times 10^3\,\text{N}\,\text{C}^{-1}$. How much work must be performed in order to move the charge a distance $c = -3.50$ cm in the x-direction? What is the potential difference between the initial and final positions of the charge? If the electric field is produced by two oppositely charged parallel plates separated by a distance $d = 5.00$ cm, what is the potential difference between the plates?

Solution: Let us denote the initial and final positions of the charge A and B, respectively. The work which we must perform in order to move the charge from A to B is minus the product of the electrostatic force on the charge due to the electric field (since the force we exert on the charge is minus this force) and the distance that the charge moves in the direction of this force [see Eq. (5.1)]. Thus,

$$W = -q\,E_x\,c = -(1.2 \times 10^{-6})\,(1.40 \times 10^{3})\,(-3.50 \times 10^{-2}) = +5.88 \times 10^{-5}\,\mathrm{J}.$$

Note that the work is positive. This makes sense, because we would have to do real work (*i.e.*, we would lose energy) in order to move a positive charge in the opposite direction to an electric field (*i.e.*, against the direction of the electrostatic force acting on the charge).

The work done on the charge goes to increase its electric potential energy, so $P_B - P_A = W$. By definition, this increase in potential energy is equal to the product of the potential difference $V_B - V_A$ between points B and A, and the magnitude of the charge q. Thus,

$$q\,(V_B - V_A) = P_B - P_A = W = -q\,E_x\,c,$$

giving

$$V_B - V_A = -E_x\,c = -(1.40 \times 10^{3})\,(-3.50 \times 10^{-2}) = 49.0\,\mathrm{V}.$$

Note that the electric field is directed from point B to point A, and that the former point is at a higher potential than the latter.

It is clear, from the above formulae, that the magnitude of the potential difference between two points in a uniform electric field is simply the product of the electric field-strength and the distance between the two points (in the direction of the field). Thus, the potential difference between the two metal plates is

$$\Delta V = E_x\,d = (1.40 \times 10^{3})\,(5.00 \times 10^{-2}) = 70.0\,\mathrm{V}.$$

If the electric field is directed from plate 1 (the positively charged plate) to plate 2 (the negatively charged plate) then the former plate is at the higher potential.

Example 5.2: Motion of an electron in an electric field

Question: An electron in a television set is accelerated from the cathode to the screen through a potential difference of +1000 V. The screen is 35 mm from the cathode. What is the net change in the potential energy of the electron during the acceleration process? How much work is done by the electric field in accelerating the electron? What is the speed of the electron when it strikes the screen?

Solution: Let call the cathode point A and the screen point B. We are told that the potential difference between points B and A is +1000 V, so

$$V_B - V_A = 1000\,\mathrm{V}.$$

By definition, the difference in electric potential energy of some charge q at points B and A is the product of the charge and the difference in electric potential between these points. Thus,

$$P_B - P_A = q\,(V_B - V_A) = (-1.6\times 10^{-19})\,(1000) = -1.6\times 10^{-16}\,\mathrm{J},$$

since $q = -1.6\times 10^{-19}\,\mathrm{C}$ for an electron. Note that the potential energy of the electron *decreases* as it is accelerated towards the screen. As we have seen, the electric potential energy of a charge is actually held in the surrounding electric field. Thus, a decrease in the potential energy of the charge corresponds to a reduction in the energy of the field. In this case, the energy of the field decreases because it does work W' on the charge. Clearly, the work done (*i.e.*, energy lost) by the field equals the decrease in potential energy of the charge,

$$W' = -\Delta P.$$

Thus,

$$W' = 1.6\times 10^{-16}\,\mathrm{J}.$$

The total energy E of the electron is made up of two components—the electric potential energy P, and the kinetic energy K. Thus,

$$E = P + K.$$

Of course,

$$K = \frac{1}{2}\,m\,v^2,$$

where $m = 9.11 \times 10^{-31}$ kg is the mass of the electron, and v its speed. By conservation of energy, E is a constant of the motion, so

$$K_B - K_A = \Delta K = -\Delta P.$$

In other words, the decrease in electric potential energy of the electron, as it is accelerated towards the screen, is offset by a corresponding increase in its kinetic energy. Assuming that the electron starts from rest (*i.e.* $v_A = 0$), it follows that

$$\frac{1}{2}\,m\,v_B^{\,2} = -\Delta P,$$

or

$$v_B = \sqrt{\frac{-2\,\Delta P}{m}} = \sqrt{\frac{-2\,(-1.6 \times 10^{-16})}{9.11 \times 10^{-31}}} = 1.87 \times 10^7\,\mathrm{m\,s^{-1}}.$$

Note that the distance between the cathode and the screen is immaterial in this problem. The final speed of the electron is entirely determined by its charge, its initial velocity, and the potential difference through which it is accelerated.

Example 5.3: Electric potential due to point charges

Question: A particle of charge $q_1 = +6.0\,\mu\mathrm{C}$ is located on the x-axis at the point $x_1 = 5.1\,\mathrm{cm}$. A second particle of charge $q_2 = -5.0\,\mu\mathrm{C}$ is placed on the x-axis at $x_2 = -3.4\,\mathrm{cm}$. What is the absolute electric potential at the origin ($x = 0$)? How much work must we perform in order to slowly move a charge of $q_3 = -7.0\,\mu\mathrm{C}$ from infinity to the origin, whilst keeping the other two charges fixed?

Solution: The absolute electric potential at the origin due to the first charge is

$$V_1 = k_e\,\frac{q_1}{x_1} = (8.988 \times 10^9)\,\frac{(6 \times 10^{-6})}{(5.1 \times 10^{-2})} = 1.06 \times 10^6\,\mathrm{V}.$$

Likewise, the absolute electric potential at the origin due to the second charge is

$$V_2 = k_e\,\frac{q_2}{|x_2|} = (8.988 \times 10^9)\,\frac{(-5 \times 10^{-6})}{(3.4 \times 10^{-2})} = -1.32 \times 10^6\,\mathrm{V}.$$

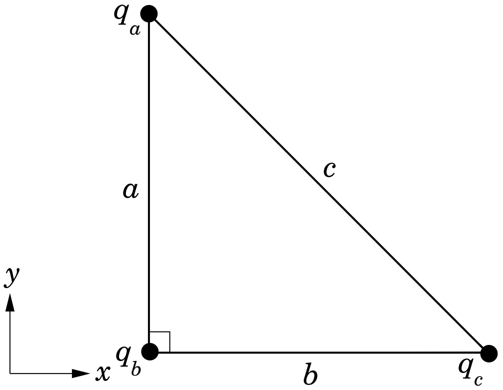
The net potential V at the origin is simply the algebraic sum of the potentials due to each charge taken in isolation. Thus,

$$V = V_1 + V_2 = -2.64 \times 10^5 \, \mathrm{V}.$$

The work W which we must perform in order to slowly moving a charge q_3 from infinity to the origin is simply the product of the charge and the potential difference V between the end and beginning points. Thus,

$$W = q_3 \, V = (-7 \times 10^{-6}) \, (-2.64 \times 10^5) = 1.85 \, \mathrm{J}.$$

Example 5.4: Electric potential due to point charges



Question: Suppose that three point charges, q_a, q_b, and q_c, are arranged at the vertices of a right-angled triangle, as shown in the diagram. What is the absolute electric potential of the third charge if $q_a = -6.0 \, \mu\mathrm{C}$, $q_b = +4.0 \, \mu\mathrm{C}$, $q_c = +2.0 \, \mu\mathrm{C}$, $a = 4.0 \, \mathrm{m}$, and $b = 3.0 \, \mathrm{m}$? Suppose that the third charge, which is initially at rest, is repelled to infinity by the combined electric field of the other two charges, which are held fixed. What is the final kinetic energy of the third charge?

Solution: The absolute electric potential of the third charge due to the presence of the first charge is

$$V_a = k_e \, \frac{q_a}{c} = (8.988 \times 10^9) \, \frac{(-6 \times 10^{-6})}{(\sqrt{4^2 + 3^2})} = -1.08 \times 10^4 \, \mathrm{V},$$

where use has been made of the Pythagorean theorem. Likewise, the absolute electric potential of the third charge due to the presence of the second charge is

$$V_b = k_e \frac{q_b}{b} = (8.988 \times 10^9) \frac{(4 \times 10^{-6})}{(3)} = 1.20 \times 10^4 \, \mathrm{V}.$$

The net absolute potential of the third charge V_c is simply the algebraic sum of the potentials due to the other two charges taken in isolation. Thus,

$$V_c = V_a + V_b = 1.20 \; \times 10^3 \, \mathrm{V}.$$

The change in electric potential energy of the third charge as it moves from its initial position to infinity is the product of the third charge, q_c, and the difference in electric potential $(-V_c)$ between infinity and the initial position. It follows that

$$\Delta P = -q_c \, V_c = -(2 \times 10^{-6}) \, (1.2 \times 10^3) = -2.40 \times 10^{-3} \, \mathrm{J}.$$

This decrease in the potential energy of the charge is offset by a corresponding increase $\Delta K = -\Delta P$ in its kinetic energy. Since the initial kinetic energy of the third charge is zero (because it is initially at rest), the final kinetic energy is simply

$$K = \Delta K = -\Delta P = 2.40 \times 10^{-3} \, \mathrm{J}.$$

6 Capacitance

6.1 Charge Storage

Consider a hollow metal sphere mounted on an insulating stand. The sphere is initially grounded so that no excess charge remains on it. Suppose that we introduce a metal ball, suspended on an insulating thread, through a small hole in the sphere, and then fill in the hole with a metal plug. Let the ball carry a charge $+Q$. What distribution of charge is induced on the hollow sphere as a result of introducing the positive charge into the cavity?

To answer this question we make use of Gauss' law (see Sect. 4.2)

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0}. \tag{6.1}$$

Assuming that the metal ball is placed at the centre of the hollow sphere, we can use symmetry arguments to deduce that the electric field depends only on the radial distance r from the centre, and is everywhere directed radially away from the ball. Let us choose a spherical gaussian surface, centred on the ball, which runs through the interior of the hollow metal sphere. We know that the electric field inside a conductor is everywhere zero (see Sect. 4.6), so the electric flux Φ_E through the surface is also zero. It follows from Gauss' law that zero net charge is enclosed by the surface. Now, there is a charge $+Q$ on the ball at the centre of the hollow sphere, so there must be an equal and opposite charge $-Q$ distributed over the interior surface of the sphere (recall that any charge carried on a conductor must reside on its surface). Furthermore, since the sphere is insulated, and was initially uncharged, a charge $+Q$ must be distributed over its exterior surface. Thus, when the charge $+Q$ is introduced into the centre of the sphere, there is a redistribution of charge in the sphere such that a positive charge $+Q$ is repelled to its exterior surface, leaving a negative charge $-Q$ on the interior surface. (In actuality, free electrons are attracted to the interior surface, exposing positive charges on the exterior surface). Further use of Gauss' law shows that the electric field between the charged ball and the interior surface of the sphere is the same as that generated by a point charge $+Q$ located at the centre of the sphere. Likewise, for the electric field exterior to the sphere. The electric field inside the conducting sphere is, of course, zero.

Suppose, finally, that the ball is moved so that it touches the inside of the hollow sphere. The charge $-Q$ on the interior surface of the sphere cancels the charge $+Q$ on the ball, leaving the charge $+Q$ distributed over its exterior surface. Thus, the effect of touching the ball to the inside of the sphere is to transfer the charge $+Q$ from the ball to the exterior surface of the sphere. In principle, we can repeat this process, again and again, until a very large amount of charge is accumulated on the outside of the sphere. The idea of transferring charge from one conductor to another by means of internal contact is the theoretical basis of the *Van de Graaff generator*. In this type of device, charge is continuously transmitted to a conducting sphere by means of a moving belt charged by friction.

6.2 Breakdown

Is there any practical limit to the charge Q which can be accumulated on the conducting sphere of a Van de Graaff generator? Well, we know that the field outside the sphere is just the same as if the charge Q were placed at the centre of the sphere. In fact, the electric field is at its most intense just above the surface of the sphere, where it has the magnitude $E = Q/(4\pi\epsilon_0\, a^2)$. Here, a is the radius of the sphere. Air (assuming that the sphere is surrounded by air) is generally a very good insulator. However, air ceases to be an insulator when the electric field-strength through it exceeds some critical value which is about $E_{\rm crit} \sim 10^6\ {\rm V\,m^{-1}}$. This phenomenon is known as *breakdown*, and is associated with the formation of sparks. The explanation of breakdown is quite straightforward. Air naturally contains a very small fraction of ionized molecules (not enough to prevent air from being an insulator). In an electric field, these ionized molecules are constantly being accelerated, and then crashing into neutral molecules. As the strength of the field is increased, the ionized molecules are accelerated to ever higher energies before crashing into the neutral molecules. Eventually, a critical field-strength $E_{\rm crit}$ is reached at which the ionized molecules are accelerated to a sufficiently high energy that they ionize the neutral molecules when they hit them. At this point, a chain reaction takes place which rapidly leads to the almost complete ionization of the air. Thus, the air makes an almost instantaneous transition from a good insulator to a good conductor. It follows that the charge Q on the conducting sphere of a Van de Graaff generator can never exceed the critical value $Q_{\rm crit} = 4\pi\epsilon_0\, a^2\, E_{\rm crit}$, because for $Q \geq Q_{\rm crit}$ the electric field around the sphere is sufficiently intense to cause breakdown. Of course, when breakdown occurs the charge on the sphere is conducted to earth.

The phenomenon of breakdown sets an upper limit on the charge which can be stored on a conductor. There is, however, another important factor which affects the onset of breakdown. This is best illustrated in the following simple example. Suppose that we have two charged conducting spheres of radii a and b, respectively, which are connected by a long conducting wire. The wire allows charge to move back and forth between the spheres until they reach the same potential (recall that the electric potential is uniform in a conductor). Let Q_a be the charge on the first sphere, and Q_b the charge on the second sphere. Of course, the total charge $Q = Q_a + Q_b$ carried by the two spheres is a conserved quantity. The electric field generated by each sphere is the same as if the charge on that sphere were concentrated at its centre. Assuming that the wire is sufficiently long that the two spheres do not affect one another very much, the absolute potential of the first sphere is $V_a = Q_a/(4\pi\epsilon_0\, a)$, whereas that of the second sphere is $V_b = Q_b/(4\pi\epsilon_0\, b)$ [see Eq. (5.22)]. Since $V_a = V_b$, we find that

$$\frac{Q_a}{Q} = \frac{a}{a+b}, \tag{6.2}$$

$$\frac{Q_b}{Q} = \frac{b}{a+b}. \tag{6.3}$$

Note that if the second sphere is much smaller than the first (*i.e.*, if $b \ll a$) then the larger sphere grabs the lion's share of the charge:

$$\frac{Q_a}{Q_b} = \frac{a}{b} \gg 1. \tag{6.4}$$

The electric field-strengths just above the surfaces of the two spheres are $E_a = Q_a/(4\pi\epsilon_0\, a^2)$ and $E_b = Q_b/(4\pi\epsilon_0\, b^2)$, respectively. Thus, the ratio of the field-strengths generated in the immediate vicinities of the two spheres is

$$\frac{E_b}{E_a} = \frac{Q_b}{Q_a}\frac{a^2}{b^2} = \frac{a}{b}. \tag{6.5}$$

Clearly, if $b \ll a$ then the field just above the smaller sphere is far stronger than that above the larger one. Suppose that the total charge Q_0 on the two spheres is gradually increased until breakdown occurs. Since $E_b \gg E_a$, it follows that breakdown always occurs above the smaller sphere.

Equation (6.5) is a special case of a far more general rule: *i.e.*, the electric field-strength above some point on the surface of a conductor is inversely proportional to the local radius of curvature of the surface. It is clear that if we wish to store significant amounts of charge on a conductor then the surface of the conductor must be made as smooth as possible. Any sharp spikes on the surface possess relatively small radii of curvature. Intense local electric fields are generated above these spikes whenever the conductor is charged. These fields can easily exceed the critical field for the breakdown of air, leading to sparking, and the eventual loss of the charge on the conductor. Sparking tends to be very destructive because of its highly localized nature, which leads inevitably to very large electric currents, and, hence, to intense heating.

Clouds can acquire very large negative charges during thunderstorms. An equal and opposite positive charge is induced on the surface of the Earth. The electric field generated between the clouds and the Earth can become sufficiently large to cause breakdown in the atmosphere, giving rise to the phenomenon which we call *lightning*. Let us consider the various factors which determine where lightning strikes. Breakdown starts at cloud level, as a so-called "dark leader" of ionized air traces out a path towards the ground. When it comes within about 10 meters of ground level, a second dark leader comes up from the ground to meet it. Once the two leaders meet, and a conducting path is established, the lightning strike proper occurs. Note that, contrary to popular opinion, the lightning strike travels *upwards* from the Earth to the clouds. It is clear that lightning "strikes" a particular object on the ground because the object emits a dark leader: *i.e.*, because breakdown takes place just above the object. In a thunderstorm, the ground, and the objects upon it, acts essentially like a charged conductor with a convoluted surface. Thus, any "spikes" on the ground (*e.g.*, a person standing in a field, a radio mast, a lightning rod) are comparatively more likely to be hit by lightning, because the electric field-strength above these points is relatively large, which facilitates breakdown.

6.3 Capacitance

As we have seen, the amount of charge which can be stored on a conductor is limited by the electric field-strength just above its surface, which is not allowed to exceed a certain critical value, $E_{\rm crit}$. Unfortunately, the field-strength varies from point to point across the surface (unless the surface possesses a constant radius of curvature). It is, therefore, generally convenient to parameterize the maximum field-strength above the surface of a conductor in terms of the voltage difference V between the conductor and either infinity or another conductor. The point is that V, unlike the electric field-strength, is a constant over the surface, and can, therefore, be specified unambiguously.

How do we tell the difference between a good and a bad charge storage device? Well, a good charge storage device must be capable of storing a large amount of charge without causing breakdown. Likewise, a bad charge storage device is only capable of storing a small amount of charge before breakdown occurs. Thus, if we place a charge Q in a good storage device then the electric fields generated just above the surface of the device should be comparatively weak. In other words, the voltage V should be relatively small. A convenient measure of the ability of a device to store electric charge is its *capacitance*, C, which is defined as the ratio of Q over V:

$$C = \frac{Q}{V}. \tag{6.6}$$

Obviously, a good charge storage device possesses a high capacitance. Note that the capacitance of a given charge storage device is a constant which depends on the dimensions of the device, but is independent of either Q or V. This follows from the *linear* nature of the laws of electrostatics: *i.e.*, if we double the charge on the device, then we double the electric fields generated around the device, and so we double the voltage difference between the device and (say) infinity.

In other words, $V \propto Q$. The units of capacitance are called *farads* (F), and are equivalent to coulombs per volt:

$$1\,\mathrm{F} \equiv 1\,\mathrm{C\,V^{-1}}. \tag{6.7}$$

A farad is actually a pretty unwieldy unit. In fact, most of the capacitors found in electronic circuits have capacitances in the micro-farad range.

Probably the simplest type of capacitor is the so-called *parallel plate capacitor*, which consists of two parallel conducting plates, one carrying a charge $+Q$ and the other a charge $-Q$, separated by a distance d. Let A be the area of the two plates. It follows that the charge densities on the plates are σ and $-\sigma$, respectively, where $\sigma = Q/A$. Now, we have already seen (in Sect. 4.5) that the electric field generated between two oppositely charged parallel plates is uniform, and of magnitude $E = \sigma/\epsilon_0$. The field is directed perpendicular to the plates, and runs from the positively to the negatively charged plate. Note that this result is only valid if the spacing between the plates is much less than their typical dimensions. According to Eq. (4.8), the potential difference V between the plates is given by

$$V = E\,d = \frac{\sigma\,d}{\epsilon_0} = \frac{Q\,d}{\epsilon_0\,A}, \tag{6.8}$$

where the positively charged plate is at the higher potential. It follows from Eq. (6.6) that the capacitance of a parallel plate capacitor takes the form

$$C = \frac{\epsilon_0\,A}{d}. \tag{6.9}$$

Note that the capacitance is proportional to the area of the plates, and inversely proportional to their perpendicular spacing. It follows that a good parallel plate capacitor possesses closely spaced plates of large surface area.

6.4 Dielectrics

Strictly speaking, the expression (6.9) for the capacitance of a parallel plate capacitor is only valid if the region between plates is a vacuum. However, this expression turns out to be a pretty good approximation if the region is filled with

Material	K
Vacuum	1
Air	1.00059
Water	80
Paper	3.5
Pyrex	4.5
Teflon	2.1

Table 6.1: *Dielectric constants of various common materials.*

air. But, what happens if the region between the plates is filled by an insulating material such as glass or plastic?

We could investigate this question experimentally. Suppose that we started with a charged parallel plate capacitor, whose plates were separated by a vacuum gap, and which was disconnected from any battery or other source of charge. We could measure the voltage difference V_0 between the plates using a voltmeter. Suppose that we inserted a slab of some insulating material (*e.g.*, glass) into the gap between the plates, and then re-measured the voltage difference between the plates. We would find that the new voltage difference V was *less* than V_0, despite that fact that the charge Q on the plates was unchanged. Let us denote the voltage ratio V_0/V as K. Since, $C = Q/V$, it follows that the capacitance of the capacitor must have increased by a factor K when the insulating slab was inserted between the plates.

An insulating material which has the effect of increasing the capacitance of a vacuum-filled parallel plate capacitor, when it is inserted between its plates, is called a *dielectric* material, and the factor K by which the capacitance is increased is called the *dielectric constant* of that material. Of course, K varies from material to material. A few sample values are given in Table 6.1. Note, however, that K is always greater than unity, so filling the gap between the plates of a parallel plate capacitor with a dielectric material always increases the capacitance of the device to some extent. On the other hand, K for air is only 0.06 percent greater than K for a vacuum (*i.e.*, $K = 1$), so an air-filled capacitor is virtually indistinguishable from a vacuum-filled capacitor.

The formula for the capacitance of a dielectric-filled parallel plate capacitor is

$$C = \frac{\epsilon\,A}{d}, \tag{6.10}$$

where

$$\epsilon = K\,\epsilon_0 \tag{6.11}$$

is called the *permittivity* of the dielectric material between the plates. Note that the permittivity ϵ of a dielectric material is always greater than the permittivity of a vacuum ϵ_0

How do we explain the reduction in voltage which occurs when we insert a dielectric between the plates of a vacuum-filled parallel plate capacitor? Well, if the voltage difference between the plates is reduced then the electric field between the plates must be reduced by the same factor. In other words, the electric field E_0 generated by the charge stored on the capacitor plates must be partially canceled out by an opposing electric field E_1 generated by the dielectric itself when it is placed in an external electric field. What is the cause of this opposing field? It turns out that the opposing field is produced by the *polarization* of the constituent molecules of the dielectric when they are placed in an electric field (see Sect. 3.4). If E_0 is sufficiently small then the degree of polarization of each molecule is *proportional to* the strength of the polarizing field E_0. It follows that the strength of the opposing field E_1 is also proportional to E_0. In fact, the constant of proportionality is $1 - 1/K$, so $E_1 = (1 - 1/K)\,E_0$. The net electric field between the plates is $E_0 - E_1 = E_0/K$. Hence, both the field and voltage between the plates are reduced by a factor K with respect to the vacuum case. In principle, the dielectric constant K of a dielectric material can be calculated from the properties of the molecules which make up the material. In practice, this calculation is too difficult to perform, except for very simple molecules. Note that the result that the degree of polarization of a polarizable molecule is proportional to the external electric field-strength E_0 breaks down if E_0 becomes too large (just as Hooke's law breaks down if we pull too hard on a spring). Fortunately, however, the field-strengths encountered in conventional laboratory experiments are not generally large enough to invalidate this result.

We have seen that when a dielectric material of dielectric constant K is placed in the uniform electric field generated between the plates of a parallel plate capacitor then the material polarizes, giving rise to a reduction of the field-strength between the plates by some factor K. Since there is nothing particularly special about the electric field between the plates of a capacitor, we surmise that this result is quite general. Thus, if space is filled with a dielectric medium then Coulomb's law is rewritten as

$$f = \frac{q\,q'}{4\pi\epsilon\, r^2}, \tag{6.12}$$

and the formula for the electric field generated by a point charge becomes

$$E = \frac{q}{4\pi\epsilon\, r^2}, \tag{6.13}$$

etc. Clearly, in a dielectric medium, the laws of electrostatics take exactly the same form as in a vacuum, except that the permittivity of free space ϵ_0 is replaced by the permittivity $\epsilon = K\,\epsilon_0$ of the medium. Dielectric materials have the general effect of reducing the electric fields and potential differences generated by electric charges. Such materials are extremely useful because they inhibit breakdown. For instance, if we fill a parallel plate capacitor with a dielectric material then we effectively increase the amount of charge we can store on the device before breakdown occurs.

6.5 Capacitors in Series and in Parallel

Capacitors are one of the standard components in electronic circuits. Moreover, complicated combinations of capacitors often occur in practical circuits. It is, therefore, useful to have a set of rules for finding the equivalent capacitance of some general arrangement of capacitors. It turns out that we can always find the equivalent capacitance by repeated application of *two* simple rules. These rules related to capacitors connected in series and in parallel.

Consider two capacitors connected in *parallel*: *i.e.*, with the positively charged plates connected to a common "input" wire, and the negatively charged plates attached to a common "output" wire—see Fig. 6.1. What is the equivalent capacitance between the input and output wires? In this case, the potential difference

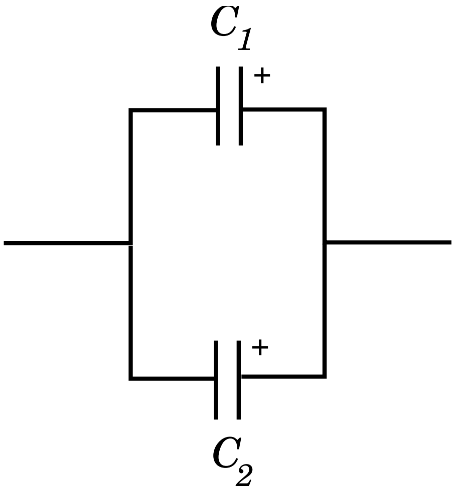


Figure 6.1: *Two capacitors connected in parallel.*

V across the two capacitors is the same, and is equal to the potential difference between the input and output wires. The total charge Q, however, stored in the two capacitors is divided between the capacitors, since it must distribute itself such that the voltage across the two is the same. Since the capacitors may have different capacitances, C_1 and C_2, the charges Q_1 and Q_2 may also be different. The equivalent capacitance C_{eq} of the pair of capacitors is simply the ratio Q/V, where $Q = Q_1 + Q_2$ is the total stored charge. It follows that

$$C_{eq} = \frac{Q}{V} = \frac{Q_1 + Q_2}{V} = \frac{Q_1}{V} + \frac{Q_2}{V}, \tag{6.14}$$

giving

$$C_{eq} = C_1 + C_2. \tag{6.15}$$

Here, we have made use of the fact that the voltage V is common to all three capacitors. Thus, the rule is:

> The equivalent capacitance of two capacitors connected in parallel is the sum of the individual capacitances.

For N capacitors connected in parallel, Eq. (6.15) generalizes to $C_{eq} = \sum_{i=1}^{N} C_i$.

Consider two capacitors connected in *series*: *i.e.*, in a line such that the positive plate of one is attached to the negative plate of the other—see Fig. 6.2. In fact,

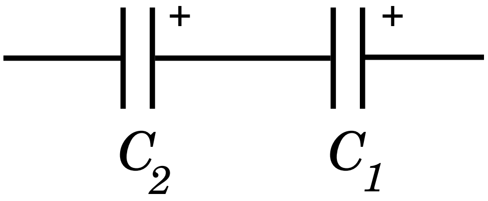


Figure 6.2: *Two capacitors connected in series.*

let us suppose that the positive plate of capacitor 1 is connected to the "input" wire, the negative plate of capacitor 1 is connected to the positive plate of capacitor 2, and the negative plate of capacitor 2 is connected to the "output" wire. What is the equivalent capacitance between the input and output wires? In this case, it is important to realize that the charge Q stored in the two capacitors is the same. This is most easily seen by considering the "internal" plates: *i.e.*, the negative plate of capacitor 1, and the positive plate of capacitor 2. These plates are physically disconnected from the rest of the circuit, so the total charge on them must remain constant. Assuming, as seems reasonable, that these plates carry zero charge when zero potential difference is applied across the two capacitors, it follows that in the presence of a non-zero potential difference the charge $+Q$ on the positive plate of capacitor 2 must be balanced by an equal and opposite charge $-Q$ on the negative plate of capacitor 1. Since the negative plate of capacitor 1 carries a charge $-Q$, the positive plate must carry a charge $+Q$. Likewise, since the positive plate of capacitor 2 carries a charge $+Q$, the negative plate must carry a charge $-Q$. The net result is that both capacitors possess the same stored charge Q. The potential drops, V_1 and V_2, across the two capacitors are, in general, different. However, the sum of these drops equals the total potential drop V applied across the input and output wires: *i.e.*, $V = V_1 + V_2$. The equivalent capacitance of the pair of capacitors is again $C_{eq} = Q/V$. Thus,

$$\frac{1}{C_{eq}} = \frac{V}{Q} = \frac{V_1 + V_2}{Q} = \frac{V_1}{Q} + \frac{V_2}{Q}, \tag{6.16}$$

giving

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}. \tag{6.17}$$

Here, we have made use of the fact that the charge Q is common to all three capacitors. Hence, the rule is:

> The reciprocal of the equivalent capacitance of two capacitors connected in series is the sum of the reciprocals of the individual capacitances.

For N capacitors connected in series, Eq. (6.17) generalizes to $1/C_{eq} = \sum_{i=1}^{N}(1/C_i)$.

6.6 Energy Stored by Capacitors

Let us consider charging an initially uncharged parallel plate capacitor by transferring a charge Q from one plate to the other, leaving the former plate with charge $-Q$ and the later with charge $+Q$. Of course, once we have transferred some charge, an electric field is set up between the plates which opposes any further charge transfer. In order to fully charge the capacitor, we must do work against this field, and this work becomes energy stored in the capacitor. Let us calculate this energy.

Suppose that the capacitor plates carry a charge q and that the potential difference between the plates is V. The work we do in transferring an *infinitesimal* amount of charge dq from the negative to the positive plate is simply

$$dW = V\,dq. \tag{6.18}$$

In order to evaluate the total work $W(Q)$ done in transferring the total charge Q from one plate to the other, we can divide this charge into many small increments dq, find the incremental work dW done in transferring this incremental charge, using the above formula, and then sum all of these works. The only complication is that the potential difference V between the plates is a function of the total transferred charge. In fact, $V(q) = q/C$, so

$$dW = \frac{q\,dq}{C}. \tag{6.19}$$

Integration yields

$$W(Q) = \int_0^Q \frac{q\,dq}{C} = \frac{Q^2}{2\,C}. \tag{6.20}$$

Note, again, that the work W done in charging the capacitor is the same as the energy stored in the capacitor. Since $C = Q/V$, we can write this stored energy in one of three equivalent forms:

$$W = \frac{Q^2}{2\,C} = \frac{C\,V^2}{2} = \frac{Q\,V}{2}. \tag{6.21}$$

These formulae are valid for any type of capacitor, since the arguments that we used to derive them do not depend on any special property of parallel plate capacitors.

Where is the energy in a parallel plate capacitor actually stored? Well, if we think about it, the only place it could be stored is in the electric field generated between the plates. This insight allows us to calculate the energy (or, rather, the energy density) of an electric field.

Consider a vacuum-filled parallel plate capacitor whose plates are of cross sectional area A, and are spaced a distance d apart. The electric field E between the plates is approximately uniform, and of magnitude σ/ϵ_0, where $\sigma = Q/A$, and Q is the charge stored on the plates. The electric field elsewhere is approximately zero. The potential difference between the plates is $V = E\,d$. Thus, the energy stored in the capacitor can be written

$$W = \frac{C\,V^2}{2} = \frac{\epsilon_0\,A\,E^2\,d^2}{2\,d} = \frac{\epsilon_0\,E^2\,A d}{2}, \tag{6.22}$$

where use has been made of Eq. (6.9). Now, $A\,d$ is the volume of the field-filled region between the plates, so if the energy is stored in the electric field then the energy per unit volume, or *energy density*, of the field must be

$$w = \frac{\epsilon_0\,E^2}{2}. \tag{6.23}$$

It turns out that this result is quite general. Thus, we can calculate the energy content of any electric field by dividing space into little cubes, applying the above formula to find the energy content of each cube, and then summing the energies thus obtained to obtain the total energy.

It is easily demonstrated that the energy density in a dielectric medium is

$$w = \frac{\epsilon\,E^2}{2}, \tag{6.24}$$

where $\epsilon = K\,\epsilon_0$ is the permittivity of the medium. This energy density consists of two elements: the energy density $\epsilon_0\,E^2/2$ held in the electric field, and the energy density $(K-1)\,\epsilon_0\,E^2/2$ held in the dielectric medium (this represents the work done on the constituent molecules of the dielectric in order to polarize them).

6.7 Worked Examples

Example 6.1: Parallel plate capacitor

Question: A parallel plate capacitor consists of two metal plates, each of area $A = 150\,\mathrm{cm}^2$, separated by a vacuum gap $d = 0.60$ cm thick. What is the capacitance of this device? What potential difference must be applied between the plates if the capacitor is to hold a charge of magnitude $Q = 1.00 \times 10^{-3}\,\mu\mathrm{C}$ on each plate?

Solution: Making use of formula (6.9), the capacitance C is given by

$$C = \frac{(8.85 \times 10^{-12})\,(150 \times 10^{-4})}{(0.6 \times 10^{-2})} = 2.21 \times 10^{-11} = 22.1\,\mathrm{pF}.$$

The voltage difference V between the plates and the magnitude of the charge Q stored on each plate are related via $C = Q/V$, or $V = Q/C$. Hence, if $Q = 1.00 \times 10^{-3}\,\mu\mathrm{C}$ then

$$V = \frac{(1.00 \times 10^{-9})}{(2.21 \times 10^{-11})} = 45.2\,\mathrm{V}.$$

Example 6.2: Dielectric filled capacitor

Question: A parallel plate capacitor has a plate area of $50\,\mathrm{cm}^2$ and a plate separation of 1.0 cm. A potential difference of $V_0 = 200$ V is applied across the plates with no dielectric present. The battery is then disconnected, and a piece of Bakelite ($K = 4.8$) is inserted which fills the region between the plates. What is the capacitance, the charge on the plates, and the potential difference between the plates, before and after the dielectric is inserted?

Answer: Before the dielectric is inserted, the space between the plates is presumably filled with air. Since the dielectric constant of air is virtually indistinguishable from that of a vacuum, let us use the vacuum formula (6.9) to calculate the initial capacitance C_0. Thus,

$$C_0 = \frac{\epsilon_0\, A}{d} = \frac{(8.85 \times 10^{-12})\,(50 \times 10^{-4})}{(1 \times 10^{-2})} = 4.4\,\text{pF}.$$

After the dielectric is inserted, the capacitance increases by a factor K, which in this case is 4.8, so the new capacitance C is given by

$$C = K\,C_0 = (4.8)\,(4.4 \times 10^{-12}) = 21\,\text{pF}.$$

Before the dielectric is inserted, the charge Q_0 on the plates is simply

$$Q_0 = C_0 V_0 = (4.4 \times 10^{-12})\,(200) = 8.8 \times 10^{-10}\,\text{C}.$$

After the dielectric is inserted, the charge Q is exactly the same, since the capacitor is disconnected, and so the charge cannot leave the plates. Hence,

$$Q = Q_0 = 8.8 \times 10^{-10}\,\text{C}.$$

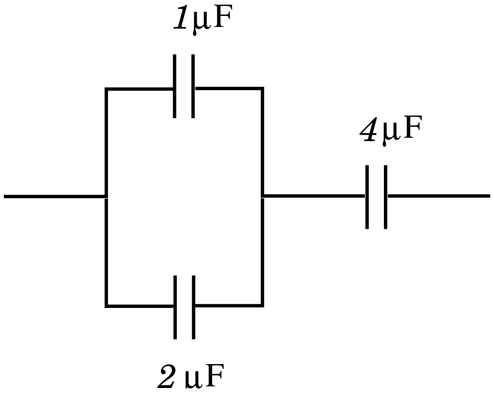
The potential difference before the dielectric is inserted is given as $V_0 = 200$ V. The potential difference V after the dielectric is inserted is simply

$$V = \frac{Q}{C} = \frac{(8.8 \times 10^{-10})}{(21 \times 10^{-12})} = 42\,\text{V}.$$

Note, of course, that $V = V_0/K$.

Example 6.3: Equivalent capacitance

Question: A 1 μF and a 2 μF capacitor are connected in parallel, and this pair of capacitors is then connected in series with a 4 μF capacitor, as shown in the diagram. What is the equivalent capacitance of the whole combination? What is



the charge on the 4 μF capacitor if the whole combination is connected across the terminals of a 6 V battery? Likewise, what are the charges on the 1 μF and 2 μF capacitors?

Answer: The equivalent capacitance of the 1 μF and 2 μF capacitors connected in parallel is $1 + 2 = 3$ μF. When a 3 μF capacitor is combined in series with a 4 μF capacitor, the equivalent capacitance of the whole combination is given by

$$\frac{1}{C_{eq}} = \frac{1}{(3 \times 10^{-6})} + \frac{1}{(4 \times 10^{-6})} = \frac{(7)}{(12 \times 10^{-6})}\ \mathrm{F}^{-1},$$

and so

$$C_{eq} = \frac{(12 \times 10^{-6})}{(7)} = 1.71\ \mu\mathrm{F}.$$

The charge delivered by the 6 V battery is

$$Q = C_{eq}V = (1.71 \times 10^{-6})\,(6) = 10.3\ \mu\mathrm{C}.$$

This is the charge on the 4 μF capacitor, since one of the terminals of the battery is connected directly to one of the plates of this capacitor.

The voltage drop across the 4 μF capacitor is

$$V_4 = \frac{Q}{C_4} = \frac{(10.3 \times 10^{-6})}{(4 \times 10^{-6})} = 2.57\,\mathrm{V}.$$

Thus, the voltage drop across the 1 μF and 2 μF combination must be $V_{12} = 6 - 2.57 = 3.43$ V. The charge stored on the 1 μF is given by

$$Q_1 = C_1\,V_{12} = (1 \times 10^{-6})\,(3.43) = 3.42\ \mu\mathrm{C}.$$

Likewise, the charge stored on the 2 μF capacitor is

$$Q_2 = C_2\,V_{12} = (2\times 10^{-6})\,(3.43) = 6.84\,\mu\text{C}.$$

Note that the total charge stored on the 1 μF and 2 μF combination is $Q_{12} = Q_1 + Q_2 = 10.3\,\mu\text{C}$, which is the same as the charge stored on the 4 μF capacitor. This makes sense because the 1 μF and 2 μF combination and the 4 μF capacitor are connected in series.

Example 6.4: Energy stored in a capacitor

Question: An air-filled parallel plate capacitor has a capacitance of 5.0 pF. A potential of 100 V is applied across the plates, which are 1.0 cm apart, using a storage battery. What is the energy stored in the capacitor? Suppose that the battery is disconnected, and the plates are moved until they are 2.0 cm apart. What now is the energy stored in the capacitor? Suppose, instead, that the battery is left connected, and the plates are again moved until they are 2.0 cm apart. What is the energy stored in the capacitor in this case?

Answer: The initial energy stored in the capacitor is

$$W = \frac{C\,V^2}{2} = \frac{(5\times 10^{-12})\,(100)^2}{2} = 2.5\times 10^{-8}\,\text{J}.$$

When the spacing between the plates is doubled, the capacitance of the capacitor is halved to 2.5 pF. If the battery is disconnected then this process takes place at constant charge Q. Thus, it follows from the formula

$$W = \frac{Q^2}{2\,C}$$

that the energy stored in the capacitor doubles. So, the new energy is 5.0×10^{-8} J. Incidentally, the increased energy of the capacitor is accounted for by the work done in pulling the capacitor plates apart (since these plates are oppositely charged, they attract one another).

If the battery is left connected, then the capacitance is still halved, but now the process takes place at constant voltage V. It follows from the formula

$$W = \frac{C\,V^2}{2}$$

that the energy stored in the capacitor is halved. So, the new energy is 1.25×10^{-8} J. Incidentally, the energy lost by the capacitor is given to the battery (in effect, it goes to re-charging the battery). Likewise, the work done in pulling the plates apart is also given to the battery.

7 Electric Current

7.1 Electric Circuits

A battery is a device possessing a *positive* and a *negative* terminal. Some process, usually a chemical reaction, takes place inside the battery which causes positive charge to migrate towards the positive terminal, and *vice versa*. This process continues until the electric field set up between the two terminals is sufficiently strong to inhibit any further charge migration.

An *electric circuit* is a conducting path, external to the battery, which allows charge to flow from one terminal to the other. A simple circuit might consist of a single strand of metal wire linking the positive and negative terminals. A more realistic circuit possesses multiple branch points, so that charge can take many different paths between the two terminals.

Suppose that a (positive) charge q is driven around the external circuit, from the positive to the negative terminal, by the electric field set up between the terminals. The work done on the charge by this field during its journey is $q\,V$, where V is the difference in electric potential between the positive and negative terminals. We usually refer to V as the *voltage* of the battery: *e.g.*, when we talk of a 6 volt battery, what we actually mean is that the potential difference between its two terminals is 6 V. Note, from Sect. 5, that the electrical work $q\,V$ done on the charge is completely independent of the route it takes between the terminals. In other words, although there are, in general, many different paths through the external circuit which the charge could take in order to get from the positive to the negative terminal of the battery, the electrical energy which the charge acquires in making this journey is always the same. Since, when analyzing electrical circuits, we are primarily interested in *energy* (*i.e.*, in the transformation of the chemical energy of the battery into heat energy in some electric heating element, or mechanical energy in some electric motor, *etc.*), it follows that the property of a battery which primarily concerns us is its *voltage*. Hence, we do not have to map out the electric field generated by a battery in order to calculate how much energy this field gives to a charge q which goes around some external circuit connected to it. All we need to know is the potential difference V between the two terminals of the battery. This is obviously an enormous simplification.

This section is only concerned with *steady-state* electric circuits powered by batteries of constant voltage. Thus, the rate at which electric charge flows out of the positive terminal of the battery into the external circuit must match the rate at which charge flows from the circuit into the negative terminal of the battery, otherwise charge would build up in either the battery or the circuit, which would not correspond to a steady-state situation. The rate at which charge flows out of the positive terminal is termed the *electric current* flowing out of the battery. Likewise, the rate at which charge flows into the negative terminal is termed the current flowing into the battery. Of course, these two currents must be the same in a steady-state. Electric current is measured in units of amperes (A), which are equivalent to coulombs per second:

$$1\,\mathrm{A} \equiv 1\,\mathrm{C\,s^{-1}}. \tag{7.1}$$

We can define the electric current I flowing at any particular point in the external circuit as follows. If an amount of charge dQ flows past this point in an infinitesimal time interval dt then

$$I = \frac{dQ}{dt}. \tag{7.2}$$

By convention, the direction of the current is taken to be the direction positive charges would have to move in order to account for the flow of charge. In a steady-state, the current at all points in the external circuit must remain constant in time. We call this type of circuit a *direct current* (DC) circuit because the current always flows in the same direction. There is a second type of circuit, called an *alternating current* (AC) circuit, in which the current periodically switches direction.

Consider a simple circuit in which a steady current I flows around a single conducting wire connecting the positive and negative terminals of a battery of voltage V. Let us suppose that the current is carried by positive charges flowing around the external circuit from the positive to the negative terminal. In reality, the current is carried by negative charges (*i.e.*, by electrons) flowing in the opposite direction, but for most purposes we can safely ignore this rather inconvenient fact. Every charge q which flows around the external circuit experiences a potential drop V. In order to flow around the circuit again, the charge must be raised to the potential of the positive terminal of the battery. This process occurs inside the battery, as the charge migrates from the negative to the positive terminal. The energy $q\,V$ required to move the charge between the two terminals is derived from the energy released by the chemical reactions taking place inside the battery.

The simple circuit described above is somewhat analogous to a small ski resort. The charges flowing around the external circuit are like people skiing down the ski-slope. The charges flow down a gradient of electric potential just as the people ski down a gradient of gravitational potential. Note that the good skiers who ski directly down the slope acquire exactly the same gravitational energy as the poor skiers who ski from side to side. In both cases, the total acquired energy depends only on the difference in height between the top and bottom of the slope. Likewise, charges flowing around an external circuit acquire the same electrical energy no matter what route they take, because the acquired energy only depends on the potential difference between the two terminals of the battery. Once the people in our ski resort reach the bottom of the slope, they must be lifted to the top in a ski-lift before they can ski down it again. Thus, the ski-lift in our resort plays an analogous role to the battery in our circuit. Of course, the ski-lift must expend non-gravitational energy in order to lift skiers to the top of the slope, in just the same manner as the battery must expend non-electrical energy to move charges up a potential gradient. If the ski-lift runs out of energy then the circulation of skiers in the resort rapidly stops. Likewise, if the battery runs out of energy (*i.e.*, if the battery "runs down") then the current in the external circuit stops flowing.

7.2 Ohm's Law

Consider, again, a simple circuit in which a steady current I flows through a single conducting wire connecting the positive and negative terminals of a battery of voltage V. What is the relationship between the current I flowing in the wire and the potential difference V applied across the two ends of the wire by the battery? If we were to investigate this relationship experimentally we would quickly conclude that the current I is *directly proportional* to the potential difference V. In other words,

$$V = I\,R, \tag{7.3}$$

where the constant of proportionality R is termed the (electrical) *resistance* of the wire. The above formula is called *Ohm's law* after its discoverer, the early nineteenth century German physicist Georg Simon Ohm. The unit of electrical resistance is the ohm (Ω), which is equivalent to a volt per ampere:

$$1\,\Omega \equiv 1\,\mathrm{V\,A^{-1}}. \tag{7.4}$$

There is a slight discrepancy between what we are saying now, and what we said earlier. In Sect. 5, we maintained that the electric field inside a conductor is zero. However, if there is a potential difference V between the beginning and the end of a conducting wire, as described above, then there must be an electric field running along the length of the wire. In fact, if the wire is straight, and the electric potential decreases uniformly with distance traveled along the wire, then the longitudinal electric field-strength is given by $E = V/L$ (see Sect. 5.3), where L is the length of the wire. The earlier result that there is zero electric field inside a conductor is equivalent to saying that conductors possess zero electrical resistance. This follows because if R is zero then the electric field, and, hence, the potential difference V, must be zero, otherwise an infinite current would flow according to Ohm's law. It turns out that good conductors (*i.e.*, copper, silver, aluminium, and most other metals) possess non-zero electrical resistances. However, these resistances are generally so small that if we were to connect the terminals of a battery together using a wire fashioned out of a good conductor then the current which would flow in the wire, according to Ohm's law, would be so large that it would damage both the wire and the battery. We usually call such a circuit a *short-circuit*. In order to prevent excessively large currents from flowing, conventional electric circuits contain components, called *resistors*, whose electrical resistance is many orders of magnitude greater than that of the conducting wires in the circuit. When we apply Ohm's law, $V = I\,R$, to a circuit, we usually only count the net resistance R of all the resistors in the circuit, and neglect the resistances of the interconnecting wires. This means that all of the major drops in electric potential, as we travel around the circuit from one terminal of the battery to the other, take place inside the resistors. The drop in potential in the conducting wires themselves is usually negligible. Thus, to all intents and purposes, good conductors, and wires made out of good conductors, act as if they have zero resistance, and contain zero electric field.

7.3 Resistance and Resistivity

Let us attempt to find a microscopic explanation for electrical resistance and Ohm's law. Now, electric current in metals, and most other conductors found in conventional electric circuits (good or bad), is carried by free electrons. Consider a uniform wire of cross-sectional area A and length L made of some conducting material. Suppose that the potential difference between the two ends of the wire is V. The longitudinal electric field inside the wire is therefore $E = V/L$. Consider a free electron of charge q and mass m inside the wire. The electric field in the wire exerts a force $f = q\,E$ on the electron, causing it to accelerate with an acceleration $a = q\,E/m$ along the direction of the wire. However, the electron does not accelerate for ever. Eventually, it crashes into one of the atoms in the wire. Since atoms are far more massive than electrons, the electron loses all forward momentum every time it hits an atom (just as we would lose all forward momentum if we ran into a brick wall). Suppose that the average time interval between collisions is τ. Of course, this characteristic time interval depends on the size and number density of the atoms in the wire. Immediately after the electron hits an atom (at $t = 0$, say) its forward velocity v is zero. The electron is then accelerated by the electric field, so $v = (q\,E/m)\,t$. The final velocity of the electron is $v = (q\,E/m)\,\tau$, and its average velocity is

$$v_d = \frac{q\,E\,\tau}{2\,m}. \tag{7.5}$$

In fact, on average, the electron acts as though it drifts along the wire with the constant velocity v_d. This velocity is therefore called the *drift velocity*.

Material	ρ (Ω m)
Silver	1.5×10^{-8}
Copper	1.7×10^{-8}
Aluminium	2.6×10^{-8}
Iron	8.85×10^{-8}

Table 7.1: *Resistivities of some common metals at* 0° *C.*

Suppose that there are N free electrons per unit volume in the wire. All of these electrons effectively drift along the wire with the drift velocity v_d. Thus, the total charge which passes any particular point on the wire in a time interval dt is $dQ = q\,N\,(A\,v_d\,dt)$. This follows because all free electrons contained in a tube of length $v_d\,dt$ and cross-sectional area A pass the point in question in the time interval dt. The electric current I flowing in the wire is given by

$$I = \frac{dQ}{dt} = \frac{q^2\,N\,\tau}{2\,m}\,\frac{A}{L}\,V. \tag{7.6}$$

This equation can be rearranged to give Ohm's law,

$$V = I\,R, \tag{7.7}$$

where

$$R = \rho\,\frac{L}{A}, \tag{7.8}$$

and

$$\rho = \frac{2\,m}{q^2\,N\,\tau}. \tag{7.9}$$

Thus, we can indeed account for Ohm's law on a microscopic level. According to Eq. (7.8), the resistance of a wire is proportional to its length, and inversely proportional to its cross-sectional area. The constant of proportionality ρ is called the *resistivity* of the material making up the wire. The units of resistivity are ohm-meters (Ω m). Table 7.1 below shows the resistivities of some common metals at 0° C.

7.4 Emf and Internal Resistance

Now, real batteries are constructed from materials which possess non-zero resistivities. It follows that real batteries are not just pure voltage sources. They also

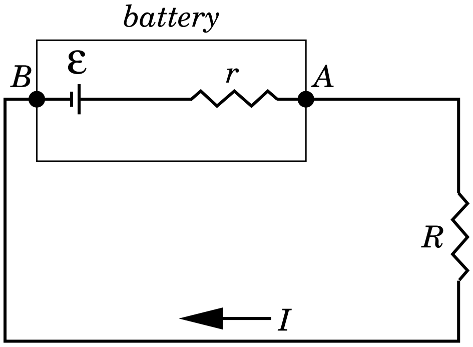


Figure 7.1: *A battery of emf $\mathcal{E}$ and internal resistance r connected to a load resistor of resistance R.*

possess *internal resistances*. Incidentally, a pure voltage source is usually referred to as an *emf* (which stands for *electromotive force*). Of course, emf is measured in units of volts. A battery can be modeled as an emf $\mathcal{E}$ connected in series with a resistor r, which represents its internal resistance. Suppose that such a battery is used to drive a current I through an external load resistor R, as shown in Fig. 7.1. Note that in circuit diagrams an emf $\mathcal{E}$ is represented as two closely spaced parallel lines of unequal length. The electric potential of the longer line is greater than that of the shorter one by $+\mathcal{E}$ volts. A resistor is represented as a zig-zag line.

Consider the battery in the figure. The voltage V of the battery is defined as the difference in electric potential between its positive and negative terminals: *i.e.*, the points A and B, respectively. As we move from B to A, the electric potential increases by $+\mathcal{E}$ volts as we cross the emf, but then decreases by $I\,r$ volts as we cross the internal resistor. The voltage drop across the resistor follows from Ohm's law, which implies that the drop in voltage across a resistor R, carrying a current I, is $I\,R$ in the direction in which the current flows. Thus, the voltage V of the battery is related to its emf $\mathcal{E}$ and internal resistance r via

$$V = \mathcal{E} - I\,r. \tag{7.10}$$

Now, we usually think of the emf of a battery as being essentially constant (since it only depends on the chemical reaction going on inside the battery, which converts chemical energy into electrical energy), so we must conclude that the voltage of a battery actually *decreases* as the current drawn from it increases. In fact, the voltage only equals the emf when the current is negligibly small. The current draw from the battery cannot normally exceed the critical value

$$I_0 = \frac{\mathcal{E}}{r}, \tag{7.11}$$

since for $I > I_0$ the voltage V becomes negative (which can only happen if the load resistor R is also negative: this is essentially impossible). It follows that if we short-circuit a battery, by connecting its positive and negative terminals together using a conducting wire of negligible resistance, the current drawn from the battery is limited by its internal resistance. In fact, in this case, the current is equal to the maximum possible current I_0.

A real battery is usually characterized in terms of its emf $\mathcal{E}$ (*i.e.*, its voltage at zero current), and the maximum current I_0 which it can supply. For instance, a standard *dry cell* (*i.e.*, the sort of battery used to power calculators and torches) is usually rated at 1.5 V and (say) 0.1 A. Thus, nothing really catastrophic is going to happen if we short-circuit a dry cell. We will run the battery down in a comparatively short space of time, but no dangerously large current is going to flow. On the other hand, a car battery is usually rated at 12 V and something like 200 A (this is the sort of current needed to operate a starter motor). It is clear that a car battery must have a much lower internal resistance than a dry cell. It follows that if we were foolish enough to short-circuit a car battery the result would be fairly catastrophic (imagine all of the energy needed to turn over the engine of a car going into a thin wire connecting the battery terminals together).

7.5 Resistors in Series and in Parallel

Resistors are probably the most commonly occurring components in electronic circuits. Practical circuits often contain very complicated combinations of resistors. It is, therefore, useful to have a set of rules for finding the equivalent resistance of some general arrangement of resistors. It turns out that we can always find the equivalent resistance by repeated application of *two* simple rules. These rules relate to resistors connected in series and in parallel.

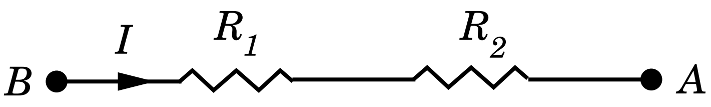


Figure 7.2: *Two resistors connected in series.*

Consider two resistors connected in *series*, as shown in Fig. 7.2. It is clear that the same current I flows through both resistors. For, if this were not the case, charge would build up in one or other of the resistors, which would not correspond to a steady-state situation (thus violating the fundamental assumption of this section). Suppose that the potential drop from point B to point A is V. This drop is the sum of the potential drops V_1 and V_2 across the two resistors R_1 and R_2, respectively. Thus,

$$V = V_1 + V_2. \tag{7.12}$$

According to Ohm's law, the equivalent resistance R_{eq} between B and A is the ratio of the potential drop V across these points and the current I which flows between them. Thus,

$$R_{eq} = \frac{V}{I} = \frac{V_1 + V_2}{I} = \frac{V_1}{I} + \frac{V_2}{I}, \tag{7.13}$$

giving

$$R_{eq} = R_1 + R_2. \tag{7.14}$$

Here, we have made use of the fact that the current I is common to all three resistors. Hence, the rule is

> The equivalent resistance of two resistors connected in series is the sum of the individual resistances.

For N resistors connected in series, Eq. (7.14) generalizes to $R_{eq} = \sum_{i=1}^{N} R_i$.

Consider two resistors connected in *parallel*, as shown in Fig. 7.3. It is clear, from the figure, that the potential drop V across the two resistors is the same.

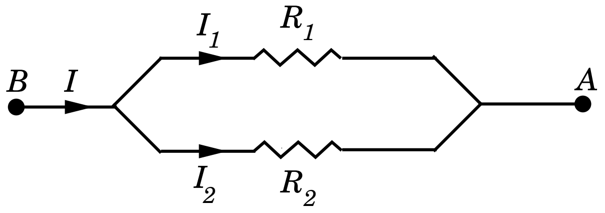


Figure 7.3: *Two resistors connected in parallel.*

In general, however, the currents I_1 and I_2 which flow through resistors R_1 and R_2, respectively, are different. According to Ohm's law, the equivalent resistance R_{eq} between B and A is the ratio of the potential drop V across these points and the current I which flows between them. This current must equal the sum of the currents I_1 and I_2 flowing through the two resistors, otherwise charge would build up at one or both of the junctions in the circuit. Thus,

$$I = I_1 + I_2. \tag{7.15}$$

It follows that

$$\frac{1}{R_{eq}} = \frac{I}{V} = \frac{I_1 + I_2}{V} = \frac{I_1}{V} + \frac{I_2}{V}, \tag{7.16}$$

giving

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}. \tag{7.17}$$

Here, we have made use of the fact that the potential drop V is common to all three resistors. Clearly, the rule is

> The reciprocal of the equivalent resistance of two resistances connected in parallel is the sum of the reciprocals of the individual resistances.

For N resistors connected in parallel, Eq. (7.17) generalizes to $1/R_{eq} = \sum_{i=1}^{N}(1/R_i)$.

7.6 Kirchhoff's Rules

We now know just about all that we need to know about emfs and resistors. However, it would be convenient if we could distill our knowledge into a number of handy rules which could then be used to analyze any DC circuit. This is essentially what the German physicist Gustav Kirchhoff did in 1845 when he proposed *two* simple rules for dealing with DC circuits.

Kirchhoff's first rule applies to *junction points* in DC circuits (*i.e.*, points at which three or more wires come together). The junction rule is:

> The sum of all the currents entering any junction point is equal to the sum of all the currents leaving that junction point.

This rule is easy to understand. As we have already remarked, if this rule were not satisfied then charge would build up at the junction points, violating our fundamental steady-state assumption.

Kirchhoff's second rule applies to *loops* in DC circuits. The loop rule is:

> The algebraic sum of the changes in electric potential encountered in a complete traversal of any closed circuit is equal to zero.

This rule is also easy to understand. We have already seen (in Sect. 5) that zero net work is done in slowly moving a charge q around some closed loop in an electrostatic field. Since the work done is equal to the product of the charge q and the difference ΔV in electric potential between the beginning and end points of the loop, it follows that this difference must be zero. Thus, if we apply this result to the special case of a loop in a DC circuit, we immediately arrive at Kirchhoff's second rule. When using this rule, we first pick a closed loop in the DC circuit that we are analyzing. Next, we decide whether we are going to traverse this loop in a clockwise or an anti-clockwise direction (the choice is arbitrary). If a source of emf $\mathcal{E}$ is traversed in the direction of increasing potential then the change in potential is $+\mathcal{E}$. However, if the emf is traversed in the opposite direction then the change in potential is $-\mathcal{E}$. If a resistor R, carrying a current I, is traversed in the direction of current flow then the change in potential is $-I\,R$. Finally, if the resistor is traversed in the opposite direction then the change in potential is $+I\,R$.

The currents flowing around a general DC circuit can always be found by applying Kirchhoff's first rule to all junction points, Kirchhoff's second rule to all loops, and then solving the simultaneous algebraic equations thus obtained. This procedure works no matter how complicated the circuit in question is (*e.g.*, Kirchhoff's rules are used in the semiconductor industry to analyze the incredibly complicated circuits, etched onto the surface of silicon wafers, which are used to construct the central processing units of computers).

7.7 Capacitors in DC Circuits

Capacitors do not play an important role in DC circuits because it is impossible for a steady current to flow across a capacitor. If an uncharged capacitor C is connected across the terminals of a battery of voltage V then a *transient* current flows as the capacitor plates charge up. However, the current stops flowing as soon as the charge Q on the positive plate reaches the value $Q = C\,V$. At this point, the electric field between the plates cancels the effect of the electric field generated by the battery, and there is no further movement of charge. Thus, if a capacitor is placed in a DC circuit then, as soon as its plates have charged up, the capacitor effectively behaves like a *break* in the circuit.

7.8 Energy in DC Circuits

Consider a simple circuit in which a battery of voltage V drives a current I through a resistor of resistance R. As we have seen, the battery is continuously doing work by raising the potentials of charges which flow into its negative terminal and then flow out of its positive terminal. How much work does the battery do per unit time? In other words, what is the power output of the battery?

Consider a (positive) charge q which flows through the battery from the negative terminal to the positive terminal. The battery raises the potential of the charge by V, so the work the battery does on the charge is $q\,V$. The total amount of charge which flows through the battery per unit time is, by definition, equal to the current I flowing through the battery. Thus, the amount of work the battery does per unit time is simply the product of the work done per unit charge, V, and the charge passing through the battery per unit time, I. In other words,

$$P = V\,I, \tag{7.18}$$

where P, of course, stands for the power output of the battery. Thus, the rule is

The power in a DC circuit is the product of the voltage and the current.

This rule does not just apply to batteries. If a current I flows through some component of a DC circuit which has a potential *drop* V in the direction of current flow then that component *gains* the energy per unit time $V\,I$ at the expense of the rest of the circuit, and *vice versa*. Incidentally, since the SI unit of power is the watt (W), it follows that

$$1\,\mathrm{W} \equiv 1\,\mathrm{V}\cdot 1\,\mathrm{A}. \tag{7.19}$$

Consider a resistor R which carries a current I. According to Ohm's law, the potential drop across the resistor is $V = I\,R$. Thus, the energy gained by the resistor per unit time is

$$P = V\,I = I^2\,R = \frac{V^2}{R}. \tag{7.20}$$

In what form does the resistor acquire this energy? In turns out that the energy is dissipated as *heat* inside the resistor. This effect is known as *Joule heating*. Thus, the above formula gives the electrical heating power of a resistor. Electrical energy is converted into heat (*i.e.*, random motion of the atoms which make up the resistor) as the electrically accelerated free electrons inside the resistor collide with the atoms and, thereby, transfer all of their kinetic energy to the atoms. It is this energy which appears as heat on a macroscopic scale (see Sect. 7.3).

Household electricity bills depend on the amount of electrical *energy* the household in question uses during a given accounting period, since the energy usage determines how much coal or gas was burnt on the household's behalf in the local power station during this period. The conventional unit of electrical energy usage employed by utility companies is the *kilowatthour*. If electrical energy is consumed for 1 hour at the rate of 1 kW (the typical rate of consumption of a single-bar electric fire) then the total energy usage is one kilowatthour (kWh). It follows that

$$1\,\text{kWh} = (1000)\,(60)\,(60) = 3.6\times 10^{6}\,\text{J}. \tag{7.21}$$

7.9 Power and Internal Resistance

Consider a simple circuit in which a battery of emf $\mathcal{E}$ and internal resistance r drives a current I through an external resistor of resistance R (see Fig. 7.1). The external resistor is usually referred to as the *load resistor*. It could stand for either an electric light, an electric heating element, or, maybe, an electric motor. The basic purpose of the circuit is to transfer energy from the battery to the load, where it actually does something useful for us (*e.g.*, lighting a light bulb, or lifting a weight). Let us see to what extent the internal resistance of the battery interferes with this process.

The equivalent resistance of the circuit is $r + R$ (since the load resistance is in series with the internal resistance), so the current flowing in the circuit is given by

$$I = \frac{\mathcal{E}}{r+R}. \tag{7.22}$$

The power output of the emf is simply

$$P_{\mathcal{E}} = \mathcal{E}\,I = \frac{\mathcal{E}^2}{r+R}. \tag{7.23}$$

The power dissipated as heat by the internal resistance of the battery is

$$P_r = I^2\,r = \frac{\mathcal{E}^2\,r}{(r+R)^2}. \tag{7.24}$$

Likewise, the power transferred to the load is

$$P_R = I^2\,R = \frac{\mathcal{E}^2\,R}{(r+R)^2}. \tag{7.25}$$

Note that

$$P_{\mathcal{E}} = P_r + P_R. \tag{7.26}$$

Thus, some of the power output of the battery is immediately lost as heat dissipated by the internal resistance of the battery. The remainder is transmitted to the load.

Let $y = P_R/(\mathcal{E}^2/r)$ and $x = R/r$. It follows from Eq. (7.25) that

$$y = \frac{x}{(1+x)^2}. \tag{7.27}$$

The function $y(x)$ increases monotonically from zero for increasing x in the range $0 < x < 1$, attains a maximum value of $1/4$ at $x = 1$, and then decreases monotonically with increasing x in the range $x > 1$. In other words, if the load resistance R is varied at constant $\mathcal{E}$ and r then the transferred power attains a maximum value of

$$(P_R)_{\rm max} = \frac{\mathcal{E}^2}{4\,r} \tag{7.28}$$

when $R = r$. This is a very important result in electrical engineering. Power transfer between a voltage source and an external load is at its most efficient when the resistance of the load matches the internal resistance of the voltage source. If the load resistance is too low then most of the power output of the voltage source is dissipated as heat inside the source itself. If the load resistance is too high then the current which flows in the circuit is too low to transfer energy to the load at an appreciable rate. Note that in the optimum case, $R = r$, only *half* of the power output of the voltage source is transmitted to the load. The other half is dissipated as heat inside the source. Incidentally, electrical engineers call the process by which the resistance of a load is matched to that of the power supply *impedance matching* (impedance is just a fancy name for resistance).

7.10 Worked Examples

Example 7.1: Ohm's law

Question: What is the resistance at 0° C of a 1.0 m long piece of no. 5 gauge copper wire (cross-sectional area 16.8 mm^2)? What voltage must be applied across the two ends of the wire to produce a current of 10 A through it?

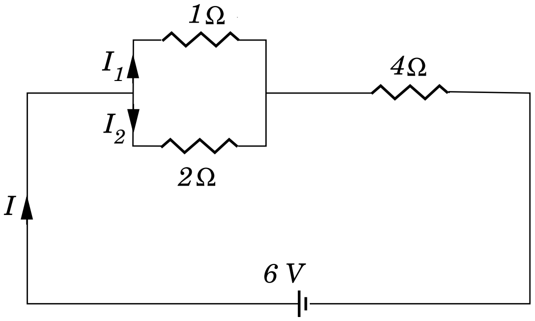
Answer: Using the basic equation $R = \rho\, L/A$, and the value of ρ for copper given in Tab. 7.1, we have

$$R = \frac{(1.7 \times 10^{-8})\,(1.0)}{(16.8 \times 10^{-6})} = 1.0 \times 10^{-3}\,\Omega.$$

Using Ohm's law $V = I\,R$, we obtain

$$V = (10)\,(1.0 \times 10^{-3}) = 1.0 \times 10^{-2}\,\mathrm{V}.$$

Example 7.2: Equivalent resistance



Question: A 1 Ω and a 2 Ω resistor are connected in parallel, and this pair of resistors is connected in series with a 4 Ω resistor. What is the equivalent resistance of the whole combination? What is the current flowing through the 4 Ω resistor if the whole combination is connected across the terminals of a 6 V battery (of negligible internal resistance)? Likewise, what are the currents flowing through the 1 Ω and 2 Ω resistors?

Answer: The equivalent resistance of the 1 Ω and 2 Ω resistors is

$$\frac{1}{R'_{eq}} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2}\,\Omega^{-1},$$

giving $R'_{eq} = 0.667\,\Omega$. When a $0.667\,\Omega$ resistor is combined in series with a $4\,\Omega$ resistor, the equivalent resistance is $R_{eq} = 0.667 + 4 = 4.667\,\Omega$.

The current driven by the 6 V battery is

$$I = \frac{V}{R_{eq}} = \frac{(6)}{(4.667)} = 1.29\,\mathrm{A}.$$

This is the current flowing through the 4 Ω resistor, since one end of this resistor is connected directly to the battery, with no intermediate junction points.

The voltage drop across the 4 Ω resistor is

$$V_4 = I\,R_4 = (1.29)\,(4) = 5.14\,\mathrm{V}.$$

Thus, the voltage drop across the 1 Ω and 2 Ω combination is $V_{12} = 6 - 5.14 = 0.857\,\mathrm{V}$. The current flowing through the 1 Ω resistor is given by

$$I_1 = \frac{V_{12}}{R_1} = \frac{(0.857)}{(1)} = 0.857\,\mathrm{A}.$$

Likewise, the current flowing through the 2 Ω resistor is

$$I_2 = \frac{V_{12}}{R_2} = \frac{(0.857)}{(2)} = 0.429\,\mathrm{A}.$$

Note that the total current flowing through the 1 Ω and 2 Ω combination is $I_{12} = I_1 + I_2 = 1.29\,\mathrm{A}$, which is the same as the current flowing through the 4 Ω resistor. This makes sense because the 1 Ω and 2 Ω combination is connected in series with the 4 Ω resistor.

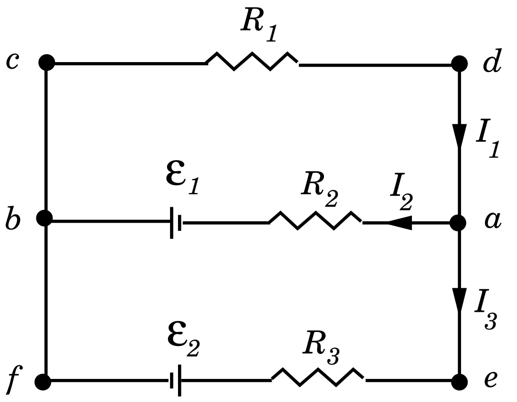
Example 7.3: Kirchhoff's rules

Question: Find the three currents I_1, I_2, and I_3 in the circuit shown in the diagram, where $R_1 = 100\,\Omega$, $R_2 = 10\,\Omega$, $R_3 = 5\,\Omega$, $\mathcal{E}_1 = 12\,\text{V}$ and $\mathcal{E}_2 = 6\,\text{V}$.

Answer: Applying the junction rule to point a, and assuming that the currents flow in the direction shown (the initial choice of directions of the currents is arbitrary), we have

$$I_1 = I_2 + I_3.$$

There is no need to apply the junction rule again at point b, since if the above equation is satisfied then this rule is automatically satisfied at b.



Let us apply the loop rule by going around the various loops in the circuit in a clockwise direction. For loop $abcd$, we have

$$-I_2\,R_2 + \mathcal{E}_1 - I_1\,R_1 = 0.$$

Note that both the terms involving resistors are negative, since we cross the resistors in question in the direction of nominal current flow. Likewise, the term involving the emf is positive since we traverse the emf in question from the negative to the positive plate. For loop $aefb$, we find

$$-I_3\,R_3 - \mathcal{E}_2 - \mathcal{E}_1 + I_2\,R_2 = 0.$$

There is no need to apply the loop rule to the full loop defc, since this loop is made up of loops abcd and aefb, and the loop rules for these two loops therefore already contain all of the information which would be obtained by applying the loop rule to defc.

Combining the junction rule with the first loop rule, we obtain

$$(R_1 + R_2)\, I_2 + R_1\, I_3 = \mathcal{E}_1.$$

The second loop rule can be rearranged to give

$$-R_2\, I_2 + R_3\, I_3 = -(\mathcal{E}_1 + \mathcal{E}_2).$$

The above two equations are a pair of simultaneous algebraic equations for the currents I_2 and I_3, and can be solved using the standard method for solving such equations. Multiplying the first equation by R_2, the second by $(R_1 + R_2)$, and adding the resulting equations, we obtain

$$(R_1 R_2 + R_2 R_3 + R_1 R_3)\, I_3 = -R_1\, \mathcal{E}_1 - (R_1 + R_2)\, \mathcal{E}_2,$$

which can be rearranged to give

$$I_3 = -\frac{R_1\, \mathcal{E}_1 + (R_1 + R_2)\, \mathcal{E}_2}{R_1 R_2 + R_2 R_3 + R_1 R_3},$$

or

$$I_3 = -\frac{(100)\,(12) + (110)\,(6)}{(1000 + 50 + 500)} = -\frac{(1860)}{(1550)} = -1.2\,\mathrm{A}.$$

Likewise, multiplying the first equation by R_3, the second by R_1, and taking the difference of the resulting equations, we obtain

$$(R_1 R_2 + R_2 R_3 + R_1 R_3)\, I_2 = (R_1 + R_3)\, \mathcal{E}_1 + R_1\, \mathcal{E}_2,$$

which can be rearranged to give

$$I_2 = \frac{(R_1 + R_3)\, \mathcal{E}_1 + R_1\, \mathcal{E}_2}{R_1 R_2 + R_2 R_3 + R_1 R_3},$$

or

$$I_2 = \frac{(105)\,(12) + (100)\,(6)}{(1000 + 50 + 500)} = \frac{(1860)}{(1550)} = 1.2\,\mathrm{A}.$$

Finally, from the junction rule,

$$I_1 = I_2 + I_3 = -1.2 + 1.2 = 0\,\mathrm{A}.$$

The fact that $I_3 = -1.2\,\mathrm{A}$ indicates that this current is of magnitude 1.2 A, but flows in the opposite direction to that which we initially guessed. In fact, we can see that a current of 1.2 A circulates in an anti-clockwise direction in the lower loop of the circuit, whereas zero current circulates in the upper loop.

Example 7.4: Energy in DC circuits

Question: A 150 W light bulb is connected to a 120 V line. What is the current drawn from the line? What is the resistance of the light bulb whilst it is burning? How much energy is consumed if the light is kept on for 6 hours? What is the cost of this energy at 8 cents/kWh?

Answer: Since power is equal to $I\,V$, it follows that

$$I = \frac{P}{V} = \frac{(150)}{(120)} = 1.25\,\mathrm{A}.$$

From Ohm's law, the resistance of the light bulb is

$$R = \frac{V}{I} = \frac{(120)}{(1.25)} = 96\,\Omega.$$

The energy W consumed is the product of the power P (the energy consumed per unit time) and the time period t for which the light is on, so

$$W = P\,t = (150)\,(6)\,(60)\,(60) = 3.24 \times 10^6\,\mathrm{J}.$$

Since, $1\,\mathrm{kWh} \equiv 3.6 \times 10^6\,\mathrm{J}$, it follows that

$$W = \frac{(3.24 \times 10^6)}{(3.6 \times 10^6)} = 0.9\,\mathrm{kWh}.$$

The cost c of the electricity is product of the number of kilowatthours used and the cost per kilowatthour, so

$$c = (0.9)\,(0.08) = 0.072 \text{ dollars} = 7.2 \text{ cents}.$$

8 Magnetism

8.1 Historical Introduction

The phenomenon of magnetism has been known to mankind for many thousands of years. Loadstone (a magnetized form of the commonly occurring iron oxide mineral magnetite) was the first permanent magnetic material to be identified and studied. The ancient Greeks were aware of the ability of loadstone to attract small pieces of iron. The Greek word *magnes* (μαγνες), which is the root of the English word *magnet*, is derived from Magnesia, the name of an ancient city in Asia Minor, which, presumably, was once a copious source of loadstones.

The magnetic compass was invented some time during the first ten centuries AD. Credit is variously given to the Chinese, the Arabs, and the Italians. What is certain is that by the 12th century magnetic compasses were in regular use by mariners to aid navigation at sea. In the 13th century, Peter Perigrinus of France discovered that the magnetic effect of a spherical loadstone is strongest at two oppositely directed points on the surface of the sphere, which he termed the *poles* of the magnet. He found that there are two types of poles, and that like poles repel one another whereas unlike poles attract. In 1600, the English physician William Gilbert concluded, quite correctly, that the reason magnets like to align themselves in a North-South direction is that the Earth itself is a magnet. Furthermore, the Earth's magnetic poles are aligned, more or less, along its axis of rotation. This insight immediately gave rise to a fairly obvious nomenclature for the two different poles of a magnet: a magnetic *north pole* (N) has the same magnetic polarity as the geographic south pole of the Earth, and a magnetic *south pole* (S) has the same polarity as the geographic north pole of the Earth. Thus, the north pole of a magnet likes to point northwards towards the geographic north pole of the Earth (which is its magnetic south pole). Another British scientist, John Michell, discovered in 1750 that the attractive and repulsive forces between the poles of magnets vary inversely as the square of the distance of separation. Thus, the inverse square law for forces between magnets was actually discovered prior to that for forces between electric charges.

8.2 Ampère's Experiments

In 1820, the Danish physicist Hans Christian Ørsted was giving a lecture demonstration of various electrical and magnetic effects. Suddenly, much to his amazement, he noticed that the needle of a compass he was holding was deflected when he moved it close to a current carrying wire. This was a very surprising observation, since, until that moment, electricity and magnetism had been thought of as two quite unrelated phenomena. Word of this discovery spread quickly along the scientific grapevine, and the French physicist Andre Marie Ampère immediately decided to investigate further. Ampère's apparatus consisted (essentially) of a long straight wire carrying an electric current current I. Ampère quickly discovered that the needle of a small compass maps out a series of concentric circular loops in the plane perpendicular to a current carrying wire—see Fig. 8.1. The direction of circulation around these magnetic loops is conventionally taken to be the direction in which the *north* pole of a compass needle points. Using this convention, the circulation of the loops is given by a *right-hand rule*. If the thumb of the right-hand points along the direction of the current, then the fingers of the right-hand circulate in the same sense as the magnetic loops.

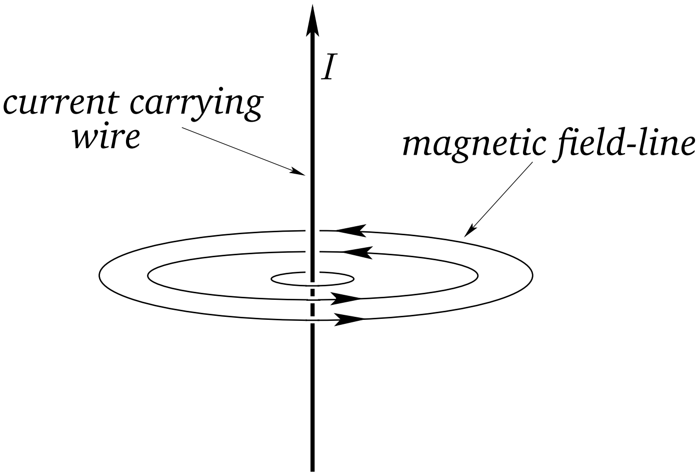


Figure 8.1: *Magnetic loops around a current carrying wire.*

Ampère's next series of experiments involved bringing a short test wire, carrying a current I′, close to the original wire, and investigating the force exerted on the test wire. This experiment is not quite as clear cut as Coulomb's experiment because, unlike electric charges, electric currents cannot exist as point entities. They have to flow in complete circuits. We must imagine that the circuit which connects with the central wire is sufficiently far away that it has no appreciable influence on the outcome of the experiment. The circuit which connects with the test wire is more problematic. Fortunately, if the feed wires are twisted around each other, as indicated in Fig. 8.2, then they effectively cancel one another out, and also do not influence the outcome of the experiment.

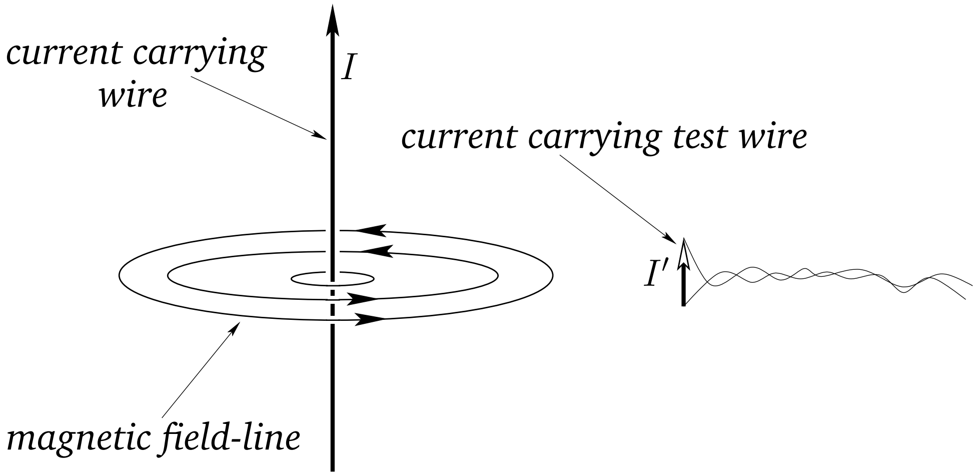


Figure 8.2: *Ampère's experiment.*

Ampère discovered that the force exerted on the test wire is directly proportional to its length. He also made the following observations. If the current in the test wire (*i.e.*, the test current) flows parallel to the current in the central wire then the two wires attract one another. If the current in the test wire is reversed then the two wires repel one another. If the test current points radially towards the central wire (and the current in the central wire flows upward) then the test wire is subject to a downward force. If the test current is reversed then the force is upward. If the test current is rotated in a single plane, so that it starts parallel to the central current and ends up pointing radially towards it, then the force on the test wire is of constant magnitude, and is always at right-angles to the test current. If the test current is parallel to a magnetic loop then there is no force exerted on the test wire. If the test current is rotated in a single plane, so that it starts parallel to the central current, and ends up pointing along a magnetic loop, then the magnitude of the force on the test wire attenuates like $\cos\theta$ (where θ is the angle the current is turned through, and $\theta = 0$ corresponds to the case where the test current is parallel to the central current), and its direction is again always at right-angles to the test current. Finally, Ampère was able to establish that the attractive force between two parallel current carrying wires is proportional to the product of the two currents, and falls off like one over the perpendicular distance between the wires.

This rather complicated force law can be summed up succinctly in vector notation provided that we define a vector field **B**, called the *magnetic field*, which fills space, and whose direction is everywhere tangential to the magnetic loops mapped out by the north pole of a small compass. The dependence of the force per unit length, **F**, acting on a test wire with the different possible orientations of the test current is described by

$$\mathbf{F} = \mathbf{I}' \times \mathbf{B}, \tag{8.1}$$

where $\mathbf{I}'$ is a vector whose direction and magnitude are the same as those of the test current.

The variation of the force per unit length acting on a test wire with the strength of the central current, and the perpendicular distance r to the central wire, is accounted for by saying that the magnetic field-strength is proportional to I, and inversely proportional to r. Thus, we can write

$$B = \frac{\mu_0\, I}{2\pi\, r}. \tag{8.2}$$

The constant of proportionality μ_0 is called the *permeability of free space*, and takes the value

$$\mu_0 = 4\pi \times 10^{-7}\, \mathrm{N\,A^{-2}}. \tag{8.3}$$

Incidentally, the SI unit of magnetic field strength is the tesla (T), which is the same as a newton per ampere per meter:

$$1\,\mathrm{T} \equiv 1\,\mathrm{N\,A^{-1}\,m^{-1}}. \tag{8.4}$$

The concept of a magnetic field which fills the space around a current carrying wire allows the calculation of the force on a test wire to be conveniently split into two parts. In the first part, we calculate the magnetic field generated by the current flowing in the central wire. This field circulates in the plane normal to the wire. Its magnitude is proportional to the central current, and inversely proportional to the perpendicular distance from the wire. In the second part, we use Eq. (8.1) to calculate the force per unit length acting on a short current carrying wire placed in the magnetic field generated by the central current. This force is perpendicular to both the direction of the magnetic field and the direction of the test current. Note that, at this stage, we have no reason to suppose that the magnetic field has any real existence. It is introduced merely to facilitate the calculation of the force exerted on the test wire by the central wire. It turns out, however, that the magnetic field *does* have a real existence, since, as we shall see, there is an energy associated with a magnetic field which fills space.

8.3 Ampère's Law

Magnetic fields, like electric fields, are completely *superposable*. So, if a field $\mathbf{B}_1$ is generated by a current I_1 flowing through some circuit, and a field $\mathbf{B}_2$ is generated by a current I_2 flowing through another circuit, then when the currents I_1 and I_2 flow through both circuits simultaneously the generated magnetic field is $\mathbf{B}_1 + \mathbf{B}_2$. This is true at all points in space.

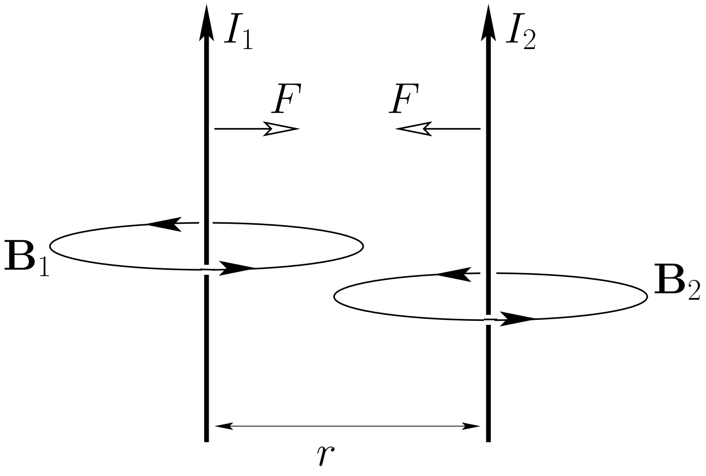


Figure 8.3: *Two parallel current carrying wires.*

Consider two parallel wires separated by a perpendicular distance r, and carrying electric currents I_1 and I_2, respectively. The magnetic field-strength at the second wire due to the current flowing in the first wire is $B = \mu_0 I_1/2\pi r$. This field is orientated at right-angles to the second wire, so the force per unit length exerted on the second wire is

$$F = \frac{\mu_0 I_1 I_2}{2\pi r}. \tag{8.5}$$

This follows from Eq. (8.1), which is valid for continuous wires as well as short test wires. The force acting on the second wire is directed radially inwards towards the first wire. The magnetic field-strength at the first wire due to the current flowing in the second wire is $B = \mu_0 I_2/2\pi r$. This field is orientated at right-angles to the first wire, so the force per unit length acting on the first wire is equal and opposite to that acting on the second wire, according to Eq. (8.1). Equation (8.5) is called *Ampère's law*.

Incidentally, Eq. (8.5) is the basis of the official SI definition of the *ampere*, which is:

> One ampere is the magnitude of the current which, when flowing in each of two long parallel wires one meter apart, results in a force between the wires of exactly 2×10^{-7} N per meter of length.

We can see that it is no accident that the constant μ_0 has the numerical value of *exactly* $4\pi \times 10^{-7}$. The SI system of units is based on four standard units: the *meter*, the *kilogram*, the *second*, and the *ampere*. Hence, the SI system is sometime referred to as the MKSA system. All other units can be derived from these four standard units. For instance, a coulomb is equivalent to an ampere-second. You may be wondering why the ampere is the standard electrical unit, rather than the coulomb, since the latter unit is clearly more fundamental than the former. The answer is simple. It is very difficult to measure charge accurately, whereas it is easy to accurately measure electric current. Clearly, it makes sense to define a standard unit in terms of something which is easily measurable, rather than something which is difficult to measure.

8.4 The Lorentz Force

The flow of an electric current down a conducting wire is ultimately due to the movement of electrically charged particles (in most cases, electrons) along the wire. It seems reasonable, therefore, that the force exerted on the wire when it is placed in a magnetic field is simply the resultant of the forces exerted on these moving charges. Let us suppose that this is the case.

Let A be the (uniform) cross-sectional area of the wire, and let n be the number density of mobile charges in the wire. Suppose that the mobile charges each have charge q and drift velocity $\mathbf{v}$. We must assume that the wire also contains stationary charges, of charge $-q$ and number density n, say, so that the net charge density in the wire is zero. In most conductors, the mobile charges are electrons, and the stationary charges are atoms. The magnitude of the electric current flowing through the wire is simply the number of coulombs per second which flow past a given point. In one second, a mobile charge moves a distance v, so all of the charges contained in a cylinder of cross-sectional area A and length v flow past a given point. Thus, the magnitude of the current is $q\,n\,A\,v$. The direction of the current is the same as the direction of motion of the charges (*i.e.*, $\mathbf{I}' \propto \mathbf{v}$), so the vector current is $\mathbf{I}' = q\,n\,A\,\mathbf{v}$. According to Eq. (8.1), the force per unit length acting on the wire is

$$\mathbf{F} = \mathbf{I}' \times \mathbf{B} = q\,n\,A\,\mathbf{v} \times \mathbf{B}. \tag{8.6}$$

However, a unit length of the wire contains $n\,A$ moving charges. So, assuming that each charge is subject to an equal force from the magnetic field (we have no reason to suppose otherwise), the magnetic force acting on an individual charge is

$$\mathbf{f} = q\,\mathbf{v} \times \mathbf{B}. \tag{8.7}$$

This formula implies that the magnitude of the magnetic force exerted on a *moving* charged particle is the product of the particle's charge, its velocity, the magnetic field-strength, and the sine of the angle subtended between the particle's direction of motion and the direction of the magnetic field. The force is directed at right-angles to both the magnetic field and the instantaneous direction of motion.

We can combine the above equation with Eq. (3.12) to give the force acting on a charge q moving with velocity $\mathbf{v}$ in an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$:

$$\mathbf{f} = q\,\mathbf{E} + q\,\mathbf{v}\times\mathbf{B}. \tag{8.8}$$

This is called the *Lorentz force law*, after the Dutch physicist Hendrick Antoon Lorentz, who first formulated it. The electric force on a charged particle is parallel to the local electric field. The magnetic force, however, is perpendicular to both the local magnetic field and the particle's direction of motion. No magnetic force is exerted on a stationary charged particle.

The equation of motion of a free particle of charge q and mass m moving in electric and magnetic fields is

$$m\,\mathbf{a} = q\,\mathbf{E} + q\,\mathbf{v}\times\mathbf{B}, \tag{8.9}$$

according to the Lorentz force law. Here, $\mathbf{a}$ is the particle's acceleration. This equation of motion was verified in a famous experiment carried out by the Cambridge physicist J.J. Thompson in 1897. Thompson was investigating *cathode rays*, a then mysterious form of radiation emitted by a heated metal element held at a large negative voltage (*i.e.*, a cathode) with respect to another metal element (*i.e.*, an anode) in an evacuated tube. German physicists maintained that cathode rays were a form of electromagnetic radiation, whereas British and French physicists suspected that they were, in reality, a stream of charged particles. Thompson was able to demonstrate that the latter view was correct. In Thompson's experiment, the cathode rays pass though a region of *crossed* electric and magnetic fields (still in vacuum). The fields are perpendicular to the original trajectory of the rays, and are also mutually perpendicular.

Let us analyze Thompson's experiment. Suppose that the rays are originally traveling in the x-direction, and are subject to a uniform electric field E in the z-direction, and a uniform magnetic field B in the $-y$-direction—see Fig. 8.4. Let us assume, as Thompson did, that cathode rays are a stream of particles of mass m and charge q. The equation of motion of the particles in the z-direction is

$$m\,a_z = q\,(E - v\,B)\,, \tag{8.10}$$

where v is the velocity of the particles in the x-direction, and a_z the acceleration of the particles in the z-direction. Thompson started off his experiment by only

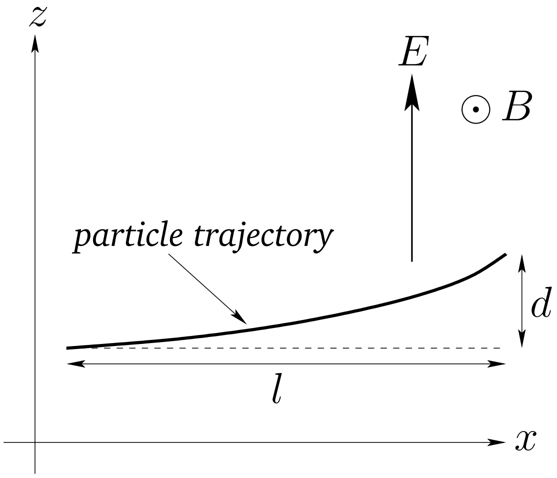


Figure 8.4: *Thompson's experiment.*

turning on the electric field in his apparatus, and measuring the deflection d of the rays in the z-direction after they had traveled a distance l through the field. Now, a particle subject to a constant acceleration a_z in the z-direction is deflected a distance $d = (1/2)\, a_z\, t^2$ in a time t. Thus,

$$d = \frac{1}{2}\frac{q\,E}{m}\,t^2 = \frac{q}{m}\frac{E\,l^2}{2\,v^2}, \tag{8.11}$$

where the *time of flight* t is replaced by l/v. This replacement is only valid if $d \ll l$ (*i.e.*, if the deflection of the rays is small compared to the distance they travel through the electric field), which is assumed to be the case. Next, Thompson turned on the magnetic field in his apparatus, and adjusted it so that the cathode rays were no longer deflected. The lack of deflection implies that the net force on the particles in the z-direction is zero. In other words, the electric and magnetic forces balance exactly. It follows from Eq. (8.10) that, with a properly adjusted magnetic field-strength,

$$v = \frac{E}{B}. \tag{8.12}$$

Thus, Eqs. (8.11) and (8.12) can be combined and rearranged to give the charge to mass ratio of the particles in terms of measured quantities:

$$\frac{q}{m} = \frac{2\,d\,E}{l^2\,B^2}. \tag{8.13}$$

Using this method, Thompson inferred that cathode rays are made up of negatively charged particles (the sign of the charge is obvious from the direction of the deflection in the electric field) with a charge to mass ratio of $-1.7 \times 10^{11}\ \mathrm{C\,kg^{-1}}$. A decade later, in 1908, the American Robert Millikan performed his famous *oil drop* experiment in which he discovered that mobile electric charges are quantized in units of -1.6×10^{-19} C. Assuming that mobile electric charges and the particles which make up cathode rays are one and the same thing, Thompson's and Millikan's experiments imply that the mass of these particles is 9.4×10^{-31} kg. Of course, this is the mass of an electron (the modern value is 9.1×10^{-31} kg), and -1.6×10^{-19} C is the charge of an electron. Thus, cathode rays are, in fact, streams of electrons which are emitted from a heated cathode, and then accelerated because of the large voltage difference between the cathode and anode.

If a particle is subject to a force **f** which causes it to displace by d**r** then the work done on the particle by the force is

$$W = \mathbf{f} \cdot d\mathbf{r} = f\, dr\, \cos\theta, \tag{8.14}$$

where θ is the angle subtended between the force and the displacement. However, this angle is always 90° for the force exerted by a magnetic field on a charged particle, since the magnetic force is always perpendicular to the particle's instantaneous direction of motion. It follows that a magnetic field is unable to do work on a charged particle. In other words, a charged particle can never gain or lose energy due to interaction with a magnetic field. On the other hand, a charged particle can certainly gain or lose energy due to interaction with an electric field. Thus, magnetic fields are often used in particle accelerators to guide charged particle motion (*e.g.*, in a circle), but the actual acceleration is always performed by electric fields.

8.5 Charged Particle in a Magnetic Field

Suppose that a particle of mass m moves in a circular orbit of radius ρ with a constant speed v. As is well-known, the acceleration of the particle is of magnitude $m\,v^2/\rho$, and is always directed towards the centre of the orbit. It follows that the acceleration is always perpendicular to the particle's instantaneous direction of motion.

We have seen that the force exerted on a charged particle by a magnetic field is always perpendicular to its instantaneous direction of motion. Does this mean that the field causes the particle to execute a circular orbit? Consider the case shown in Fig. 8.5. Suppose that a particle of positive charge q and mass m moves in a plane perpendicular to a uniform magnetic field B. In the figure, the field points into the plane of the paper. Suppose that the particle moves, in an anti-clockwise manner, with constant speed v (remember that the magnetic field cannot do work on the particle, so it cannot affect its speed), in a circular orbit of radius ρ. The magnetic force acting on the particle is of magnitude $f = q\,v\,B$ and, according to Eq. (8.7), this force is always directed towards the centre of the orbit. Thus, if

$$f = q\,v\,B = \frac{m\,v^2}{\rho}, \tag{8.15}$$

then we have a self-consistent picture. It follows that

$$\rho = \frac{m\,v}{q\,B}. \tag{8.16}$$

The angular frequency of rotation of the particle (*i.e.*, the number of radians the particle rotates through in one second) is

$$\omega = \frac{v}{\rho} = \frac{q\,B}{m}. \tag{8.17}$$

Note that this frequency, which is known as the *Larmor frequency*, does not depend on the velocity of the particle. For a negatively charged particle, the picture is exactly the same as described above, except that the particle moves in a clockwise orbit.

It is clear, from Eq. (8.17), that the angular frequency of gyration of a charged particle in a known magnetic field can be used to determine its charge to mass ratio. Furthermore, if the speed of the particle is known, then the radius of the orbit can also be used to determine q/m, via Eq. (8.16). This method is employed in High Energy Physics to identify particles from photographs of the tracks which

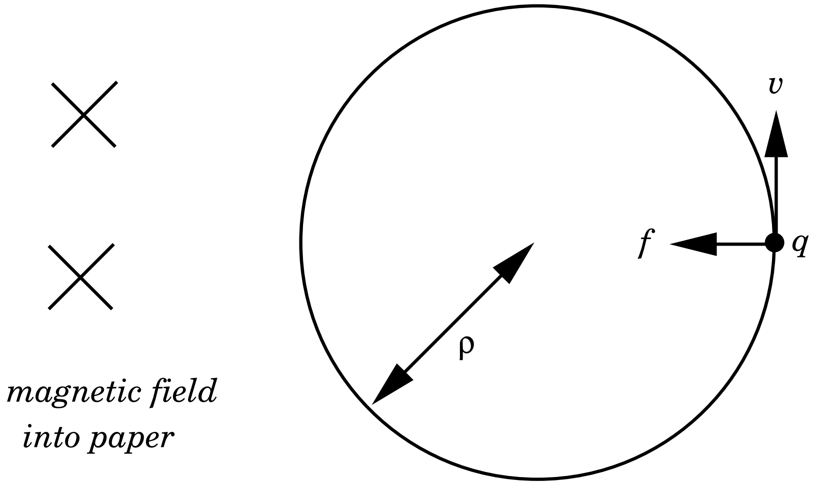


Figure 8.5: *Circular motion of a charged particle in a magnetic field.*

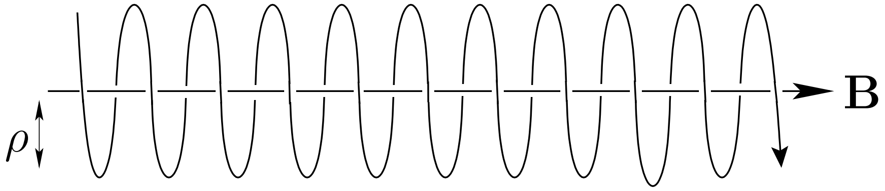


Figure 8.6: *Spiral trajectory of a charged particle in a uniform magnetic field.*

they leave in magnetized cloud chambers or bubble chambers. It is, of course, easy to differentiate positively charged particles from negatively charged ones using the direction of deflection of the particles in the magnetic field.

We have seen that a charged particle placed in a magnetic field executes a circular orbit in the plane perpendicular to the direction of the field. Is this the most general motion of a charged particle in a magnetic field? Not quite. We can also add an arbitrary drift along the direction of the magnetic field. This follows because the force $q\,\mathbf{v}\times\mathbf{B}$ acting on the particle only depends on the component of the particle's velocity which is *perpendicular* to the direction of magnetic field (the cross product of two parallel vectors is always zero because the angle θ they subtend is zero). The combination of circular motion in the plane perpendicular to the magnetic field, and uniform motion along the direction of the field, gives rise to a *spiral* trajectory of a charged particle in a magnetic field, where the field forms the axis of the spiral—see Fig. 8.6.

8.6 The Hall Effect

We have repeatedly stated that the mobile charges in conventional conducting materials are negatively charged (they are, in fact, electrons). Is there any direct experimental evidence that this is true? Actually, there is. We can use a phenomenon called the *Hall effect* to determine whether the mobile charges in a given conductor are positively or negatively charged. Let us investigate this effect.

Consider a thin, flat, uniform, ribbon of some conducting material which is orientated such that its flat side is perpendicular to a uniform magnetic field B—see Fig. 8.7. Suppose that we pass a current I along the length of the ribbon. There are two alternatives. Either the current is carried by positive charges moving from left to right (in the figure), or it is carried by negative charges moving in the opposite direction.

Suppose that the current is carried by positive charges moving from left to right. These charges are deflected *upward* (in the figure) by the magnetic field. Thus, the upper edge of the ribbon becomes positively charged, whilst the lower edge becomes negatively charged. Consequently, there is a *positive* potential difference V_H between the upper and lower edges of the ribbon. This potential difference is called the *Hall voltage*.

Suppose, now, that the current is carried by negative charges moving from right to left. These charges are also deflected *upward* by the magnetic field. Thus, the upper edge of the ribbon becomes negatively charged, whilst the lower edge becomes positively charged. It follows that the Hall voltage (*i.e.*, the potential difference between the upper and lower edges of the ribbon) is *negative* in this case.

Clearly, it is possible to determine the sign of the mobile charges in a current carrying conductor by measuring the Hall voltage. If the voltage is positive then the mobile charges are positive (assuming that the magnetic field and the current are orientated as shown in the figure), whereas if the voltage is negative then the mobile charges are negative. If we were to perform this experiment we would

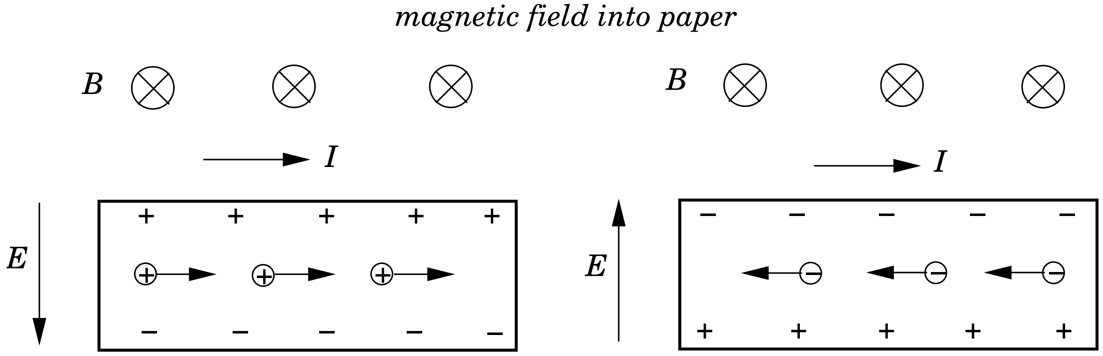


Figure 8.7: *Hall effect for positive charge carriers (left) and negative charge carriers (right).*

discover that the the mobile charges in metals are always negative (because they are electrons). However, in some types of semiconductor the mobile charges turn out to be positive. These positive charge carriers are called *holes*. Holes are actually missing electrons in the atomic lattice of the semiconductor, but they act essentially like positive charges.

Let us investigate the magnitude of the Hall voltage. Suppose that the mobile charges each possess a charge q and move along the ribbon with the drift velocity v_d. The magnetic force on a given mobile charge is of magnitude $q\,v_d\,B$, since the charge moves essentially at right-angles to the magnetic field. In a steady-state, this force is balanced by the electric force due to the build up of charges on the upper and lower edges of the ribbon. If the Hall voltage is V_H, and the width of the ribbon is w, then the electric field pointing from the upper to the lower edge of the ribbon is of magnitude $E = V_H/w$. Now, the electric force on a mobile charge is $q\,E$. This force acts in opposition to the magnetic force. In a steady-state,

$$q\,E = \frac{q\,V_H}{w} = q\,v_d\,B, \tag{8.18}$$

giving

$$V_H = v_d\,w\,B. \tag{8.19}$$

Note that the Hall voltage is directly proportional to the magnitude of the magnetic field. In fact, this property of the Hall voltage is exploited in instruments, called *Hall probes*, which are used to measure magnetic field-strength.

Suppose that the thickness of the conducting ribbon is d, and that it contains n mobile charge carriers per unit volume. It follows that the total current flowing through the ribbon can be written

$$I = q\,n\,w\,d\,v_d, \tag{8.20}$$

since all mobile charges contained in a rectangular volume of length v_d, width w, and thickness d, flow past a given point on the ribbon in one second. Combining Eqs. (8.19) and (8.20), we obtain

$$V_H = \frac{I\,B}{q\,n\,d}. \tag{8.21}$$

It is clear that the Hall voltage is proportional to the current flowing through the ribbon, and the magnetic field-strength, and is inversely proportional to the number density of mobile charges in the ribbon, and the thickness of the ribbon. Thus, in order to construct a sensitive Hall probe (*i.e.*, one which produces a large Hall voltage in the presence of a small magnetic field), we need to take a thin ribbon of some material which possesses relatively few mobile charges per unit volume (*e.g.*, a semiconductor), and then run a large current through it.

8.7 Ampère's Circuital Law

Consider a long thin wire carrying a steady current I. Suppose that the wire is orientated such that the current flows along the z-axis. Consider some closed loop C in the x-y plane which circles the wire in an anti-clockwise direction, looking down the z-axis. Suppose that $d\mathbf{r}$ is a short straight-line element of this loop. Let us form the dot product of this element with the local magnetic field $\mathbf{B}$. Thus,

$$dw = \mathbf{B}\cdot d\mathbf{r} = B\,dr\,\cos\theta, \tag{8.22}$$

where θ is the angle subtended between the direction of the line element and the direction of the local magnetic field. We can calculate a dw for every line element which makes up the loop C. If we sum all of the dw values thus obtained, and take the limit as the number of elements goes to infinity, we obtain the *line integral*

$$w = \oint_C \mathbf{B}\cdot d\mathbf{r}. \tag{8.23}$$

What is the value of this integral? In general, this is a difficult question to answer. However, let us consider a special case. Suppose that C is a circle of radius r centred on the wire. In this case, the magnetic field-strength is the same at all points on the loop. In fact,

$$B = \frac{\mu_0\, I}{2\pi\, r}. \tag{8.24}$$

Moreover, the field is everywhere parallel to the line elements which make up the loop. Thus,

$$w = 2\pi\, r\, B = \mu_0\, I, \tag{8.25}$$

or

$$\oint_C \mathbf{B}\cdot d\mathbf{r} = \mu_0\, I. \tag{8.26}$$

In other words, the line integral of the magnetic field around some circular loop C, centred on a current carrying wire, and in the plane perpendicular to the wire, is equal to μ_0 times the current flowing in the wire. Note that this answer is independent of the radius r of the loop: *i.e.*, the same result is obtained by taking the line integral around *any* circular loop centred on the wire.

In 1826, Ampère demonstrated that Eq. (8.26) holds for *any* closed loop which circles around *any* distribution of currents. Thus, Ampère's circuital law can be written:

> The line integral of the magnetic field around some closed loop is equal to the μ_0 times the algebraic sum of the currents which pass through the loop.

In forming the algebraic sum of the currents passing through the loop, those currents which the loop circles in an anti-clockwise direction (looking against the direction of the current) count as positive currents, whereas those which the loop circles in a clockwise direction (looking against the direction of the current) count as negative currents.

Ampère's circuital law is to magnetostatics (the study of the magnetic fields generated by steady currents) what Gauss' law is to electrostatics (the study of the electric fields generated by stationary charges). Like Gauss' law, Ampère's circuital law is particularly useful in situations which possess a high degree of symmetry.

8.8 Magnetic Field of a Solenoid

A *solenoid* is a tightly wound helical coil of wire whose diameter is small compared to its length. The magnetic field generated in the centre, or *core*, of a current carrying solenoid is essentially *uniform*, and is directed along the axis of the solenoid. Outside the solenoid, the magnetic field is far weaker. Figure 8.8 shows (rather schematically) the magnetic field generated by a typical solenoid. The solenoid is wound from a single helical wire which carries a current I. The winding is sufficiently tight that each turn of the solenoid is well approximated as a circular wire loop, lying in the plane perpendicular to the axis of the solenoid, which carries a current I. Suppose that there are n such turns per unit axial length of the solenoid. What is the magnitude of the magnetic field in the core of the solenoid?

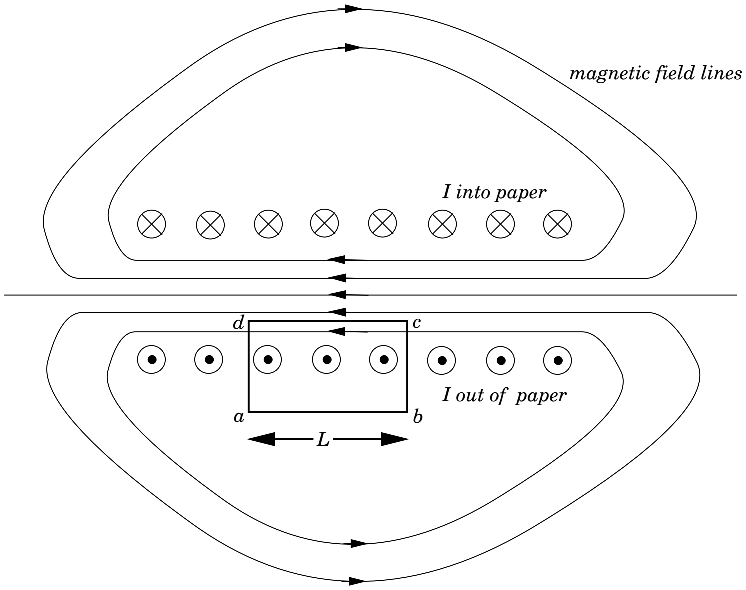


Figure 8.8: *A solenoid.*

In order to answer this question, let us apply Ampère's circuital law to the rectangular loop $abcd$. We must first find the line integral of the magnetic field around $abcd$. Along bc and da the magnetic field is essentially perpendicular to the loop, so there is no contribution to the line integral from these sections of the loop. Along cd the magnetic field is approximately uniform, of magnitude B, say, and is directed parallel to the loop. Thus, the contribution to the line integral from this section of the loop is $B\,L$, where L is the length of cd. Along ab the magnetic field-strength is essentially negligible, so this section of the loop makes no contribution to the line integral. It follows that the line integral of the magnetic field around $abcd$ is simply

$$w = B\,L. \tag{8.27}$$

By Ampère's circuital law, this line integral is equal to μ_0 times the algebraic sum of the currents which flow through the loop $abcd$. Since the length of the loop along the axis of the solenoid is L, the loop intersects $n\,L$ turns of the solenoid, each carrying a current I. Thus, the total current which flows through the loop is $n\,L\,I$. This current counts as a positive current since if we look against the direction of the currents flowing in each turn (*i.e.*, into the page in the figure), the loop $abcd$ circulates these currents in an anti-clockwise direction. Ampère's circuital law yields

$$B\,L = \mu_0\,n\,L\,I, \tag{8.28}$$

which reduces to

$$B = \mu_0\,n\,I. \tag{8.29}$$

Thus, the magnetic field in the core of a solenoid is directly proportional to the product of the current flowing around the solenoid and the number of turns per unit length of the solenoid. This, result is *exact* in the limit in which the length of the solenoid is very much greater than its diameter.

8.9 Origin of Permanent Magnetism

We now know of two distinct methods of generating a magnetic field. We can either use a permanent magnet, such as a piece of loadstone, or we can run a current around an electric circuit. Are these two methods fundamentally different, or are they somehow related to one another? Let us investigate further.

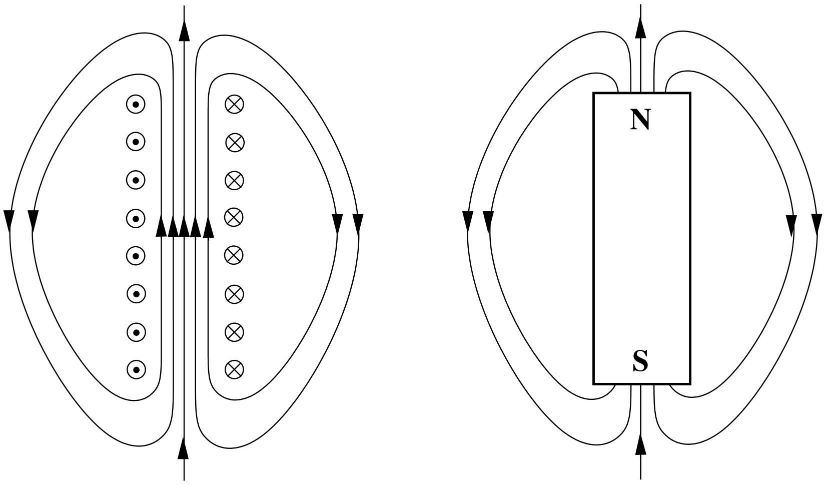


Figure 8.9: *Magnetic fields of a solenoid (left) and a bar magnet (right).*

As illustrated in Fig. 8.9, the *external* magnetic fields generated by a solenoid and a conventional bar magnet are remarkably similar in appearance. Incidentally, these fields can easily be mapped out using iron filings. The above observation allows us to formulate *two* alternative hypotheses for the origin of the magnetic field of a bar magnet. The first hypotheses is that the field of a bar magnet is produced by solenoid-like currents which flow around the outside of the magnet, in an anti-clockwise direction as we look along the magnet from its north to its south pole. There is no doubt, by analogy with a solenoid, that such currents would generate the correct sort of field outside the magnet. The second hypothesis is that the field is produced by a positive *magnetic monopole* located close to the north pole of the magnet, in combination with a negative monopole of equal magnitude located close to the south pole of the magnet. What is a magnetic monopole? Well, it is basically the magnetic equivalent of an electric charge. A positive magnetic monopole is an isolated magnetic north pole. We would expect magnetic field-lines to radiate away from such an object, just as electric field-lines radiate away from a positive electric charge. Likewise, a negative magnetic monopole is an isolated magnetic south pole. We would expect magnetic field-lines to radiate towards such an object, just as electric field-lines radiate towards a negative electric charge. The magnetic field patterns generated by both types of monopole are sketched in Fig. 8.10. If we place a positive monopole close to the north pole of a bar magnet, and a negative monopole of the same magnitude close to the south pole, then the resultant magnetic field pattern is obtained by *superposing* the fields generated by the two monopoles individually. As is easily demonstrated, the field generated outside the magnet is indistinguishable from that of a solenoid.

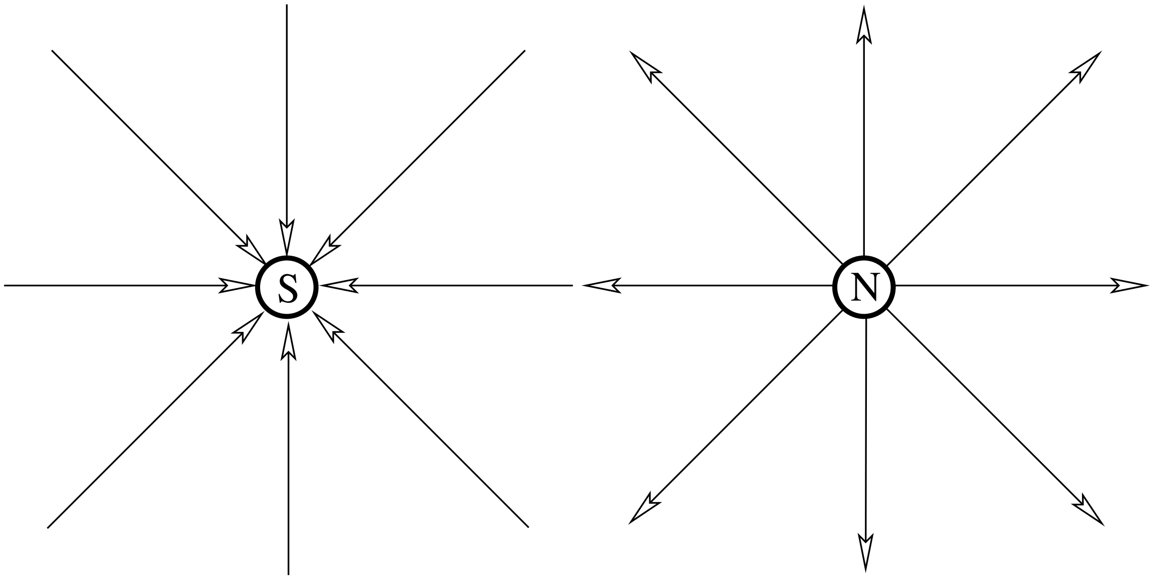


Figure 8.10: *Magnetic field-lines generated by magnetic monopoles.*

We now have two alternative hypotheses to explain the origin of the magnetic field of a bar magnet. What experiment could we perform in order to determine which of these two hypotheses is correct? Well, suppose that we snap our bar magnet in two. What happens according to each hypothesis? If we cut a solenoid in two then we just end up with two shorter solenoids. So, according to our first hypothesis, if we snap a bar magnet in two then we just end up with two smaller bar magnets. However, our second hypothesis predicts that if we snap a bar magnet in two then we end up with two equal and opposite magnetic monopoles. Needless to say, the former prediction is in accordance with experiment, whereas the latter most certainly is not. Indeed, we can break a bar magnetic into as many separate pieces as we like. Each piece will still act like a little bar magnet. No matter how small we make the pieces, we cannot produce a magnetic monopole. In fact, nobody has ever observed a magnetic monopole experimentally, which leads most physicists to conclude that *magnetic monopoles do not exist*. Thus, we can conclude that the magnetic field of a bar magnet is produced by solenoid-like currents flowing over the surface of the magnet. But, what is the origin of these currents?

In order to answer the last question, let us adopt a somewhat simplistic model of the atomic structure of a bar magnet. Suppose that the north-south axis of the magnet is aligned along the z-axis, such that the z-coordinate of the magnet's north pole is larger than that of its south pole. Suppose, further, that the atoms which make up the magnet are identical *cubes* which are packed very closely together. Each atom carries a *surface current* which circulates in the x-y plane in an anti-clockwise direction (looking down the z-axis). When the atoms are arranged in a uniform lattice, so as to form the magnet, the interior surface currents cancel out, leaving a current which flows only on the outer surface of the magnet. This is illustrated in Fig. 8.11. Thus, the solenoid-like currents which must flow over the surface of a magnet in order to account for its associated magnetic field are, in fact, just the *resultant* of currents which circulate in every constituent atom of the magnet. But, what is the origin of these atomic currents?

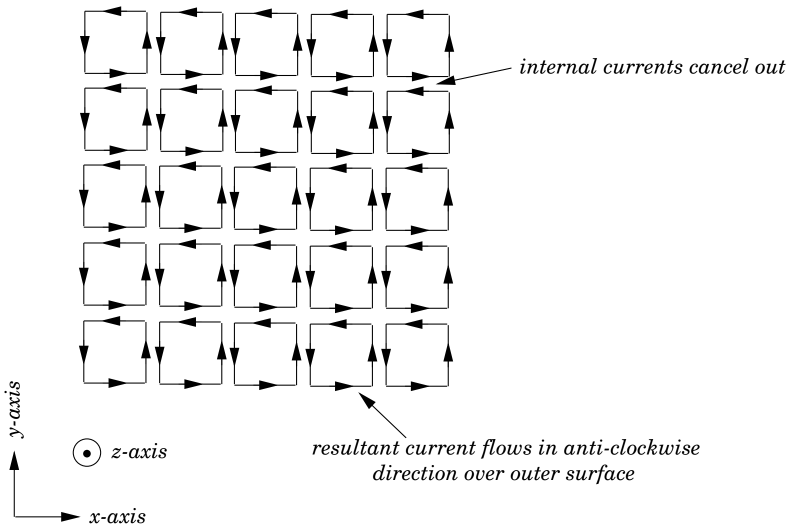

internal currents cancel out
y-axis
z-axis
x-axis
resultant current flows in anti-clockwise direction over outer surface

Figure 8.11: *A schematic diagram of the current pattern in a permanent magnet.*

Well, atoms consist of negatively charged electrons in orbit around positively charged nuclei. A moving electric charge constitutes an electric current, so there must be a current associated with every electron in an atom. In most atoms, these currents cancel one another out, so that the atom carries zero net current. However, in the atoms of *ferromagnetic* materials (*i.e.*, iron, cobalt, and nickel) this cancellation is not complete, so these atoms do carry a net current. Usually, the atomic currents are all jumbled up (*i.e.*, they are not aligned in any particular plane) so that they average to zero on a macroscopic scale. However, if a ferromagnetic material is placed in a *strong* magnetic field then the currents circulating in each atom become *aligned* such that they flow predominately in the plane perpendicular to the field. In this situation, the currents can combine together to form a macroscopic magnetic field which reinforces the alignment field. In some ferromagnetic materials, the atomic currents remain aligned after the alignment field is switched off, so the macroscopic field generated by these currents also remains. We call such materials *permanent magnets*.

In conclusion, *all* magnetic fields encountered in nature are generated by *circulating currents*. There is no fundamental difference between the fields generated by permanent magnets and those generated by currents flowing around conventional electric circuits. In the former, case the currents which generate the fields circulate on the atomic scale, whereas, in the latter case, the currents circulate on a macroscopic scale (*i.e.*, the scale of the circuit).

8.10 Gauss' Law for Magnetic Fields

Recall (from Sect. 4.2) that the electric flux through a closed surface S is written

$$\Phi_{\rm E} = \oint_S \mathbf{E}\cdot d\mathbf{S}. \tag{8.30}$$

Similarly, we can also define the *magnetic flux* through a closed surface as

$$\Phi_{\rm B} = \oint_S \mathbf{B}\cdot d\mathbf{S}. \tag{8.31}$$

According to Gauss' law (see Sect. 4.2), the electric flux through any closed surface is directly proportional to the net electric charge enclosed by that surface. Given the very direct analogy which exists between an electric charge and a magnetic monopole, we would expect to be able to formulate a second law which states that the magnetic flux through any closed surface is directly proportional to the number of magnetic monopoles enclosed by that surface. However, as we have already discussed, magnetic monopoles do not exist. It follows that the equivalent of Gauss' law for magnetic fields reduces to:

> The magnetic flux though any closed surface is zero.

This is just another way of saying that magnetic monopoles do not exist, and that all magnetic fields are actually generated by circulating currents.

An immediate corollary of the above law is that the number of magnetic field-lines which enter a closed surface is always equal to the number of field-lines which leave the surface. In other words:

> Magnetic field-lines form closed loops which never begin or end.

Thus, magnetic field-lines behave in a quite different manner to electric field-lines, which begin on positive charges, end on negative charges, and never form closed loops. Incidentally, the statement that electric field-lines never form closed loops follows from the result that the work done in taking an electric charge around a closed loop is always zero (see Sect. 5). This clearly cannot be true if it is possible to take a charge around the path of a closed electric field-line. Note, however, that this conclusion regarding electric field-lines only holds for the electric fields generated by stationary charges.

8.11 Galvanometers

We have talked a lot about potential differences, currents, and resistances, but we have not talked much about how these quantities can be measured. Let us now investigate this topic.

Broadly speaking, only electric currents can be measured directly. Potential differences and resistances are usually *inferred* from measurements of electric currents. The most accurate method of measuring an electric current is by using a device called a *galvanometer*.

A galvanometer consists of a rectangular conducting coil which is free to pivot vertically in an approximately uniform horizontal magnetic field B—see Fig. 8.12. The magnetic field is usually generated by a permanent magnet. Suppose that a current I runs through the coil. What are the forces exerted on the coil by the magnetic field? According to Eq. (8.1), the forces exerted on those sections of the coil in which the current runs in the horizontal plane are directed vertically up or down. These forces are irrelevant, since they are absorbed by the support structure of the coil, which does not allow the coil to move vertically. Equation (8.1) also implies that the force exerted on the section of the coil in which the current flows downward is of magnitude $F = I\,B\,L$, where L the length of this section, and is directed out of the page (in the figure). Likewise, the force exerted on the section of the coil in which the current flows upward is also of magnitude $F = I\,B\,L$, and is directed into the page. These two forces exert a *torque* on the coil which tries to twist it about its vertical axis in an anti-clockwise direction (looking from above). Using the usual definition of torque (*i.e.*, torque is the product of the force and the perpendicular distance from the line of action of the force to the axis of rotation), the net torque τ acting on the coil is

$$\tau = 2\,F\,\frac{D}{2} = I\,B\,L\,D = I\,B\,A. \tag{8.32}$$

where D is the horizontal width of the coil, and A is its area. Note that the two vertical sections of the coil give rise to equal contributions to the torque. Strictly speaking, the above expression is only valid when the coil lies in the plane of the magnetic field. However, galvanometers are usually constructed with curved magnetic pole pieces in order to ensure that, as the coil turns, it always remains in the plane of the magnetic field. It follows that, for fixed magnetic field-strength, and fixed coil area, the torque exerted on the coil is directly proportional to the current I.

The coil in a galvanometer is usually suspended from a torsion wire. The wire exerts a restoring torque on the coil which tries to twist it back to its original position. The strength of this restoring torque is directly proportional to the angle of twist $\Delta\theta$. It follows that, in equilibrium, where the magnetic torque

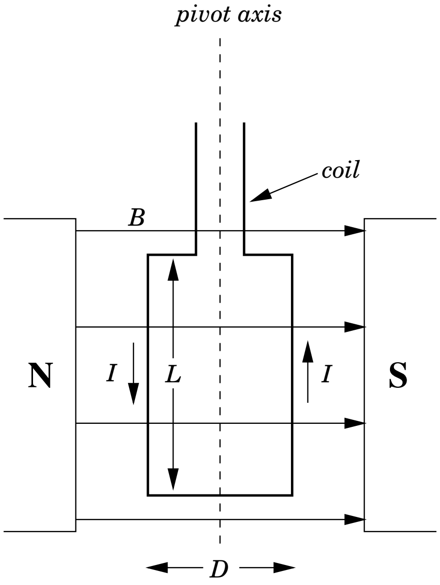


Figure 8.12: *A galvanometer.*

balances the restoring torque, the angle of twist $\Delta\theta$ is directly proportional to the current I flowing around the coil. The angle of twist can be measured by attaching a pointer to the coil, or, even better, by mounting a mirror on the coil, and reflecting a light beam off the mirror. Since $\Delta\theta \propto I$, the device can easily be calibrated by running a known current through it.

There is, of course, a practical limit to how large the angle of twist $\Delta\theta$ can become in a galvanometer. If the torsion wire is twisted through too great an angle then it will deform permanently, and will eventually snap. In order to prevent this from happening, most galvanometers are equipped with a "stop" which physically prevents the coil from twisting through more than (say) 90°. Thus, there is a maximum current I_{fsd} which a galvanometer can measure. This is usually referred to as the *full-scale-deflection current*. The full-scale-deflection current in conventional galvanometers is usually pretty small: *e.g.*, $10\,\mu A$. So, what do we do if we want to measure a large current?

What we do is to connect a *shunt resistor* in parallel with the galvanometer, so that most of the current flows through the resistor, and only a small fraction of the current flows through the galvanometer itself. This is illustrated in Fig. 8.13. Let the resistance of the galvanometer be R_G, and the resistance of the shunt resistor be R_S. Suppose that we want to be able to measure the total current I flowing through the galvanometer and the shunt resistor up to a maximum value of I_{max}. We can achieve this if the current I_G flowing through the galvanometer equals the full-scale-deflection current I_{fsd} when $I = I_{max}$. In this case, the current $I_S = I - I_G$ flowing through the shunt resistor takes the value $I_{max} - I_{fsd}$. The potential drop across the shunt resistor is therefore $(I_{max} - I_{fsd})\,R_S$. This potential drop must match the potential drop $I_{fsd}\,R_G$ across the galvanometer, since the galvanometer is connected in parallel with the shunt resistor. It follows that

$$(I_{max} - I_{fsd})\,R_S = I_{fsd}\,R_G, \tag{8.33}$$

which reduces to

$$R_S = \frac{I_{fsd}}{I_{max} - I_{fsd}}\,R_G. \tag{8.34}$$

Using this formula, we can always choose an appropriate shunt resistor to allow a galvanometer to measure any current, no matter how large. For instance, if the full-scale-deflection current is $I_{fsd} = 10\,\mu$A, the maximum current we wish to measure is $I_{max} = 1$ A, and the resistance of the galvanometer is $R_G = 40\,\Omega$, then the appropriate shunt resistance is

$$R_S = \frac{1\times 10^{-5}}{1 - 1\times 10^{-5}}\,40 \simeq 4.0\times 10^{-4}\,\Omega. \tag{8.35}$$

Most galvanometers are equipped with a dial which allows us to choose between various alternative ranges of currents which the device can measure: *e.g.*, 0–100 mA, 0–1 A, or 0–10 A. All the dial does is to switch between different shunt resistors connected in parallel with the galvanometer itself. Note, finally, that the equivalent resistance of the galvanometer and its shunt resistor is

$$R_{eq} = \frac{1}{(1/R_G) + (1/R_S)} = \frac{I_{fsd}}{I_{max}}\,R_G. \tag{8.36}$$

Clearly, if the full-scale-deflection current I_{fsd} is much less than the maximum current I_{max} which we wish to measure then the equivalent resistance is very small indeed. Thus, there is an advantage to making the full-scale-deflection current of a galvanometer small. A small full-scale-deflection current implies a small equivalent resistance of the galvanometer, which means that the galvanometer can be

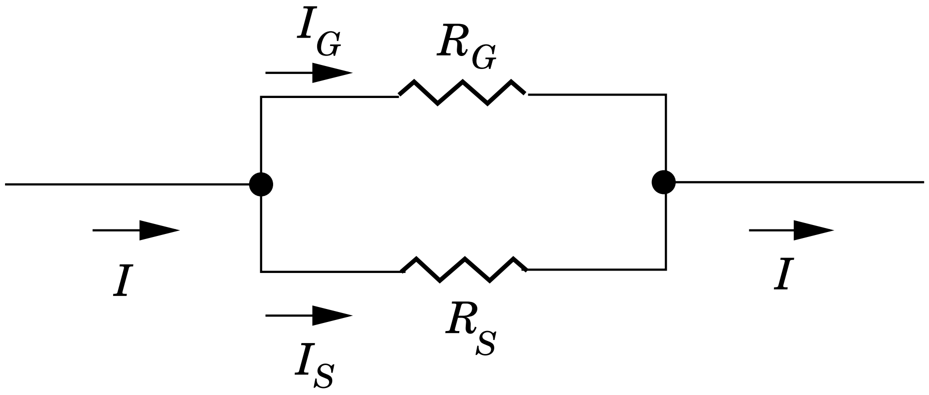


Figure 8.13: *Circuit diagram for a galvanometer measuring current.*

connected into a circuit without seriously disturbing the currents flowing around that circuit.

A galvanometer can be used to measure potential difference as well as current (although, in the former case, it is really measuring current). In order to measure the potential difference V between two points a and b in some circuit, we connect a galvanometer, in series with a shunt resistor, across these two points—see Fig. 8.14. The galvanometer draws a current I from the circuit. This current is, of course, proportional to the potential difference between a and b, which enables us to relate the reading on the galvanometer to the voltage we are trying to measure. Suppose that we wish to measure voltages in the range 0 to $V_{\rm max}$. What is an appropriate choice of the shunt resistance R_S? Well, the equivalent resistance of the shunt resistor and the galvanometer is $R_S + R_G$, where R_G is the resistance of the galvanometer. Thus, the current flowing through the galvanometer is $I = V/(R_S + R_G)$. We want this current to equal the full-scale-deflection current $I_{\rm fsd}$ of the galvanometer when the potential difference between points a and b attains its maximum allowed value $V_{\rm max}$. It follows that

$$I_{\rm fsd} = \frac{V_{\rm max}}{R_S + R_G}, \tag{8.37}$$

which yields

$$R_S = \frac{V_{\rm max}}{I_{\rm fsd}} - R_G. \tag{8.38}$$

Using this formula, we can always choose an appropriate shunt resistor to allow a galvanometer to measure any voltage, no matter how large. For instance, if the full-scale-deflection current is $I_{\rm fsd} = 10\,\mu\text{A}$, the maximum voltage we wish to measure is 1000 V, and the resistance of the galvanometer is $R_G = 40\,\Omega$, then the appropriate shunt resistance is

$$R_S = \frac{1000}{1 \times 10^{-5}} - 40 \simeq 10^8\,\Omega. \tag{8.39}$$

Again, there is an advantage in making the full-scale-deflection current of a galvanometer used as a voltmeter small, because, when it is properly set up, the galvanometer never draws more current from the circuit than its full-scale-deflection current. If this current is small then the galvanometer can measure potential differences in a circuit without significantly perturbing the currents flowing around that circuit.

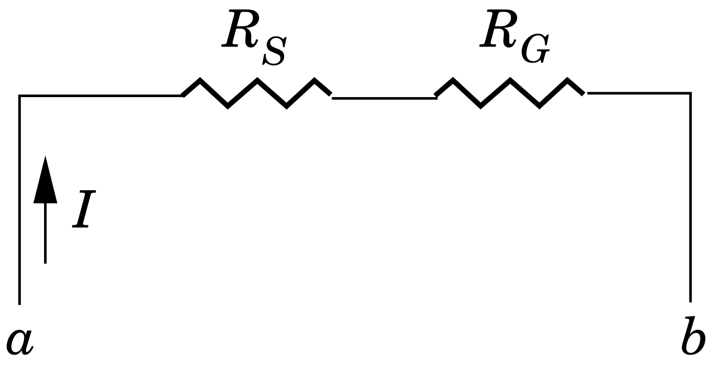
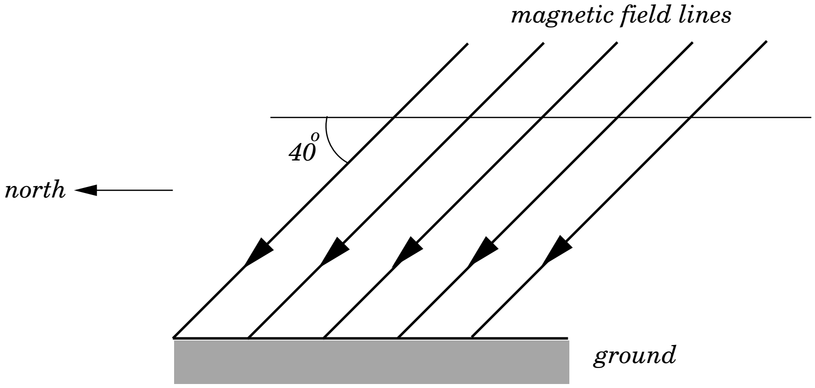


Figure 8.14: *Circuit diagram for a galvanometer measuring potential difference.*

8.12 Worked Examples

Example 8.1: Earth's magnetic field

Question: In Texas, the Earth's magnetic field is approximately uniform, and of magnitude $B = 10^{-4}$ T. The horizontal component of the field is directed northward. The field also has a vertical component which is directed into the ground. The angle the field lines dip below the horizontal is 40°. A metal bar of length $l = 1.2$ m carries a current of $I' = 1.7$ A. Suppose that the bar is held horizontally



such that the current flows from East to West. What is the magnitude and direction of the magnetic force on the bar? Suppose that the direction of the current is reversed. What, now, is the magnitude and direction of the magnetic force on the bar? Suppose that the bar is held vertically such the current flows upward. What is the magnitude and direction of the magnetic force on the bar? Suppose, finally, that the direction of the current is reversed. What, now, is the magnitude and direction of the magnetic force on the bar?

Answer: If the current in the bar flows horizontally from East to West then the direction of the current makes an angle of 90° with the direction of the magnetic field. So, from Eq. (8.1), the magnetic force *per unit length* acting on the bar is

$$F = I' B \sin 90^\circ = I' B = (1.7)\,(10^{-4}) = 1.7 \times 10^{-4}\,\mathrm{N\,m^{-1}}.$$

Thus, the total force acting on the bar is

$$f = F\,l = (1.7 \times 10^{-4})\,(1.2) = 2.04 \times 10^{-4}\,\mathrm{N}.$$

Using the right-hand rule, if the index finger of a right-hand points horizontally from East to West, and the middle finger points northward, but dips 40° below the horizontal, then the thumb points southward, but dips 50° below the horizontal. Thus, the force on the bar is directed southward, and dips 50° below the horizontal. If the current in the bar is reversed, so that it now flows horizontally from West to East, then the angle subtended between the direction of current flow and the direction of the magnetic field is still 90°, so the magnitude of the force on the bar remains unchanged. According to the right-hand rule, if the index finger of a right-hand points horizontally from West to East, and the middle finger points northward, but dips 40° below the horizontal, then the thumb points northward, but is directed 50° above the horizontal. Thus, the force on the bar is directed northward at an angle of 50° above the horizontal. In other words, the new force points in exactly the opposite direction to the old one.

If the current in the bar flows vertically upward then the direction of current flow subtends an angle of 50° degrees with the direction of the magnetic field. So, from Eq. (8.1), the magnetic force *per unit length* acting on the bar is

$$F = I' B \sin 50^\circ = (1.7)\,(10^{-4})\,(0.7660) = 1.30 \times 10^{-4}\,\mathrm{N\,m^{-1}}.$$

Thus, the total force acting on the bar is

$$f = F\,l = (1.30 \times 10^{-4})\,(1.2) = 1.56 \times 10^{-4}\,\mathrm{N}.$$

Using the right-hand rule, if the index finger of a right-hand points vertically upward, and the middle finger points northward, but dips 40° below the horizontal, then the thumb points horizontally westward. Thus, the force on the bar is directed horizontally westward. If the current in the bar is reversed, so that it flows vertically downward, then the force on the bar is of the same magnitude, but points in the opposite direction, which means that the new force points horizontally eastward.

Example 8.2: Charged particle in magnetic field

Question: Suppose that an electron is accelerated from rest through a voltage difference of $V = 10^3$ volts and then passes into a region containing a uniform magnetic field of magnitude $B = 1.2\,\mathrm{T}$. The electron subsequently executes a closed circular orbit in the plane perpendicular to the field. What is the radius of this orbit? What is the angular frequency of gyration of the electron?

Answer: If an electron of mass $m_e = 9.11 \times 10^{-31}$ kg and charge $e = 1.60 \times 10^{-19}$ C is accelerated from rest through a potential difference V then its final kinetic energy is

$$\frac{1}{2}\, m_e\, v^2 = e\, V.$$

Thus, the final velocity v of the electron is given by

$$v = \sqrt{\frac{2\, e\, V}{m_e}} = \sqrt{\frac{(2)\,(1.6\times 10^{-19})\,(10^3)}{(9.11\times 10^{-31})}} = 1.87\times 10^7\,\mathrm{m\,s^{-1}}.$$

The initial direction of motion of the electron is at right-angles to the direction of the magnetic field, otherwise the orbit of the electron would be a spiral instead of a closed circle. Thus, we can use Eq. (8.16) to calculate the radius ρ of the orbit. We obtain

$$\rho = \frac{m_e\, v}{e\, B} = \frac{(9.11\times 10^{-31})\,(1.87\times 10^7)}{(1.6\times 10^{-19})\,(1.2)} = 8.87\times 10^{-5}\,\mathrm{m}.$$

The angular frequency of gyration ω of the electron comes from Eq. (8.17):

$$\omega = \frac{e\, B}{m_e} = \frac{(1.6\times 10^{-19})\,(1.2)}{(9.11\times 10^{-31})} = 2.11\times 10^{11}\,\mathrm{rad.\,s^{-1}}.$$

Example 8.3: Ampère's circuital law

Question: A z-directed wire of radius a carries a total z-directed current I. What is the magnetic field distribution, both inside and outside the wire, if the current is evenly distributed throughout the wire? What is the magnetic field distribution if the current is concentrated in a thin layer at the surface of the wire?

Answer: Since the current distribution possesses cylindrical symmetry, it is reasonable to suppose that the magnetic field it generates also possesses cylindrical symmetry. By analogy with the magnetic field generated by an infinitely thin z-directed wire, we expect the magnetic field to circulate in the x-y plane in an anti-clockwise direction (looking against the direction of the current). Let us apply Ampère's circuital law to a circular loop in the x-y plane which is centred on the centre of the wire, and is of radius $r > a$. The magnetic field is everywhere tangential to the loop, so the line integral of the magnetic field (taken in an anti-clockwise sense, looking against the direction of the current) is

$$w(r) = 2\pi\, r\, B(r),$$

where $B(r)$ is the magnetic field-strength at radius r. According to Ampère's circuital law, this line integral is equal to μ_0 times the total current enclosed by the loop. The total current is clearly I, since the loop lies outside the wire. Thus,

$$w(r) = 2\pi\, r\, B(r) = \mu_0\, I,$$

giving

$$B(r) = \frac{\mu_0\, I}{2\pi\, r}$$

for $r > a$. This is exactly the same field distribution as that generated by an infinitely thin wire carrying the current I. Thus, we conclude that the magnetic field generated outside a cylindrically symmetric z-directed current distribution is the same as if all of the current were concentrated at the centre of the distribution. Let us now apply Ampère's circuital law to a circular loop which is of radius $r < a$. The line integral of the magnetic field around this loop is simply $w(r) = 2\pi\, r\, B(r)$. However, the current enclosed by the loop is equal to I times the ratio of the area of the loop to the cross-sectional area of the wire (since the current is evenly distributed throughout the wire). Thus, Ampère's law yields

$$2\pi\, r\, B(r) = \mu_0\, I\, \frac{r^2}{a^2},$$

which gives

$$B(r) = \frac{\mu_0\, I\, r}{2\pi\, a^2}.$$

Clearly, the field inside the wire increases linearly with increasing distance from the centre of the wire.

If the current flows along the outside of the wire then the magnetic field distribution exterior to the wire is exactly the same as that described above. However, there is no field inside the wire. This follows immediately from Ampère's circuital law because the current enclosed by a circular loop whose radius is less than the radius of the wire is clearly zero.

9 Magnetic Induction

9.1 Faraday's Law

The phenomenon of magnetic induction plays a crucial role in three very useful electrical devices: the *electric generator*, the *electric motor*, and the *transformer*. Without these devices, modern life would be impossible in its present form. Magnetic induction was discovered in 1830 by the English physicist Michael Faraday. The American physicist Joseph Henry independently made the same discovery at about the same time. Both physicists were intrigued by the fact that an electric current flowing around a circuit can generate a magnetic field. Surely, they reasoned, if an electric current can generate a magnetic field then a magnetic field must somehow be able to generate an electric current. However, it took many years of fruitless experimentation before they were able to find the essential ingredient which allows a magnetic field to generate an electric current. This ingredient is *time variation*.

Consider a planar loop C of conducting wire of cross-sectional area A. Let us place this loop in a magnetic field whose strength B is approximately uniform over the extent of the loop. Suppose that the direction of the magnetic field subtends an angle θ with the normal direction to the loop. The *magnetic flux* Φ_B through the loop is defined as the product of the area of the loop and the component of the magnetic field perpendicular to the loop. Thus,

$$\Phi_B = A\,B_\perp = A\,B\,\cos\theta. \tag{9.1}$$

If the loop is wrapped around itself N times (*i.e.*, if the loop has N *turns*) then the magnetic flux through the loop is simply N times the magnetic flux through a single turn:

$$\Phi_B = N\,A\,B_\perp. \tag{9.2}$$

Finally, if the magnetic field is not uniform over the loop, or the loop does not lie in one plane, then we must evaluate the magnetic flux as a surface integral

$$\Phi_B = \int_S \mathbf{B}\cdot d\mathbf{S}. \tag{9.3}$$

Here, S is some surface attached to C. If the loop has N turns then the flux is N times the above value. The SI unit of magnetic flux is the weber (Wb). One tesla is equivalent to one weber per meter squared:

$$1\,\mathrm{T} \equiv 1\,\mathrm{Wb\,m^{-2}}. \tag{9.4}$$

Faraday discovered that if the magnetic field through a loop of wire *varies in time* then an emf is induced around the loop. Faraday was able to observe this effect because the emf gives rise to a current circulating in the loop. Faraday found that the magnitude of the emf is directly proportional to the time rate of change of the magnetic field. He also discovered that an emf is generated when a loop of wire *moves* from a region of low magnetic field-strength to one of high magnetic field-strength, and *vice versa*. The emf is directly proportional to the velocity with which the loop moves between the two regions. Finally, Faraday discovered that an emf is generated around a loop which *rotates* in a uniform magnetic field of constant strength. In this case, the emf is directly proportional to the rate at which the loop rotates. Faraday was eventually able to propose a single law which could account for all of his many and varied observations. This law, which is known as *Faraday's law of magnetic induction*, is as follows:

> The emf induced in a circuit is proportional to the time rate of change of the magnetic flux linking that circuit.

SI units have been fixed so that the constant of proportionality in this law is *unity*. Thus, if the magnetic flux through a circuit changes by an amount $d\Phi_B$ in a time interval dt then the emf $\mathcal{E}$ generated in the circuit is

$$\mathcal{E} = \frac{d\Phi_B}{dt}. \tag{9.5}$$

There are many different ways in which the magnetic flux linking an electric circuit can change. Either the magnetic field-strength can change, or the direction of the magnetic field can change, or the position of the circuit can change, or the shape of the circuit can change, or the orientation of the circuit can change. Faraday's law states that all of these ways are completely *equivalent* as far as the generation of an emf around the circuit is concerned.

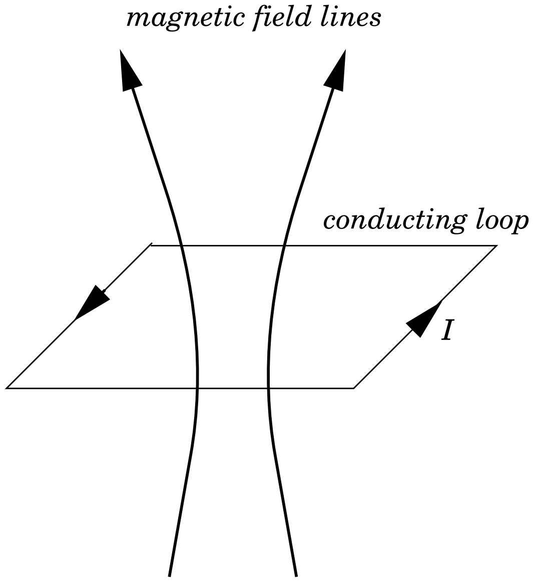


Figure 9.1: *Magnetic field generated by a planar current-carrying loop.*

9.2 Lenz's Law

We still have not specified in which direction the emf generated by a time-varying magnetic flux linking an electric circuit acts. In order to help specify this direction, we need to make use of a right-hand rule. Suppose that a current I circulates around a planar loop of conducting wire, and, thereby, generates a magnetic field **B**. What is the direction of this magnetic field as it passes through the middle of the loop? Well, if the fingers of a right-hand circulate in the same direction as the current, then the thumb indicates the direction of the magnetic field as it passes through the centre of the loop. This is illustrated in Fig. 9.1.

Consider a plane loop of conducting wire which is linked by magnetic flux. By convention, the direction in which current would have to flow around the loop in order to *increase* the magnetic flux linking the loop is termed the *positive* direction. Likewise, the direction in which current would have to flow around the loop in order to *decrease* the magnetic flux linking the loop is termed the *negative* direction. Suppose that the magnetic flux linking the loop is increased. In accordance with Faraday's law, an emf is generated around the loop. Does this emf act in the positive direction, so as to drive a current around the loop which further increases the magnetic flux, or does it act in the negative direction, so as to drive a current around the loop which decreases the magnetic flux? It is easily demonstrated experimentally that the emf acts in the negative direction. Thus:

> The emf induced in an electric circuit always acts in such a direction that the current it drives around the circuit opposes the change in magnetic flux which produces the emf.

This result is known as *Lenz's law*, after the nineteenth century Russian scientist Heinrich Lenz, who first formulated it. Faraday's law, combined with Lenz's law, is usually written

$$\mathcal{E} = -\frac{d\Phi_B}{dt}. \tag{9.6}$$

The minus sign is to remind us that the emf always acts to oppose the change in magnetic flux which generates the emf.

9.3 Magnetic Induction

Consider a one-turn loop of conducting wire C which is placed in a magnetic field **B**. The magnetic flux Φ_B linking loop C can be written

$$\Phi_B = \int_S \mathbf{B}\cdot d\mathbf{S} \tag{9.7}$$

where S is any surface attached to the loop.

Suppose that the magnetic field changes in time, causing the magnetic flux Φ_B linking circuit C to vary. Let the flux change by an amount $d\Phi_B$ in the time interval dt. According to Faraday's law, the emf $\mathcal{E}$ induced around loop C is given by

$$\mathcal{E} = -\frac{d\Phi_B}{dt}. \tag{9.8}$$

If $\mathcal{E}$ is *positive* then the emf acts around the loop in the *same* sense as that indicated by the fingers of a right-hand, when the thumb points in the direction of the mean magnetic field passing through the loop. Likewise, if $\mathcal{E}$ is *negative* then the emf acts around the loop in the *opposite* sense to that indicated by the fingers of a right-hand, when the thumb points in the direction of the mean magnetic field passing through the loop. In the former case, we say that the emf acts in the *positive* direction, whereas in the latter case we say it acts in the *negative* direction.

Suppose that $\mathcal{E} > 0$, so that the emf acts in the positive direction. How, exactly, is this emf produced? In order to answer this question, we need to remind ourselves what an emf actually is. When we say that an emf $\mathcal{E}$ acts around the loop C in the positive direction, what we really mean is that a charge q which circulates once around the loop in the positive direction acquires the energy $q\,\mathcal{E}$. How does the charge acquire this energy? Clearly, either an electric field or a magnetic field, or some combination of the two, must perform the work $q\,\mathcal{E}$ on the charge as it circulates around the loop. However, we have already seen, from Sect. 8.4, that a magnetic field is unable to do work on a charged particle. Thus, the charge must acquire the energy $q\,\mathcal{E}$ from an *electric* field as it circulates once around the loop in the positive direction.

According to Sect. 5, the work that the electric field does on the charge as it goes around the loop is

$$W = q \oint_C \mathbf{E}\cdot d\mathbf{r}, \tag{9.9}$$

where $d\mathbf{r}$ is a line element of the loop. Hence, by energy conservation, we can write $W = q\,\mathcal{E}$, or

$$\mathcal{E} = \oint_C \mathbf{E}\cdot d\mathbf{r}. \tag{9.10}$$

The term on the right-hand side of the above expression can be recognized as the *line integral* of the electric field around loop C in the positive direction. Thus, the emf generated around the circuit C in the positive direction is equal to the line integral of the electric field around the circuit in the same direction.

Equations (9.8) and (9.10) can be combined to give

$$\oint_C \mathbf{E}\cdot d\mathbf{r} = -\frac{d\Phi_B}{dt}. \tag{9.11}$$

Thus, Faraday's law implies that the line integral of the electric field around circuit C (in the positive direction) is equal to minus the time rate of change of the magnetic flux linking this circuit. Does this law just apply to conducting circuits, or can we apply it to an arbitrary closed loop in space? Well, the difference between a conducting circuit and an arbitrary closed loop is that electric current can flow around a circuit, whereas current cannot, in general, flow around an arbitrary loop. In fact, the emf $\mathcal{E}$ induced around a conducting circuit drives a current $I = \mathcal{E}/R$ around that circuit, where R is the resistance of the circuit. However, we can make this resistance arbitrarily large without invalidating Eq. (9.11). In the limit in which R tends to infinity, no current flows around the circuit, so the circuit becomes indistinguishable from an arbitrary loop. Since we can place such a circuit anywhere in space, and Eq. (9.11) still holds, we are forced to the conclusion that Eq. (9.11) is valid for *any* closed loop in space, and not just for conducting circuits.

Equation (9.11) describes how a time-varying magnetic field *generates* an electric field which fills space. The strength of the electric field is directly proportional to the rate of change of the magnetic field. The field-lines associated with this electric field form loops in the plane perpendicular to the magnetic field. If the magnetic field is increasing then the electric field-lines circulate in the opposite sense to the fingers of a right-hand, when the thumb points in the direction of the field. If the magnetic field is decreasing then the electric field-lines circulate in the same sense as the fingers of a right-hand, when the thumb points in the direction of the field. This is illustrated in Fig. 9.2.

We can now appreciate that when a conducting circuit is placed in a time-varying magnetic field, it is the electric field induced by the changing magnetic field which gives rise to the emf around the circuit. If the loop has a finite resistance then this electric field also drives a current around the circuit. Note, however, that the electric field is generated irrespective of the presence of a conducting circuit. The electric field generated by a time-varying magnetic field is quite different in nature to that generated by a set of stationary electric charges. In the latter case, the electric field-lines begin on positive charges, end on negative charges, and *never* form closed loops in free space. In the former case, the electric field-lines *never* begin or end, and *always* form closed loops in free space. In fact, the electric field-lines generated by magnetic induction behave in much the same manner as magnetic field-lines. Recall, from Sect. 5.1, that an elec-

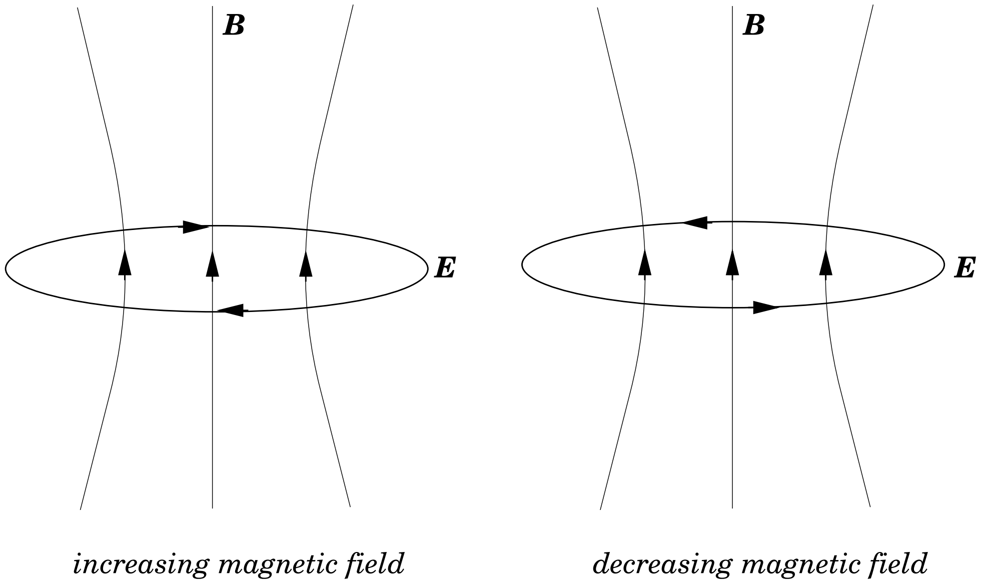


Figure 9.2: *Inductively generated electric fields*

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tric field generated by fixed charges is unable to do net work on a charge which circulates in a closed loop. On the other hand, an electric field generated by magnetic induction certainly can do work on a charge which circulates in a closed loop. This is basically how a current is induced in a conducting loop placed in a time-varying magnetic field. One consequence of this fact is that the work done in slowly moving a charge between two points in an inductive electric field *does* depend on the path taken between the two points. It follows that we cannot calculate a *unique* potential difference between two points in an inductive electric field. In fact, the whole idea of electric potential breaks down in a such a field (fortunately, there is a way of salvaging the idea of electric potential in an inductive field, but this topic lies beyond the scope of this course). Note, however, that it is still possible to calculate a *unique* value for the emf generated around a conducting circuit by an inductive electric field, because, in this case, the path taken by electric charges is uniquely specified: *i.e.*, the charges have to follow the circuit.

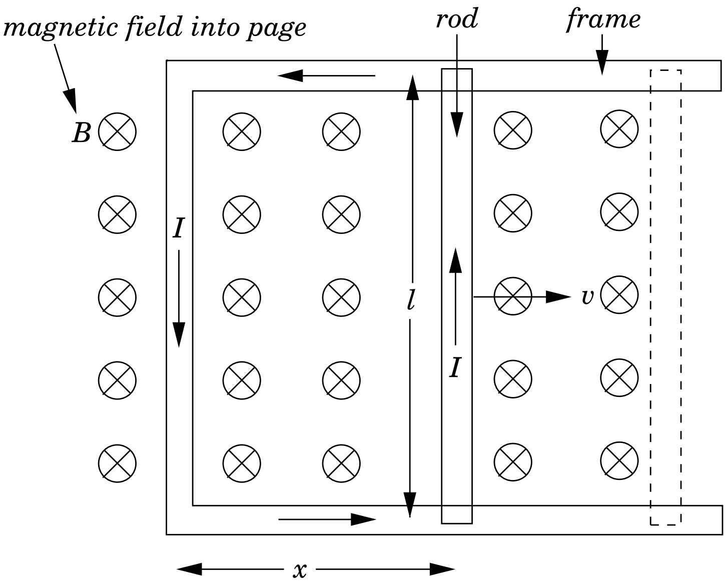

Figure 9.3: *Motional emf.*

9.4 Motional Emf

We now understand how an emf is generated around a *fixed* circuit placed in a time-varying magnetic field. But, according to Faraday's law, an emf is also generated around a *moving* circuit placed in a magnetic field which does not vary in time. According to Eq. (9.11), no space-filling inductive electric field is generated in the latter case, since the magnetic field is steady. So, how do we account for the emf in the latter case?

In order to help answer this question, let us consider a simple circuit in which a conducting rod of length l slides along a U-shaped conducting frame in the presence of a uniform magnetic field. This circuit is illustrated in Fig. 9.3. Suppose, for the sake of simplicity, that the magnetic field is directed perpendicular to the plane of the circuit. To be more exact, the magnetic field is directed into the page in the figure. Suppose, further, that we move the rod to the right with the constant velocity v.

The magnetic flux linked by the circuit is simply the product of the perpendicular magnetic field-strength, B, and the area of the circuit, $l\,x$, where x determines the position of the sliding rod. Thus,

$$\Phi_B = B\,l\,x. \tag{9.12}$$

Now, the rod moves a distance $dx = v\,dt$ in a time interval dt, so in the same time interval the magnetic flux linking the circuit *increases* by

$$d\Phi_B = B\,l\,dx = B\,l\,v\,dt. \tag{9.13}$$

It follows, from Faraday's law, that the magnitude of the emf $\mathcal{E}$ generated around the circuit is given by

$$\mathcal{E} = \frac{d\Phi_B}{dt} = B\,l\,v. \tag{9.14}$$

Thus, the emf generated in the circuit by the moving rod is simply the product of the magnetic field-strength, the length of the rod, and the velocity of the rod. If the magnetic field is not perpendicular to the circuit, but instead subtends an angle θ with respect to the normal direction to the plane of the circuit, then it is easily demonstrated that the *motional emf* generated in the circuit by the moving rod is

$$\mathcal{E} = B_\perp\,l\,v, \tag{9.15}$$

where $B_\perp = B\,\cos\theta$ is the component of the magnetic field which is perpendicular to the plane of the circuit.

Since the magnetic flux linking the circuit *increases* in time, the emf acts in the *negative* direction (*i.e.*, in the opposite sense to the fingers of a right-hand, if the thumb points along the direction of the magnetic field). The emf, $\mathcal{E}$, therefore, acts in the *anti-clockwise* direction in the figure. If R is the total resistance of the circuit, then this emf drives an anti-clockwise electric current of magnitude $I = \mathcal{E}/R$ around the circuit.

But, where does the emf come from? Let us again remind ourselves what an emf is. When we say that an emf $\mathcal{E}$ acts around the circuit in the anti-clockwise direction, what we really mean is that a charge q which circulates once around the circuit in the anti-clockwise direction acquires the energy $q\,\mathcal{E}$. The only way in which the charge can acquire this energy is if something does *work* on it as it circulates. Let us assume that the charge circulates very *slowly*. The magnetic field exerts a negligibly small force on the charge when it is traversing the non-moving part of the circuit (since the charge is moving very slowly). However, when the charge is traversing the moving rod it experiences an *upward* (in the figure) magnetic force of magnitude $f = q\,v\,B$ (assuming that $q > 0$). The net work done on the charge by this force as it traverses the rod is

$$W' = q\,v\,B\,l = q\,\mathcal{E}, \tag{9.16}$$

since $\mathcal{E} = B\,l\,v$. Thus, it would appear that the motional emf generated around the circuit can be accounted for in terms of the magnetic force exerted on charges traversing the moving rod.

But, if we think carefully, we can see that there is something seriously wrong with the above explanation. We seem to be saying that the charge acquires the energy $q\,\mathcal{E}$ from the *magnetic field* as it moves around the circuit once in the anti-clockwise direction. But, this is impossible, because a magnetic field *cannot* do work on an electric charge.

Let us look at the problem from the point of view of a charge q traversing the moving rod. In the frame of reference of the rod, the charge only moves very slowly, so the magnetic force on it is negligible. In fact, only an electric field can exert a significant force on a slowly moving charge. In order to account for the motional emf generated around the circuit, we need the charge to experience an upward force of magnitude $q\,v\,B$. The only way in which this is possible is if the charge sees an upward pointing *electric field* of magnitude

$$E_0 = v\,B. \tag{9.17}$$

In other words, although there is no electric field in the laboratory frame, there is an electric field in the frame of reference of the moving rod, and it is this field which does the necessary amount of work on charges moving around the circuit to account for the existence of the motional emf, $\mathcal{E} = B\,l\,v$.

More generally, if a conductor moves in the laboratory frame with velocity $\mathbf{v}$ in the presence of a magnetic field $\mathbf{B}$ then a charge q inside the conductor experiences a magnetic force $\mathbf{f} = q\,\mathbf{v}\times\mathbf{B}$. In the frame of the conductor, in which the charge is essentially stationary, the same force takes the form of an electric force $\mathbf{f} = q\,\mathbf{E}_0$, where $\mathbf{E}_0$ is the electric field in the frame of reference of the conductor. Thus, if a conductor moves with velocity $\mathbf{v}$ through a magnetic field $\mathbf{B}$ then the electric field $\mathbf{E}_0$ which appears in the rest frame of the conductor is given by

$$\mathbf{E}_0 = \mathbf{v} \times \mathbf{B}. \tag{9.18}$$

This electric field is the ultimate origin of the motional emfs which are generated whenever circuits move with respect to magnetic fields.

We can now appreciate that Faraday's law is due to a combination of two apparently distinct effects. The first is the space-filling electric field generated by a changing magnetic field. The second is the electric field generated inside a conductor when it moves through a magnetic field. In reality, these effects are two aspects of the same basic phenomenon, which explains why no real distinction is made between them in Faraday's law.

9.5 Eddy Currents

We have seen, in the above example, that when a conductor is moved in a magnetic field a motional emf is generated. Moreover, according to Worked Example 9.3, this emf drives a current which heats the conductor, and, when combined with the magnetic field, also gives rise to a magnetic force acting on the conductor which opposes its motion. In turns out that these results are quite general. Incidentally, the induced currents which circulate inside a moving conductor in a static magnetic field, or a stationary conductor in a time-varying magnetic field, are usually called *eddy currents*.

Consider a metal disk which rotates in a perpendicular magnetic field which only extends over a small rectangular portion of the disk, as shown in Fig. 9.4. Such a field could be produced by the pole of a horseshoe magnetic. The motional emf induced in the disk, as it moves through the field-containing region, acts in the direction $\mathbf{v} \times \mathbf{B}$, where $\mathbf{v}$ is the velocity of the disk, and $\mathbf{B}$ the magnetic field. It follows from Fig. 9.4 that the emf acts downward. The emf drives currents which are also directed downward. However, these currents must form closed loops, and, hence, they are directed upward in those regions of the disk immediately

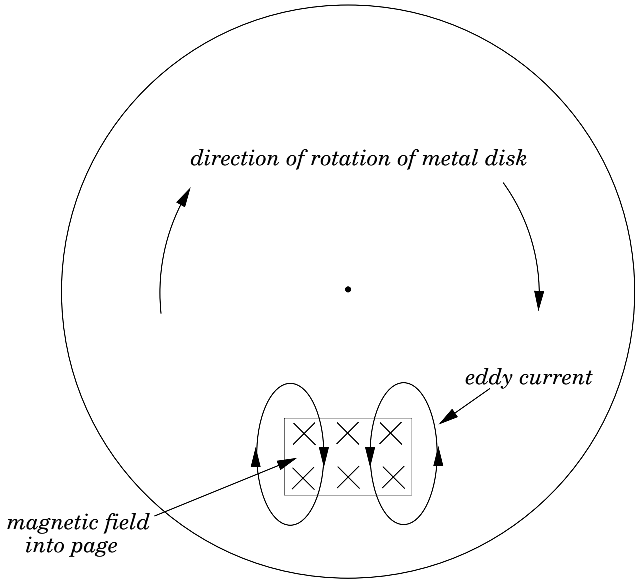

Figure 9.4: *Eddy currents*

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adjacent to the field-containing region, as shown in the figure. It can be seen that the induced currents flow in little eddies. Hence, the name “eddy currents.” According to the right-hand rule, the downward currents in the field-containing region give rise to a magnetic force on the disk which acts to the right. In other words, the magnetic force acts to prevent the rotation of the disk. Clearly, external work must be done on the disk in order to keep it rotating at a constant angular velocity. This external work is ultimately dissipated as heat by the eddy currents circulating inside the disk.

Eddy currents can be very useful. For instance, some cookers work by using eddy currents. The cooking pots, which are usually made out of aluminium, are placed on plates which generate oscillating magnetic fields. These fields induce eddy currents in the pots which heat them up. The heat is then transmitted to the food inside the pots. This type of cooker is particularly useful for food which needs to be cooked gradually over a long period of time: *i.e.*, over many hours, or even days. Eddy currents can also be used to heat small pieces of metal until they become white-hot by placing them in a very rapidly oscillating magnetic field. This technique is sometimes used in brazing. Heating conductors by means of eddy currents is called *inductive heating*. Eddy currents can also be used to damp

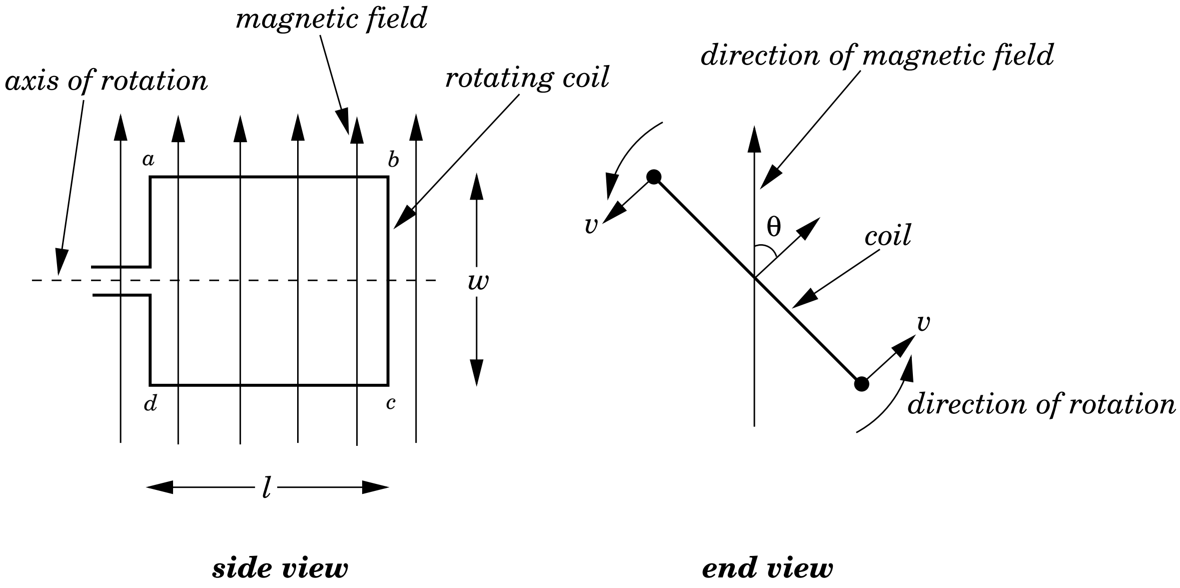


Figure 9.5: *An alternating current generator.*

motion. This technique, which is called *eddy current damping*, is often employed in galvanometers.

9.6 The Alternating Current Generator

An electric generator, or dynamo, is a device which converts mechanical energy into electrical energy. The simplest practical generator consists of a rectangular coil rotating in a uniform magnetic field. The magnetic field is usually supplied by a permanent magnet. This setup is illustrated in Fig. 9.5.

Let l be the length of the coil along its axis of rotation, and w the width of the coil perpendicular to this axis. Suppose that the coil rotates with constant angular velocity ω in a uniform magnetic field of strength B. The velocity v with which the the two long sides of the coil (*i.e.*, sides ab and cd) move through the magnetic field is simply the product of the angular velocity of rotation ω and the distance $w/2$ of each side from the axis of rotation, so $v = \omega\, w/2$. The motional emf induced in each side is given by $\mathcal{E} = B_\perp\, l\, v$, where $B_\perp$ is the component of the magnetic field perpendicular to instantaneous direction of motion of the side in question. If the direction of the magnetic field subtends an angle θ with the normal direction to the coil, as shown in the figure, then $B_\perp = B\,\sin\theta$. Thus, the magnitude of the motional emf generated in sides ab and cd is

$$\mathcal{E}_{ab} = \frac{B\,w\,l\,\omega\,\sin\theta}{2} = \frac{B\,A\,\omega\,\sin\theta}{2}, \tag{9.19}$$

where $A = w\,l$ is the area of the coil. The emf is zero when $\theta = 0^\circ$ or 180°, since the direction of motion of sides ab and cd is *parallel* to the direction of the magnetic field in these cases. The emf attains its maximum value when $\theta = 90^\circ$ or 270°, since the direction of motion of sides ab and cd is *perpendicular* to the direction of the magnetic field in these cases. Incidentally, it is clear, from symmetry, that no net motional emf is generated in sides bc and da of the coil.

Suppose that the direction of rotation of the coil is such that side ab is moving into the page in Fig. 9.5 (side view), whereas side cd is moving out of the page. The motional emf induced in side ab acts from a to b. Likewise, the motional emf induce in side cd acts from c to d. It can be seen that both emfs act in the clockwise direction around the coil. Thus, the net emf $\mathcal{E}$ acting around the coil is $2\,\mathcal{E}_{ab}$. If the coil has N turns then the net emf becomes $2\,N\,\mathcal{E}_{ab}$. Thus, the general expression for the emf generated around a steadily rotating, multi-turn coil in a uniform magnetic field is

$$\mathcal{E} = N\,B\,A\,\omega\,\sin(\omega\,t), \tag{9.20}$$

where we have written $\theta = \omega\,t$ for a steadily rotating coil (assuming that $\theta = 0$ at $t = 0$). This expression can also be written

$$\mathcal{E} = \mathcal{E}_{\rm max}\,\sin(2\pi\,f\,t), \tag{9.21}$$

where

$$\mathcal{E}_{\rm max} = 2\pi\,N\,B\,A\,f \tag{9.22}$$

is the peak emf produced by the generator, and $f = \omega/2\pi$ is the number of complete rotations the coils executes per second. Thus, the peak emf is directly proportional to the area of the coil, the number of turns in the coil, the rotation frequency of the coil, and the magnetic field-strength.

Figure 9.6 shows the emf specified in Eq. (9.21) plotted as a function of time. It can be seen that the variation of the emf with time is *sinusoidal* in nature. The

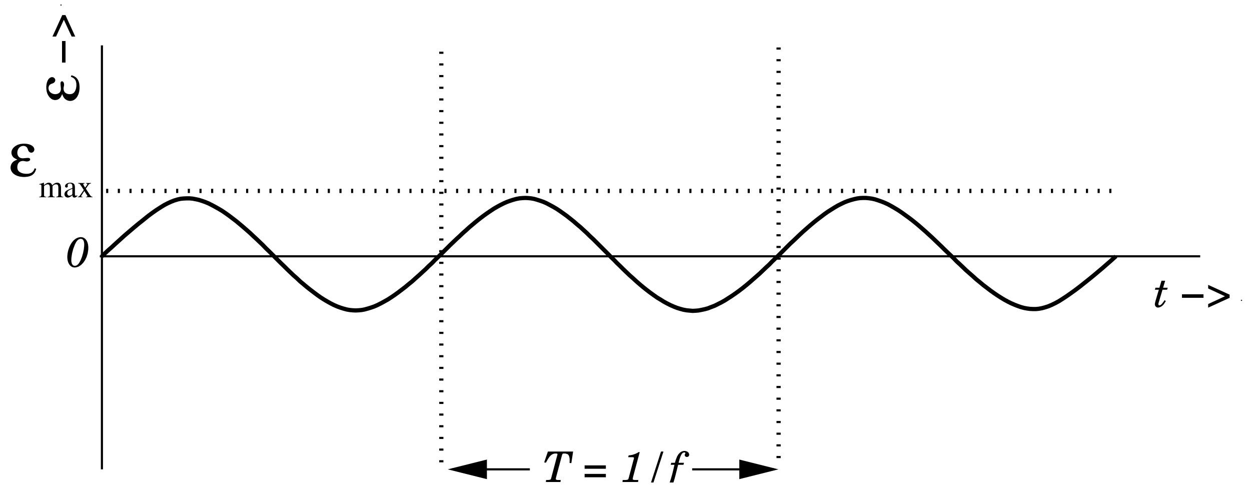


Figure 9.6: *Emf generated by a steadily rotating AC generator.*

emf attains its peak values when the plane of the coil is parallel to the plane of the magnetic field, passes through zero when the plane of the coil is perpendicular to the magnetic field, and reverses sign every half period of revolution of the coil. The emf is periodic (*i.e.*, it continually repeats the same pattern in time), with period $T = 1/f$ (which is, of course, the rotation period of the coil).

Suppose that some load (*e.g.*, a light-bulb, or an electric heating element) of resistance R is connected across the terminals of the generator. In practice, this is achieved by connecting the two ends of the coil to rotating rings which are then connected to the external circuit by means of metal brushes. According to Ohm's law, the current I which flows in the load is given by

$$I = \frac{\mathcal{E}}{R} = \frac{\mathcal{E}_{\rm max}}{R}\,\sin(2\pi\,f\,t). \tag{9.23}$$

Note that this current is constantly changing direction, just like the emf of the generator. Hence, the type of generator described above is usually termed an *alternating current*, or AC, generator.

The current I which flows through the load must also flow around the coil. Since the coil is situated in a magnetic field, this current gives rise to a torque on the coil which, as is easily demonstrated, acts to slow down its rotation. According to Sect. 8.11, the braking torque τ acting on the coil is given by

$$\tau = N\,I\,B_{\parallel}\,A, \tag{9.24}$$

where $B_{\parallel} = B\,\sin\theta$ is the component of the magnetic field which lies in the plane of the coil. It follows from Eq. (9.20) that

$$\tau = \frac{\mathcal{E}\,I}{\omega}, \tag{9.25}$$

since $\mathcal{E} = N\,B_{\parallel}\,A\,\omega$. An external torque which is equal and opposite to the breaking torque must be applied to the coil if it is to rotate *uniformly*, as assumed above. The rate P at which this external torque does work is equal to the product of the torque τ and the angular velocity ω of the coil. Thus,

$$P = \tau\,\omega = \mathcal{E}\,I. \tag{9.26}$$

Not surprisingly, the rate at which the external torque performs works exactly matches the rate $\mathcal{E}\,I$ at which electrical energy is generated in the circuit comprising the rotating coil and the load.

Equations (9.20), (9.23), and (9.25) yield

$$\tau = \tau_{\rm max}\,\sin^2(2\pi\,f\,t), \tag{9.27}$$

where $\tau_{\rm max} = (\mathcal{E}_{\rm max})^2/(2\pi\,f\,R)$. Figure 9.7 shows the breaking torque τ plotted as a function of time t, according to Eq. (9.27). It can be seen that the torque is always of the same sign (*i.e.*, it always acts in the same direction, so as to continually oppose the rotation of the coil), but is not constant in time. Instead, it *pulsates* periodically with period T. The breaking torque attains its maximum value whenever the plane of the coil is parallel to the plane of the magnetic field, and is zero whenever the plane of the coil is perpendicular to the magnetic field. It is clear that the external torque needed to keep the coil rotating at a constant angular velocity must also pulsate in time with period T. A constant external torque would give rise to a non-uniformly rotating coil, and, hence, to an alternating emf which varies with time in a more complicated manner than $\sin(2\pi\,f\,t)$.

Virtually all commercial power stations generate electricity using AC generators. The external power needed to turn the generating coil is usually supplied by a steam turbine (steam blasting against fan-like blades which are forced into rotation). Water is vaporized to produce high pressure steam by burning coal, or

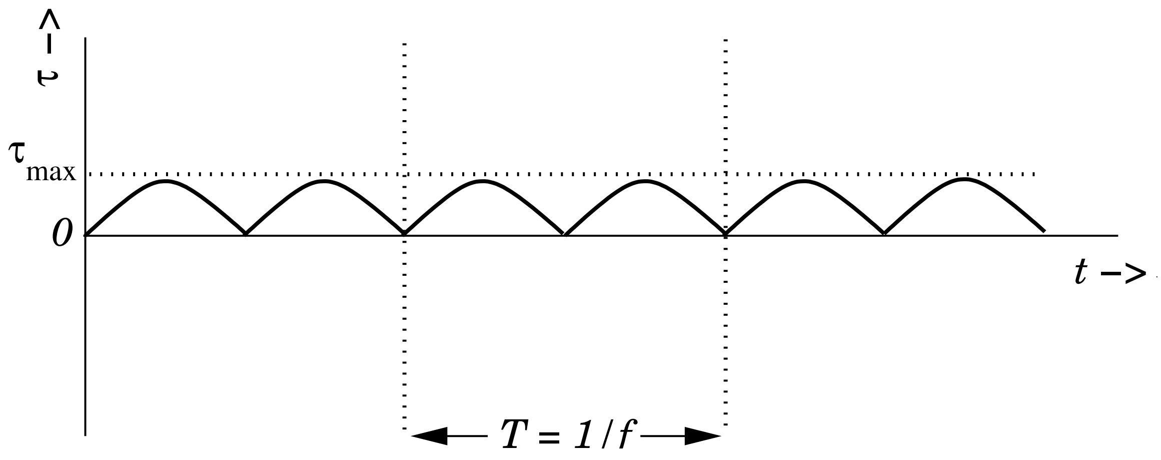


Figure 9.7: *The braking torque in a steadily rotating AC generator.*

by using the energy released inside a nuclear reactor. Of course, in hydroelectric power stations, the power needed to turn the generator coil is supplied by a water turbine (which is similar to a steam turbine, except that falling water plays the role of the steam). Recently, a new type of power station has been developed in which the power needed to rotate the generating coil is supplied by a gas turbine (basically, a large jet engine which burns natural gas). In the United States and Canada, the alternating emf generated by power stations oscillates at $f = 60\,\text{Hz}$, which means that the generator coils in power stations rotate exactly sixty times a second. In Europe, and much of the rest of the world, the oscillation frequency of commercially generated electricity is $f = 50\,\text{Hz}$.

9.7 The Direct Current Generator

Most common electrical appliances (*e.g.*, electric light-bulbs, and electric heating elements) work fine on AC electrical power. However, there are some situations in which DC power is preferable. For instance, small electric motors (*e.g.*, those which power food mixers and vacuum cleaners) work very well on AC electricity, but very large electric motors (*e.g.*, those which power subway trains) generally work much better on DC electricity. Let us investigate how DC electricity can be generated.

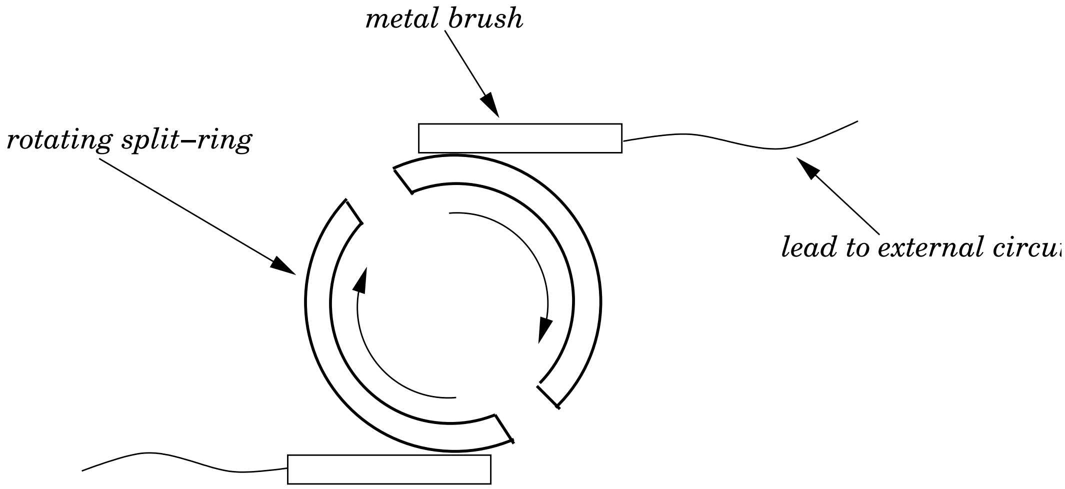


Figure 9.8: *A split-ring commutator.*

A simple DC generator consists of the same basic elements as a simple AC generator: *i.e.*, a multi-turn coil rotating uniformly in a magnetic field. The main difference between a DC generator and an AC generator lies in the manner in which the rotating coil is connected to the external circuit containing the load. In an AC generator, both ends of the coil are connected to separate slip-rings which co-rotate with the coil, and are connected to the external circuit via wire brushes. In this manner, the emf $\mathcal{E}_{\rm ext}$ seen by the external circuit is always the same as the emf $\mathcal{E}$ generated around the rotating coil. In a DC generator, the two ends of the coil are attached to different halves of a single split-ring which co-rotates with the coil. The split-ring is connected to the external circuit by means of metal brushes—see Fig. 9.8. This combination of a rotating split-ring and stationary metal brushes is called a *commutator*. The purpose of the commutator is to ensure that the emf $\mathcal{E}_{\rm ext}$ seen by the external circuit is equal to the emf $\mathcal{E}$ generated around the rotating coil for *half* the rotation period, but is equal to minus this emf for the other half (since the connection between the external circuit and the rotating coil is reversed by the commutator every half-period of rotation). The positions of the metal brushes can be adjusted such that the connection between the rotating coil and the external circuit reverses whenever the emf $\mathcal{E}$ generated around the coil goes through zero. In this special case, the emf seen in the external circuit is simply

$$\mathcal{E}_{\rm ext} = |\mathcal{E}| = \mathcal{E}_{\rm max}\,|\sin(2\pi\,f\,t)|. \tag{9.28}$$

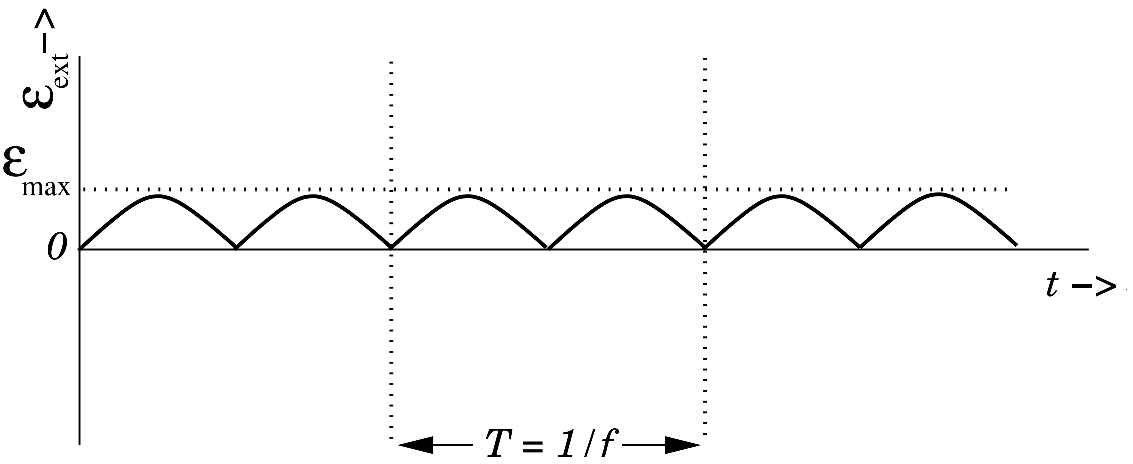


Figure 9.9: *Emf generated in a steadily rotating DC generator.*

Figure 9.9 shows $\mathcal{E}_{\rm ext}$ plotted as a function of time, according to the above formula. The variation of the emf with time is very similar to that of an AC generator, except that whenever the AC generator would produce a negative emf the commutator in the DC generator reverses the polarity of the coil with respect to the external circuit, so that the negative half of the AC signal is reversed and made positive. The result is a bumpy direct emf which rises and falls but never changes direction. This type of pulsating emf can be smoothed out by using more than one coil rotating about the same axis, or by other electrical techniques, to give a good imitation of the direct current delivered by a battery. The *alternator* in a car (*i.e.*, the DC generator which recharges the battery) is a common example of a DC generator of the type discussed above. Of course, in an alternator, the external torque needed to rotate the coil is provided by the engine of the car.

9.8 The Alternating Current Motor

The first electric dynamo was constructed in 1831 by Michael Faraday. An electric dynamo is, of course, a device which transforms mechanical energy into electrical energy. An electric motor, on the other hand, is a device which transforms electrical energy into mechanical energy. In other words, an electric motor is an electric dynamo run in *reverse*. It took a surprisingly long time for scientists in the nineteenth century to realize this. In fact, the message only really sank home after a fortuitous accident during the 1873 Vienna World Exposition. A large hall was filled with modern gadgets. One of these gadgets, a steam engine driven dynamo, was producing electric power when a workman unwittingly connected the output leads from another dynamo to the energized circuit. Almost immediately, the latter dynamo started to whirl around at great speed. The dynamo was, in effect, transformed into an electric motor.

An AC electric motor consists of the same basic elements as an AC electric generator: *i.e.*, a multi-turn coil which is free to rotate in a constant magnetic field. Furthermore, the rotating coil is connected to the external circuit in just the same manner as in an AC generator: *i.e.*, via two slip-rings attached to metal brushes. Suppose that an external voltage source of emf V is connected across the motor. It is assumed that V is an *alternating* emf, so that

$$V = V_{\rm max}\,\sin(2\pi\,f\,t), \tag{9.29}$$

where $V_{\rm max}$ is the peak voltage, and f is the alternation frequency. Such an emf could be obtained from an AC generator, or, more simply, from the domestic mains supply. For the case of the mains, $V_{\rm max} = 110\,{\rm V}$ and $f = 60\,{\rm Hz}$ in the U.S. and Canada, whereas $V_{\rm max} = 220\,{\rm V}$ and $f = 50\,{\rm Hz}$ in Europe and Asia. The external emf drives an alternating current

$$I = I_{\rm max}\,\sin(2\pi\,f\,t) \tag{9.30}$$

around the external circuit, and through the motor. As this current flows around the coil, the magnetic field exerts a torque on the coil, which causes it to rotate. The motor eventually attains a steady-state in which the rotation frequency of the coil matches the alternation frequency of the external emf. In other words, the steady-state rotation frequency of the coil is f. Now a coil rotating in a magnetic field generates an emf $\mathcal{E}$. It is easily demonstrated that this emf acts to *oppose* the circulation of the current around the coil: *i.e.*, the induced emf acts in the opposite direction to the external emf. For an N-turn coil of cross-sectional area A, rotating with frequency f in a magnetic field B, the back-emf $\mathcal{E}$ is given by

$$\mathcal{E} = \mathcal{E}_{\rm max}\,\sin(2\pi\,f\,t), \tag{9.31}$$

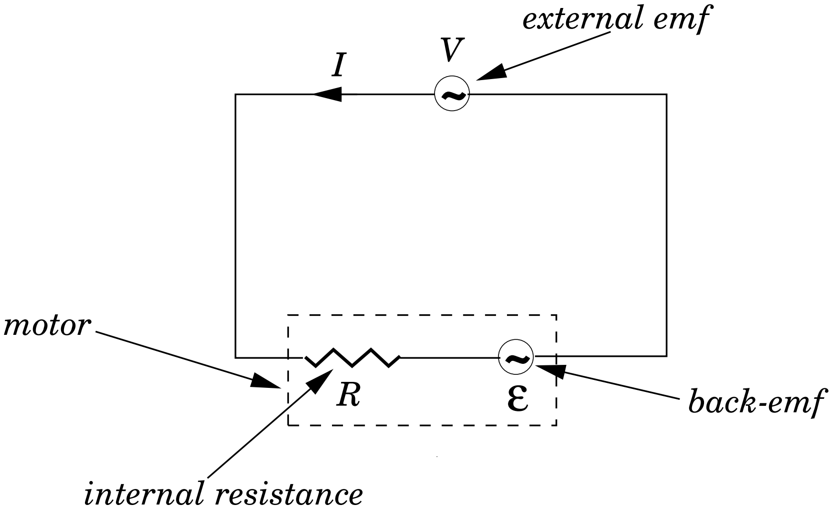


Figure 9.10: *Circuit diagram for an AC motor connected to an external AC emf source.*

where

$$\mathcal{E}_{\rm max} = 2\pi\,N\,B\,A\,f, \tag{9.32}$$

and use has been made of the results of Sect. 9.6.

Figure 9.10 shows the circuit in question. A circle with a wavy line inside is the conventional way of indicating an AC voltage source. The motor is modeled as a resistor R, which represents the internal resistance of the motor, in series with the back-emf $\mathcal{E}$. Of course, the back-emf acts in the opposite direction to the external emf V. Application of Ohm's law around the circuit gives

$$V = I\,R + \mathcal{E}, \tag{9.33}$$

or

$$V_{\rm max}\,\sin(2\pi\,f\,t) = I_{\rm max}\,R\,\sin(2\pi\,f\,t) + \mathcal{E}_{\rm max}\,\sin(2\pi\,f\,t), \tag{9.34}$$

which reduces to

$$V_{\rm max} = I_{\rm max}\,R + \mathcal{E}_{\rm max}. \tag{9.35}$$

The rate P at which the motor *gains* electrical energy from the external circuit is given by

$$P = \mathcal{E}\,I = P_{\rm max}\,\sin^2(2\pi\,f\,t), \tag{9.36}$$

where

$$P_{\rm max} = \mathcal{E}_{\rm max}\, I_{\rm max} = \frac{\mathcal{E}_{\rm max}\,(V_{\rm max}-\mathcal{E}_{\rm max})}{R}. \tag{9.37}$$

By conservation of energy, P is also the rate at which the motor performs mechanical work. Note that the rate at which the motor does mechanical work is not constant in time, but, instead, pulsates at the rotation frequency of the coil. It is possible to construct a motor which performs work at a more uniform rate by employing more than one coil rotating about the same axis.

As long as $V_{\rm max} > \mathcal{E}_{\rm max}$, the rate at which the motor performs mechanical work is positive (*i.e.*, the motor does useful work). However, if $V_{\rm max} < \mathcal{E}_{\rm max}$ then the rate at which the motor performs work becomes negative. This means that we must do mechanical work on the motor in order to keep it rotating, which is another way of saying that the motor does not do useful work. Clearly, in order for an AC motor to do useful work, the external emf V must be able to overcome the back-emf $\mathcal{E}$ induced in the motor (*i.e.*, $V_{\rm max} > \mathcal{E}_{\rm max}$).

9.9 The Direct Current Motor

In steady-state, an AC motor always rotates at the alternation frequency of its power supply. Thus, an AC motor powered by the domestic mains supply rotates at 60 Hz in the U.S. and Canada, and at 50 Hz in Europe and Asia. Suppose, however, that we require a *variable speed* electric motor. We could always use an AC motor driven by a variable frequency AC power supply, but such power supplies are very expensive. A far cheaper alternative is to use a DC motor driven by a DC power supply. Let us investigate DC motors.

A DC motor consists of the same basic elements as a DC electric generator: *i.e.*, a multi-turn coil which is free to rotate in a constant magnetic field. Furthermore, the rotating coil is connected to the external circuit in just the same manner as in a DC generator: *i.e.*, via a split-ring commutator which reverses the polarity of the coil with respect to the external circuit whenever the coil passes through the plane perpendicular to the direction of the magnetic field. Suppose that an external DC voltage source (*e.g.*, a battery, or a DC generator) of emf V is connected across the

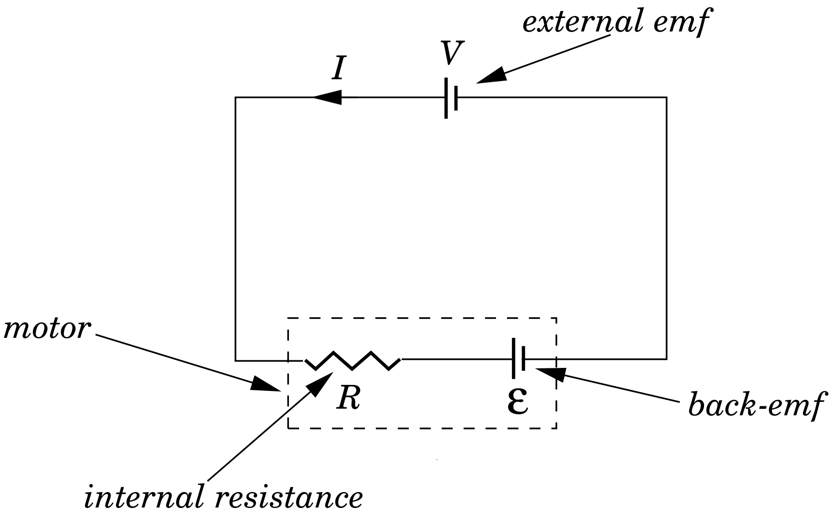


Figure 9.11: *Circuit diagram for an DC motor connected to an external DC emf source.*

motor. The voltage source drives a steady current I around the external circuit, and through the motor. As the current flows around the coil, the magnetic field exerts a torque on the coil, which causes it to rotate. Let us suppose that the motor eventually attains a steady-state rotation frequency f. The rotating coil generates a back-emf $\mathcal{E}$ whose magnitude is directly proportional to the frequency of rotation [see Eq. (9.32)].

Figure 9.11 shows the circuit in question. The motor is modeled as a resistor R, which represents the internal resistance of the motor, in series with the back-emf $\mathcal{E}$. Of course, the back-emf acts in the opposite direction to the external emf V. Application of Ohm's law around the circuit gives

$$V = I\,R + \mathcal{E}, \tag{9.38}$$

which yields

$$I = \frac{V - \mathcal{E}}{R}. \tag{9.39}$$

The rate at which the motor performs mechanical work is

$$P = \mathcal{E}\,I = \frac{\mathcal{E}\,(V - \mathcal{E})}{R}. \tag{9.40}$$

Suppose that a DC motor is subject to a light external load, so that it only has to perform mechanical work at a relatively low rate. In this case, the motor will spin up until its back-emf $\mathcal{E}$ is slightly less than the external emf V, so that very little current flows through the motor [according to Eq. (9.39)], and, hence, the mechanical power output of the motor is relatively low [according to Eq. (9.40)]. If the load on the motor is increased then the motor will slow down, so that its back-emf is reduced, the current flowing through the motor is increased, and, hence, the mechanical power output of the motor is raised until it matches the new load. Note that the current flowing through a DC motor is generally limited by the back-emf, rather than the internal resistance of the motor. In fact, conventional DC motors are designed on the assumption that the back-emf will always limit the current flowing through the motor to a relatively small value. If the motor jams, so that the coil stops rotating and the back-emf falls to zero, then the current $I = V/R$ which flows through the motor is generally so large that it will burn out the motor if allowed to flow for any appreciable length of time. For this reason, the power to an electric motor should always be shut off immediately if the motor jams. When a DC motor is started up, the coil does not initially spin fast enough to generate a substantial back-emf. Thus, there is a short time period, just after the motor is switched on, in which the motor draws a relatively large current from its power supply. This explains why the lights in a house sometimes dim transiently when a large motor, such as an air conditioner motor, is switched on.

Suppose that a DC motor is subject to a constant, but relatively light, load. As mentioned above, the motor will spin up until its back emf almost matches the external emf. If the external emf is increased then the motor will spin up further, until its back-emf matches the new external emf. Likewise, if the external emf is decreased then the motor will spin down. It can be seen that the rotation rate of a DC motor is controlled by the emf of the DC power supply to which the motor is attached. The higher the emf, the higher the rate of rotation. Thus, it is relatively easy to vary the speed of a DC motor, unlike an AC motor, which is essentially a fixed speed motor.

9.10 Worked Examples

Example 9.1: Faraday's law

Question: A plane circular loop of conducting wire of radius $r = 10\,\mathrm{cm}$ which possesses $N = 15$ turns is placed in a uniform magnetic field. The direction of the magnetic field makes an angle of 30° with respect to the normal direction to the loop. The magnetic field-strength B is increased at a constant rate from $B_1 = 1\,\mathrm{T}$ to $B_2 = 5\,\mathrm{T}$ in a time interval of $\Delta t = 10\,\mathrm{s}$. What is the emf generated around the loop? If the electrical resistance of the loop is $R = 15\,\Omega$, what current flows around the loop as the magnetic field is increased?

Answer: The area of the loop is

$$A = \pi\, r^2 = \pi\,(0.1)^2 = 0.0314\,\mathrm{m}^2.$$

The component of the magnetic field perpendicular to the loop is

$$B_\perp = B\ \cos\theta = B\ \cos 30^\circ = 0.8660\,B,$$

where B is the magnetic field-strength. Thus, the initial magnetic flux linking the loop is

$$\Phi_{B\,1} = N\,A\,B_1\ \cos\theta = (15)\,(0.0314)\,(1)\,(0.8660) = 0.408\,\mathrm{Wb}.$$

Likewise, the final flux linking the loop is

$$\Phi_{B\,2} = N\,A\,B_2\ \cos\theta = (15)\,(0.0314)\,(5)\,(0.8660) = 2.039\,\mathrm{Wb}.$$

The time rate of change of the flux is

$$\frac{d\Phi_B}{dt} = \frac{\Phi_{B\,2} - \Phi_{B\,1}}{\Delta t} = \frac{(2.039 - 0.408)}{(10)} = 0.163\,\mathrm{Wb\,s^{-1}}.$$

Thus, the emf generated around the loop is

$$\mathcal{E} = \frac{d\Phi_B}{dt} = 0.163\,\mathrm{V}.$$

Note, incidentally, that one weber per second is equivalent to one volt.

According to Ohm's law, the current which flows around the loop in response to the emf is

$$I = \frac{\mathcal{E}}{R} = \frac{(0.163)}{(15)} = 0.011\,\mathrm{A}.$$

Worked Example 2: Lenz's law

Question: A long solenoid with an air core has $n_1 = 400$ turns per meter and a cross-sectional area of $A_1 = 10\,\mathrm{cm}^2$. The current I_1 flowing around the solenoid increases from 0 to 50 A in 2.0 s. A plane loop of wire consisting of $N_2 = 10$ turns, which is of cross-sectional area $A_2 = 100\,\mathrm{cm}^2$ and resistance $R_2 = 0.050\,\Omega$, is placed around the solenoid close to its centre. The loop is orientated such that it lies in the plane perpendicular to the axis of the solenoid. What is the magnitude $\mathcal{E}_2$ of the emf induced in the coil? What current I_2 does does this emf drive around the coil? Does this current circulate in the same direction as the current flowing in the solenoid, or in the opposite direction?

Answer: We must, first of all, calculate the magnetic flux linking the coil. The magnetic field is confined to the region inside the solenoid (the field generated outside a *long* solenoid is essentially negligible). The magnetic field runs along the axis of the solenoid, so it is directed perpendicular to the plane of the coil. Thus, the magnetic flux linking a single turn of the coil is the product of the area A_1 of the magnetic-field-containing region and the strength B of the perpendicular field. Note that, in this case, the magnetic flux does not depend on the area A_2 of the coil, as long as the magnetic-field-containing region lies completely within the coil. The magnetic flux Φ_B linking the whole coil is the flux linking a single turn times the number N_2 of turns in the coil. Thus,

$$\Phi_B = N_2\,A_1\,B.$$

Now, the magnitude of the magnetic field generated by the solenoid is given by (see Sect. 8.8)

$$B = \mu_0\,n_1\,I_1,$$

so the magnetic flux linking the coil can be written

$$\Phi_B = N_2\, A_1\, \mu_0\, n_1\, I_1.$$

This magnetic flux increases because the current I_1 flowing in the solenoid increases. Thus, the time rate of change of the magnetic flux is given by

$$\begin{aligned}\frac{d\Phi_B}{dt} = N_2\, A_1\, \mu_0\, n_1\, \frac{dI_1}{dt} &= (10)\,(10\times 10^{-4})\,(4\pi\times 10^{-7})\,(400)\,\frac{(50)}{(2)}\\ &= 1.26\times 10^{-4}\,\text{Wb s}^{-1}.\end{aligned}$$

By Faraday's induction law, the emf generated around the coil is

$$\mathcal{E}_2 = -\frac{d\Phi_B}{dt} = -1.26\times 10^{-4}\,\text{V}.$$

Ohm's law gives

$$I_2 = \frac{\mathcal{E}_2}{R_2} = \frac{(-1.26\times 10^{-4})}{(0.050)} = -2.6\,\text{mA},$$

as the current induced in the coil.

According to Lenz's law, the current induced in the coil is such as to oppose the increase in the magnetic flux linking the coil. Thus, the current in the coil must circulate in the *opposite* direction to the current in the solenoid, so that the magnetic field generated by the the former current, in the middle of the coil, is oppositely directed to that generated by the latter current. The fact that the current I_2 in the above formula is *negative* is indicative of the fact that this current runs in the opposite direction to the current flowing around the solenoid.

Worked Example 3: Motional emf

Question: Consider the circuit described in Sect. 9.4. Suppose that the length of the moving rod is $l = 0.2\,\text{m}$, its speed is $v = 0.1\,\text{m}\,\text{s}^{-1}$, the magnetic field-strength is $B = 1.0\,\text{T}$ (the field is directed into the page—see Fig. 9.3), and the resistance of the circuit is $R = 0.020\,\Omega$. What is the emf generated around the circuit? What current flows around the circuit? What is the magnitude and direction of the force acting on the moving rod due to the fact that a current is flowing along it? What is the rate at which work must be performed on the rod in order to keep it moving at constant velocity against this force? What is the rate at which electrical energy is generated? What is the rate at which energy is converted into heat due to the resistivity of the circuit?

Answer: The emf is generated by the motion of the rod. According to Eq. (9.14), the magnitude of the motional emf is

$$\mathcal{E} = B\,l\,v = (1)\,(0.2)\,(0.1) = 0.020\,\mathrm{V}.$$

The emf acts in the anti-clockwise direction in Fig. 9.3.

The anti-clockwise current driven around the circuit by the motional emf follows from Ohm's law:

$$I = \frac{\mathcal{E}}{R} = \frac{(0.020)}{(0.020)} = 1.0\,\mathrm{A}.$$

Since the rod carries a current I which flows perpendicular to a magnetic field B, the force per unit length acting on the rod is $F = I\,B$ (see Sect. 8.2). Thus, the total force acting on the rod is of magnitude

$$f = I\,B\,l = (1)\,(1)\,(0.2) = 0.20\,\mathrm{N}.$$

This force is directed parallel to the vector $\mathbf{I}\times\mathbf{B}$. It follows that the force is to the left in Fig. 9.3. In other words, the force *opposes* the motion producing the emf.

In order to keep the rod moving at a constant velocity, some external agent must apply a force to the rod which is equal and opposite to the magnetic force described above. Thus, the externally applied force acts to the right. The rate P at which work is done by this force is the product of the force and the velocity of the rod in the direction of this force. Thus,

$$P = f\,v = (0.20)\,(0.10) = 0.020\,\mathrm{W}.$$

Every charge q which circulates around the circuit in the anti-clockwise direction acquires the energy $q\,\mathcal{E}$. The amount of charge per unit time which circulates around the circuit is, by definition, equal to the current I flowing around the circuit. Thus, the rate at which electric charges acquire energy in the circuit is

$$P = \mathcal{E}\, I = (0.020)\,(1) = 0.020\,\mathrm{W}.$$

Now, the rate at which electric charges acquire energy in the circuit is equal to the rate at which mechanical work is done on the rod by the external force, as must be the case if energy is to be conserved. Thus, we can think of this circuit as constituting a primitive generator which transforms mechanical into electrical energy.

The rate at which electrical energy is converted into heat energy in the circuit is

$$P = I^2\, R = (1)\,(1)\,(0.020) = 0.020\,\mathrm{W}.$$

Thus, all of the mechanical work done on the rod eventually ends up as heat dissipated in the circuit.

Worked Example 4: AC generators

Question: A simple AC generator consists of an $N = 10$ turn coil of area $A = 1200\,\mathrm{cm}^2$ which rotates at a constant frequency of $f = 60\,\mathrm{Hz}$ in a $B = 0.40\,\mathrm{T}$ magnetic field. What is the peak emf of the device?

Answer: The peak emf $\mathcal{E}_{\rm max}$ is given by [see Eq. (9.22)]

$$\mathcal{E}_{\rm max} = 2\pi\, N\, B\, A\, f = (6.283)\,(10)\,(0.40)\,(0.12)\,(60) = 181\,\mathrm{V}.$$

Worked Example 5: AC motors

Question: An AC motor has an internal resistance of $R = 4.0\,\Omega$. When powered by a 50 Hz AC supply of peak voltage $V = 120\,\mathrm{V}$ it draws a peak current of $I = 5.0\,\mathrm{A}$. What is the peak back-emf produced by the motor? What is the peak power delivered to the motor by the AC supply? What is the peak rate of energy loss as heat in the motor? What is the peak useful power produced by the motor? What is the efficiency (*i.e.*, the ratio of the peak useful power output to the peak power delivered) of such a motor?

Answer: If V is the peak applied voltage, and $\mathcal{E}$ the peak back-emf, then the peak applied voltage must equal the sum of the peak voltage drops across the motor, or $V = \mathcal{E} + I\,R$. It follows that

$$\mathcal{E} = V - I\,R = (120) - (5.0)\,(4.0) = 100\,\text{V}.$$

The peak power delivered by the AC supply is

$$P_1 = V\,I = (120)\,(5.0) = 600\,\text{W}.$$

Energy is lost as heat in the motor at the peak rate

$$P_2 = I^2\,R = (5.0)^2\,(4.0) = 100\,\text{W}.$$

The peak useful power produced by the motor is the difference between the peak power supplied to the motor and the peak power dissipated as heat:

$$P = P_1 - P_2 = (600) - (100) = 500\,\text{W}.$$

The peak useful power is also given by the product of the peak back-emf and the peak current flowing through the motor [see Eq. (9.36)],

$$P = \mathcal{E}\,I = (100)\,(5.0) = 500\,\text{W}.$$

The efficiency η is the ratio of the peak useful power output of the motor to the peak power supplied, or

$$\eta = \frac{P}{P_1} = \frac{500}{600} = 0.83 = 83\,\%.$$

10 Inductance

10.1 Mutual Inductance

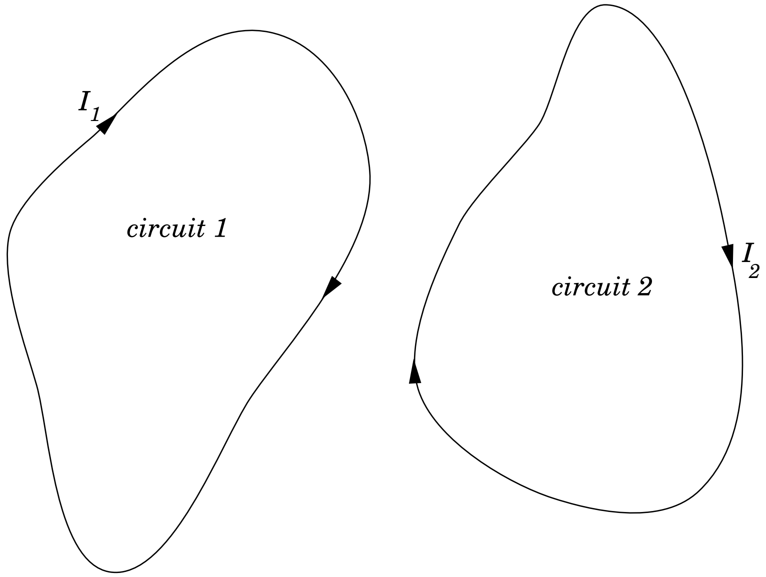


Figure 10.1: *Two inductively coupled circuits.*

Consider two arbitrary conducting circuits, labelled 1 and 2. Suppose that I_1 is the instantaneous current flowing around circuit 1. This current generates a magnetic field $\mathbf{B}_1$ which links the second circuit, giving rise to a magnetic flux Φ_2 through that circuit. If the current I_1 doubles, then the magnetic field $\mathbf{B}_1$ doubles in strength at all points in space, so the magnetic flux Φ_2 through the second circuit also doubles. This conclusion follows from the *linearity* of the laws of magnetostatics, plus the definition of magnetic flux. Furthermore, it is obvious that the flux through the second circuit is zero whenever the current flowing around the first circuit is zero. It follows that the flux Φ_2 through the second circuit is *directly proportional* to the current I_1 flowing around the first circuit. Hence, we can write

$$\Phi_2 = M_{21}\, I_1, \tag{10.1}$$

where the constant of proportionality M_{21} is called the *mutual inductance* of circuit 2 with respect to circuit 1. Similarly, the flux Φ_1 through the first circuit due to the instantaneous current I_2 flowing around the second circuit is directly proportional to that current, so we can write

$$\Phi_1 = M_{12}\, I_2, \tag{10.2}$$

where M_{12} is the *mutual inductance* of circuit 1 with respect to circuit 2. It is possible to demonstrate mathematically that $M_{12} = M_{21}$. In other words, the flux linking circuit 2 when a certain current flows around circuit 1 is exactly the same as the flux linking circuit 1 when the same current flows around circuit 2. This is true irrespective of the size, number of turns, relative position, and relative orientation of the two circuits. Because of this, we can write

$$M_{12} = M_{21} = M, \tag{10.3}$$

where M is termed the *mutual inductance* of the two circuits. Note that M is a purely geometric quantity, depending only on the size, number of turns, relative position, and relative orientation of the two circuits. The SI units of mutual inductance are called *Henries* (H). One henry is equivalent to a volt-second per ampere:

$$1\,\mathrm{H} \equiv 1\,\mathrm{V\,s\,A^{-1}}. \tag{10.4}$$

It turns out that a henry is a rather unwieldy unit. The mutual inductances of the circuits typically encountered in laboratory experiments are measured in millihenries.

Suppose that the current flowing around circuit 1 changes by an amount dI_1 in a time interval dt. It follows from Eqs. (10.1) and (10.3) that the flux linking circuit 2 changes by an amount $d\Phi_2 = M\, dI_1$ in the same time interval. According to Faraday's law, an emf

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} \tag{10.5}$$

is generated around the second circuit due to the changing magnetic flux linking that circuit. Since, $d\Phi_2 = M\, dI_1$, this emf can also be written

$$\mathcal{E}_2 = -M\,\frac{dI_1}{dt}. \tag{10.6}$$

Thus, the emf generated around the second circuit due to the current flowing around the first circuit is directly proportional to the rate at which that current changes. Likewise, if the current I_2 flowing around the second circuit changes by an amount dI_2 in a time interval dt then the emf generated around the first circuit is

$$\mathcal{E}_1 = -M\,\frac{dI_2}{dt}. \tag{10.7}$$

Note that there is no direct physical coupling between the two circuits. The coupling is due entirely to the magnetic field generated by the currents flowing around the circuits.

As a simple example, suppose that two insulated wires are wound on the same cylindrical former, so as to form two solenoids sharing a common air-filled core. Let l be the length of the core, A the cross-sectional area of the core, N_1 the number of times the first wire is wound around the core, and N_2 the number of times the second wire is wound around the core. If a current I_1 flows around the first wire then a uniform axial magnetic field of strength $B_1 = \mu_0\,N_1\,I_1/l$ is generated in the core (see Sect. 8.8). The magnetic field in the region outside the core is of negligible magnitude. The flux linking a single turn of the second wire is $B_1\,A$. Thus, the flux linking all N_2 turns of the second wire is $\Phi_2 = N_2\,B_1\,A = \mu_0\,N_1\,N_2\,A\,I_1/l$. From Eq. (10.1), the mutual inductance of the second wire with respect to the first is

$$M_{21} = \frac{\Phi_2}{I_1} = \frac{\mu_0\,N_1\,N_2\,A}{l}. \tag{10.8}$$

Now, the flux linking the second wire when a current I_2 flows in the first wire is $\Phi_1 = N_1\,B_2\,A$, where $B_2 = \mu_0\,N_2\,I_2/l$ is the associated magnetic field generated in the core. It follows from Eq. (10.2) that the mutual inductance of the first wire with respect to the second is

$$M_{12} = \frac{\Phi_1}{I_2} = \frac{\mu_0\,N_1\,N_2\,A}{l}. \tag{10.9}$$

Note that $M_{21} = M_{12}$, in accordance with Eq. (10.3). Thus, the mutual inductance of the two wires is given by

$$M = \frac{\mu_0\,N_1\,N_2\,A}{l}. \tag{10.10}$$

As described previously, M is a geometric quantity depending on the dimensions of the core, and the manner in which the two wires are wound around the core, but not on the actual currents flowing through the wires.

10.2 Self Inductance

We do not necessarily need two circuits in order to have inductive effects. Consider a single conducting circuit around which a current I is flowing. This current generates a magnetic field $\mathbf{B}$ which gives rise to a magnetic flux Φ linking the circuit. We expect the flux Φ to be directly proportional to the current I, given the linear nature of the laws of magnetostatics, and the definition of magnetic flux. Thus, we can write

$$\Phi = L\,I, \tag{10.11}$$

where the constant of proportionality L is called the *self inductance* of the circuit. Like mutual inductance, the self inductance of a circuit is measured in units of henries, and is a purely geometric quantity, depending only on the shape of the circuit and number of turns in the circuit.

If the current flowing around the circuit changes by an amount dI in a time interval dt then the magnetic flux linking the circuit changes by an amount $d\Phi = L\,dI$ in the same time interval. According to Faraday's law, an emf

$$\mathcal{E} = -\frac{d\Phi}{dt} \tag{10.12}$$

is generated around the circuit. Since $d\Phi = L\,dI$, this emf can also be written

$$\mathcal{E} = -L\,\frac{dI}{dt}. \tag{10.13}$$

Thus, the emf generated around the circuit due to its own current is directly proportional to the rate at which the current changes. Lenz's law, and common sense, demand that if the current is increasing then the emf should always act to reduce the current, and *vice versa*. This is easily appreciated, since if the emf acted to increase the current when the current was increasing then we would clearly get an unphysical positive feedback effect in which the current continued to increase without limit. It follows, from Eq. (10.13), that the self inductance L of a circuit is necessarily a *positive* number. This is not the case for mutual inductances, which can be either positive or negative.

Consider a solenoid of length l and cross-sectional area A. Suppose that the solenoid has N turns. When a current I flows in the solenoid, a uniform axial field of magnitude

$$B = \frac{\mu_0\, N\, I}{l} \tag{10.14}$$

is generated in the core of the solenoid. The field-strength outside the core is negligible. The magnetic flux linking a single turn of the solenoid is $\Phi = B\,A$. Thus, the magnetic flux linking all N turns of the solenoid is

$$\Phi = N\,B\,A = \frac{\mu_0\, N^2\, A\, I}{l}. \tag{10.15}$$

According to Eq. (10.11), the self inductance of the solenoid is given by $L = \Phi/I$, which reduces to

$$L = \frac{\mu_0\, N^2\, A}{l}. \tag{10.16}$$

Note that L is positive. Furthermore, L is a geometric quantity depending only on the dimensions of the solenoid, and the number of turns in the solenoid.

Engineers like to reduce all pieces of electrical apparatus, no matter how complicated, to an *equivalent circuit* consisting of a network of just *four* different types of component. These four basic components are *emfs*, *resistors*, *capacitors*, and *inductors*. An inductor is simply a pure self inductance, and is usually represented a little solenoid in circuit diagrams. In practice, inductors generally consist of short air-cored solenoids wound from enameled copper wire.

10.3 Energy Stored in an Inductor

Suppose that an inductor of inductance L is connected to a variable DC voltage supply. The supply is adjusted so as to increase the current i flowing through the inductor from zero to some final value I. As the current through the inductor is ramped up, an emf $\mathcal{E} = -L\,di/dt$ is generated, which acts to oppose the increase in the current. Clearly, work must be done against this emf by the voltage source in order to establish the current in the inductor. The work done by the voltage source during a time interval dt is

$$dW = P\,dt = -\mathcal{E}\,i\,dt = i\,L\,\frac{di}{dt}\,dt = L\,i\,di. \tag{10.17}$$

Here, $P = -\mathcal{E}\,i$ is the instantaneous rate at which the voltage source performs work. To find the total work W done in establishing the final current I in the inductor, we must integrate the above expression. Thus,

$$W = L\int_0^I i\,di, \tag{10.18}$$

giving

$$W = \frac{1}{2}\,L\,I^2. \tag{10.19}$$

This energy is actually stored in the magnetic field generated by the current flowing through the inductor. In a pure inductor, the energy is stored without loss, and is returned to the rest of the circuit when the current through the inductor is ramped down, and its associated magnetic field collapses.

Consider a simple solenoid. Equations (10.14), (10.16), and (10.19) can be combined to give

$$W = \frac{1}{2}\,L\,I^2 = \frac{\mu_0\,N^2\,A}{2\,l}\left(\frac{B\,l}{\mu_0\,N}\right)^2, \tag{10.20}$$

which reduces to

$$W = \frac{B^2}{2\,\mu_0}\,l\,A. \tag{10.21}$$

This represents the energy stored in the magnetic field of the solenoid. However, the volume of the field-filled core of the solenoid is $l\,A$, so the magnetic energy density (*i.e.*, the energy per unit volume) inside the solenoid is $w = W/(l\,A)$, or

$$w = \frac{B^2}{2\,\mu_0}. \tag{10.22}$$

It turns out that this result is quite general. Thus, we can calculate the energy content of any magnetic field by dividing space into little cubes (in each of which the magnetic field is approximately uniform), applying the above formula to find the energy content of each cube, and summing the energies thus obtained to find the total energy.

When electric and magnetic fields exist together in space, Eqs. (6.23) and (10.22) can be combined to give an expression for the total energy stored in the combined fields per unit volume:

$$w = \frac{\epsilon_0\,E^2}{2} + \frac{B^2}{2\,\mu_0}. \tag{10.23}$$

10.4 The RL Circuit

Consider a circuit in which a battery of emf V is connected in series with an inductor of inductance L and a resistor of resistance R. For obvious reasons, this type of circuit is usually called an RL *circuit*. The resistance R includes the resistance of the wire loops of the inductor, in addition to any other resistances in the circuit.

In steady-state, the current I flowing around the the circuit has the magnitude

$$I = \frac{V}{R} \tag{10.24}$$

specified by Ohm's law. Note that, in a steady-state, or DC, circuit, zero back-emf is generated by the inductor, according to Eq. (10.13), so the inductor effectively disappears from the circuit. In fact, inductors have no effect whatsoever in DC circuits. They just act like pieces of conducting wire.

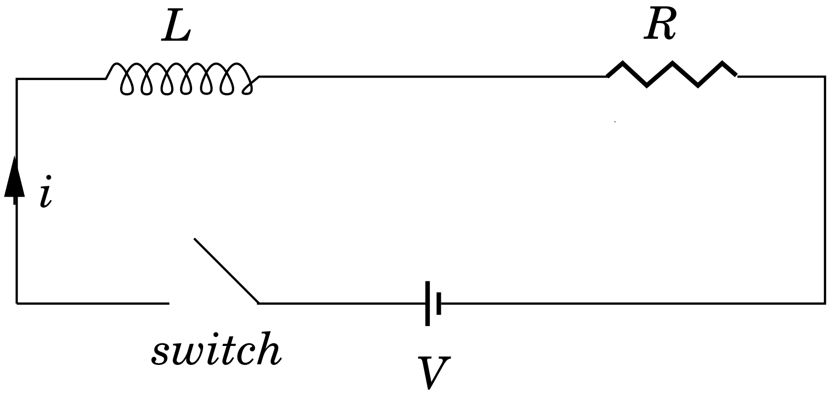


Figure 10.2: *An* RL *circuit with a switch.*

Let us now slightly modify our RL circuit by introducing a switch. The new circuit is shown in Fig. 10.2. Suppose that the switch is initially open, but is suddenly closed at $t = 0$. Obviously, we expect the instantaneous current i which flows around the circuit, once the switch is thrown, to eventually settle down to the steady-state value $I = V/R$. But, how long does this process take? Note that as the current flowing around the circuit is building up to its final value, a non-zero back-emf is generated in the inductor, according to Eq. (10.13). Thus, although the inductor does not affect the final steady-state value of the current flowing around the circuit, it certainly does affect how long after the switch is closed it takes for this final current to be established.

If the instantaneous current i flowing around the circuit changes by an amount di in a short time interval dt, then the emf generated in the inductor is given by [see Eq. (10.13)]

$$\mathcal{E} = -L\,\frac{di}{dt}. \tag{10.25}$$

Applying Ohm's law around the circuit, we obtain

$$V + \mathcal{E} = i\,R, \tag{10.26}$$

which yields

$$-\,L\,\frac{di}{dt} = i\,R - V. \tag{10.27}$$

Let

$$i' = i - I, \tag{10.28}$$

where $I = V/R$ is the steady-state current. Equation (10.27) can be rewritten

$$\frac{di'}{dt} = -i'\,\frac{R}{L}, \tag{10.29}$$

since $di' = di$ (because I is non-time-varying). At $t = 0$, just after the switch is closed, we expect the current i flowing around the circuit to be zero. It follows from Eq. (10.28) that

$$i'(t = 0) = -I. \tag{10.30}$$

Integration of Eq. (10.29), subject to the initial condition (10.30), yields

$$i'(t) = -I\,e^{-R\,t/L}. \tag{10.31}$$

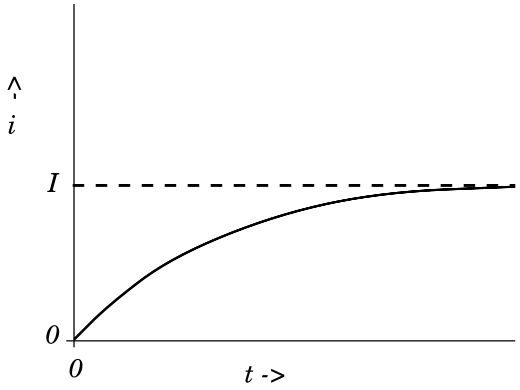


Figure 10.3: *Sketch of the current rise phase in an* RL *circuit switched on at* $t = 0$.

Thus, it follows from Eq. (10.28) that

$$i(t) = I\,(1 - e^{-R\,t/L}). \tag{10.32}$$

The above expression specifies the current i flowing around the circuit a time interval t after the switch is closed (at time $t = 0$). The variation of the current with time is sketched in Fig. 10.3. It can be seen that when the switch is closed the current i flowing in the circuit does not suddenly jump up to its final value, $I = V/R$. Instead, the current increases smoothly from zero, and gradually asymptotes to its final value. The current has risen to approximately 63% of its final value a time

$$\tau = \frac{L}{R} \tag{10.33}$$

after the switch is closed (since $e^{-1} \simeq 0.37$). By the time $t = 5\,\tau$, the current has risen to more than 99% of its final value (since $e^{-5} < 0.01$). Thus, $\tau = L/R$ is a good measure of how long after the switch is closed it takes for the current flowing in the circuit to attain its steady-state value. The quantity τ is termed the *time-constant*, or, somewhat unimaginatively, the *L over R time*, of the circuit.

Suppose that the current flowing in the circuit discussed above has settled down to its steady-state value $I = V/R$. Consider what would happen if we were to suddenly (at $t = 0$, say) switch the battery out of the circuit, and replace it by a conducting wire. Obviously, we would expect the current to eventually decay away to zero, since there is no longer a steady emf in the circuit to maintain a steady current. But, how long does this process take?

Applying Ohm's law around the circuit, in the absence of the battery, we obtain

$$\mathcal{E} = i\,R, \tag{10.34}$$

where $\mathcal{E} = -L\,di/dt$ is the back-emf generated by the inductor, and i is the instantaneous current flowing around the circuit. The above equation reduces to

$$\frac{di}{dt} = -i\,\frac{R}{L}. \tag{10.35}$$

At $t = 0$, immediately after the battery is switched out of the circuit, we expect the current i flowing around the circuit to equal its steady-state value I, so that

$$i(t=0) = I. \tag{10.36}$$

Integration of Eq. (10.35), subject to the boundary condition (10.36), yields

$$i(t) = I\,e^{-R\,t/L}. \tag{10.37}$$

According to the above formula, once the battery is switched out of the circuit, the current decays smoothly to zero. After one L/R time (*i.e.*, $t = L/R$), the current has decayed to 37% of its initial value. After five L/R times, the current has decayed to less than 1% of its initial value.

We can now appreciate the significance of self inductance. The back-emf generated in an inductor, as the current flowing through it tries to change, effectively prevents the current from rising (or falling) much faster than the L/R time of the circuit. This effect is sometimes advantageous, but is often a great nuisance. All circuits possess some self inductance, as well as some resistance, so all have a finite L/R time. This means that when we power up a DC circuit, the current does not jump up instantaneously to its steady-state value. Instead, the rise is spread out over the L/R time of the circuit. This is a good thing. If the current were to rise instantaneously then extremely large inductive electric fields would be generated by the sudden jump in the magnetic field, leading, inevitably, to breakdown and electric arcing. So, if there were no such thing as self inductance then every time we switched a DC electric circuit on or off there would be a big blue flash due to arcing between conductors. Self inductance can also be a bad thing. Suppose that we possess a fancy power supply, and wish to use it to send

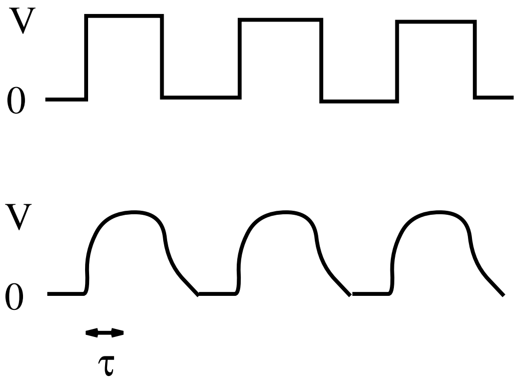


Figure 10.4: *Typical difference between the input wave-form (top) and the output wave-form (bottom) when a square-wave is sent down a line with finite* L/R *time,* τ.

an electric signal down a wire. Of course, the wire will possess both resistance and inductance, and will, therefore, have some characteristic L/R time. Suppose that we try to send a square-wave signal down the wire. Since the current in the wire cannot rise or fall faster than the L/R time, the leading and trailing edges of the signal get smoothed out over an L/R time. The typical difference between the signal fed into the wire (upper trace) and that which comes out of the other end (lower trace) is illustrated in Fig. 10.4. Clearly, there is little point in us having a fancy power supply unless we also possess a low inductance wire, so that the signal from the power supply can be transmitted to some load device without serious distortion.

10.5 The RC Circuit

Let us now discuss a topic which, admittedly, has nothing whatsoever to do with inductors, but is mathematically so similar to the topic just discussed that it seems sensible to consider it at this point.

Consider a circuit in which a battery of emf V is connected in series with a capacitor of capacitance C, and a resistor of resistance R. For fairly obvious reasons, such a circuit is generally referred to as an RC *circuit*. In steady-state, the charge on the positive plate of the capacitor is given by $Q = C\,V$, and zero current flows around the circuit (since current cannot flow across the insulating

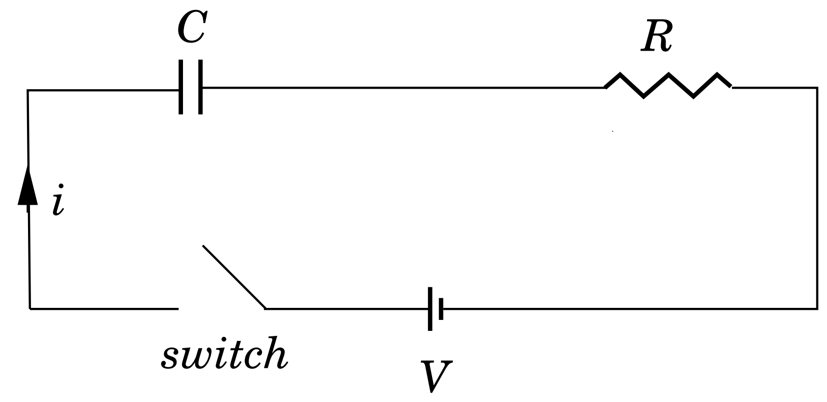


Figure 10.5: *An* RC *circuit with a switch.*

gap between the capacitor plates).

Let us now introduce a switch into the circuit, as shown in Fig. 10.5. Suppose that the switch is initially open, but is suddenly closed at $t = 0$. It is assumed that the capacitor plates are uncharged when the switch is thrown. We expect a transient current i to flow around the circuit until the charge q on the positive plate of the capacitor attains its final steady-state value $Q = C\,V$. But, how long does this process take?

The potential difference v between the positive and negative plates of the capacitor is given by

$$v = V - i\,R. \tag{10.38}$$

In other words, the potential difference between the plates is the emf of the battery minus the potential drop across the resistor. The charge q on the positive plate of the capacitor is written

$$q = C\,v = Q - i\,R\,C, \tag{10.39}$$

where $Q = C\,V$ is the final charge. Now, if i is the instantaneous current flowing around the circuit, then in a short time interval dt the charge on the positive plate of the capacitor increases by a small amount $dq = i\,dt$ (since all of the charge which flows around the circuit must accumulate on the plates of the capacitor). It follows that

$$i = \frac{dq}{dt}. \tag{10.40}$$

Thus, the instantaneous current flowing around the circuit is numerically equal to the rate at which the charge accumulated on the positive plate of the capacitor increases with time. Equations (10.39) and (10.40) can be combined together to give

$$\frac{dq'}{dt} = -\frac{q'}{RC}, \tag{10.41}$$

where

$$q' = q - Q. \tag{10.42}$$

At $t = 0$, just after the switch is closed, the charge on the positive plate of the capacitor is zero, so

$$q'(t = 0) = -Q. \tag{10.43}$$

Integration of Eq. (10.41), subject to the boundary condition (10.43), yields

$$q'(t) = -Q\, e^{-t/RC}. \tag{10.44}$$

It follows from Eq. (10.42) that

$$q(t) = Q\,(1 - e^{-t/RC}). \tag{10.45}$$

The above expression specifies the charge q on the positive plate of the capacitor a time interval t after the switch is closed (at time $t = 0$). The variation of the charge with time is sketched in Fig. 10.6. It can be seen that when the switch is closed the charge q on the positive plate of the capacitor does not suddenly jump up to its final value, $Q = C\,V$. Instead, the charge increases smoothly from zero, and gradually asymptotes to its final value. The charge has risen to approximately 63% of its final value a time

$$\tau = RC \tag{10.46}$$

after the switch is closed. By the time $t = 5\,\tau$, the charge has risen to more than 99% of its final value. Thus, $\tau = RC$ is a good measure of how long after the switch is closed it takes for the capacitor to fully charge up. The quantity τ is termed the *time-constant*, or the RC *time*, of the circuit.

According to Eqs. (10.40) and (10.41),

$$i = \frac{dq}{dt} = \frac{dq'}{dt} = -\frac{q'}{RC}. \tag{10.47}$$

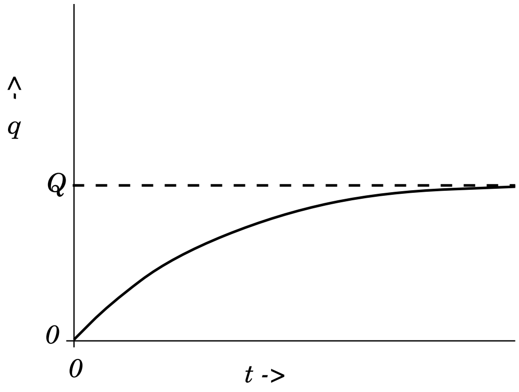


Figure 10.6: *Sketch of the charging phase in an* RC *circuit switched on at* $t = 0$.

It follows from Eq. (10.44) that

$$i(t) = I\, e^{-t/RC}, \tag{10.48}$$

where $I = V/R$. The above expression specifies the current i flowing around the circuit a time interval t after the switch is closed (at time $t = 0$). It can be seen that, immediately after the switch is thrown, the current $I = V/R$ which flows in the circuit is that which would flow if the capacitor were replaced by a conducting wire. However, this current is only transient, and rapidly decays away to a negligible value. After one RC time, the current has decayed to 37% of its initial value. After five RC times, the current has decayed to less than 1% of its initial value. It is interesting to note that for a short instant of time, just after the switch is closed, the current in the circuit acts as if there is no insulating gap between the capacitor plates. It essentially takes an RC time for the information about the break in the circuit to propagate around the circuit, and cause the current to stop flowing.

Suppose that we take a capacitor of capacitance C, which is charged to a voltage V, and discharge it by connecting a resistor of resistance R across its terminals at $t = 0$. How long does it take for the capacitor to discharge? By analogy with the previous analysis, the charge q on the positive plate of the capacitor at time t is given by

$$q(t) = Q\, e^{-t/RC}, \tag{10.49}$$

where $Q = C\,V$ is the initial charge on the positive plate. It can be seen that it takes a few RC times for the capacitor to fully discharge. The current i which flows through the resistor is

$$i(t) = I\, e^{-t/RC}, \tag{10.50}$$

where $I = V/R$ is the initial current. It can be seen that the capacitor initially acts like a battery of emf V (since it drives the current $I = V/R$ across the resistor), but that, as it discharges, its effective emf decays to a negligible value on a few RC times.

10.6 Transformers

A transformer is a device for stepping-up, or stepping-down, the voltage of an alternating electric signal. Without efficient transformers, the transmission and distribution of AC electric power over long distances would be impossible. Figure 10.7 shows the circuit diagram of a typical transformer. There are two circuits. Namely, the *primary circuit*, and the *secondary circuit*. There is no direct electrical connection between the two circuits, but each circuit contains a coil which links it *inductively* to the other circuit. In real transformers, the two coils are wound onto the same iron core. The purpose of the iron core is to channel the magnetic flux generated by the current flowing around the primary coil, so that as much of it as possible also links the secondary coil. The common magnetic flux linking the two coils is conventionally denoted in circuit diagrams by a number of parallel straight lines drawn between the coils.

Let us consider a particularly simple transformer in which the primary and secondary coils are *solenoids* sharing the same air-filled core. Suppose that l is the length of the core, and A is its cross-sectional area. Let N_1 be the total number of turns in the primary coil, and let N_2 be the total number of turns in the secondary coil. Suppose that an alternating voltage

$$v_1 = V_1 \cos(\omega\, t) \tag{10.51}$$

is fed into the primary circuit from some external AC power source. Here, V_1 is the peak voltage in the primary circuit, and ω is the alternation frequency (in radians per second). The current driven around the primary circuit is written

$$i_1 = I_1 \sin(\omega\, t), \tag{10.52}$$

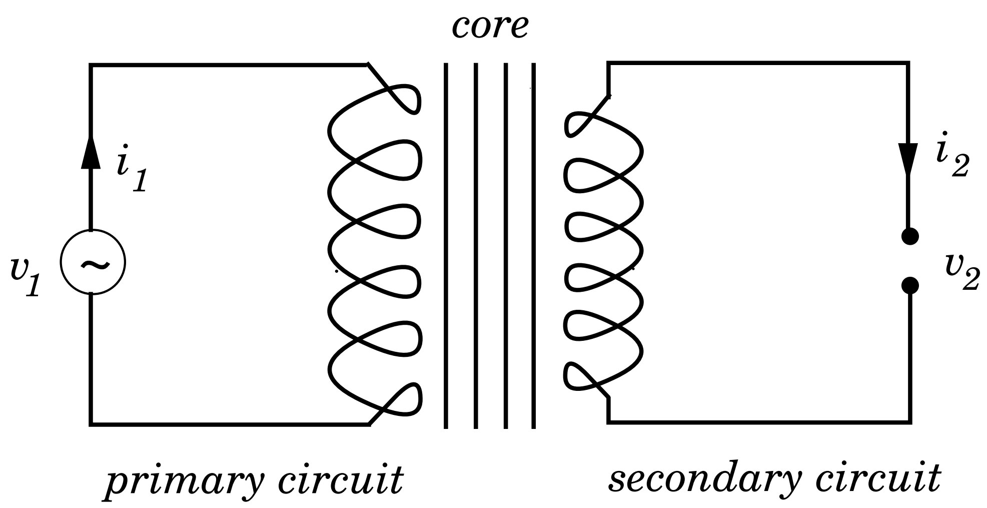


Figure 10.7: *Circuit diagram of a transformer.*

where I_1 is the peak current. This current generates a changing magnetic flux, in the core of the solenoid, which links the secondary coil, and, thereby, inductively generates the alternating emf

$$v_2 = V_2 \cos(\omega t) \tag{10.53}$$

in the secondary circuit, where V_2 is the peak voltage. Suppose that this emf drives an alternating current

$$i_2 = I_2 \sin(\omega t) \tag{10.54}$$

around the secondary circuit, where I_2 is the peak current.

The circuit equation for the primary circuit is written

$$v_1 - L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} = 0, \tag{10.55}$$

assuming that there is negligible resistance in this circuit. The first term in the above equation is the externally generated emf. The second term is the back-emf due to the self inductance L_1 of the primary coil. The final term is the emf due to the mutual inductance M of the primary and secondary coils. In the absence of any significant resistance in the primary circuit, these three emfs must add up to zero. Equations (10.51), (10.52), (10.54), and (10.55) can be combined to give

$$V_1 = \omega (L_1 I_1 + M I_2), \tag{10.56}$$

since

$$\frac{d\sin(\omega\,t)}{dt} = \omega\,\cos(\omega\,t). \tag{10.57}$$

The alternating emf generated in the secondary circuit consists of the emf generated by the self inductance L_2 of the secondary coil, plus the emf generated by the mutual inductance of the primary and secondary coils. Thus,

$$v_2 = L_2\,\frac{di_2}{dt} + M\,\frac{di_1}{dt}. \tag{10.58}$$

Equations (10.52), (10.53), (10.54), (10.57), and (10.58) yield

$$V_2 = \omega\,(L_2\,I_2 + M\,I_1). \tag{10.59}$$

Now, the instantaneous power output of the external AC power source which drives the primary circuit is

$$P_1 = i_1\,v_1. \tag{10.60}$$

Likewise, the instantaneous electrical energy per unit time transfered inductively from the primary to the secondary circuit is

$$P_2 = i_2\,v_2. \tag{10.61}$$

If resistive losses in the primary and secondary circuits are negligible, as is assumed to be the case, then, by energy conservation, these two powers must equal one another at all times. Thus,

$$i_1\,v_1 = i_2\,v_2, \tag{10.62}$$

which easily reduces to

$$I_1\,V_1 = I_2\,V_2. \tag{10.63}$$

Equations (10.56), (10.59), and (10.63) yield

$$I_1\,V_1 = \omega\,(L_1\,I_1^{\,2} + M\,I_1\,I_2) = \omega\,(L_2\,I_2^{\,2} + M\,I_1\,I_2) = I_2\,V_2, \tag{10.64}$$

which gives

$$\omega\,L_1\,I_1^{\,2} = \omega\,L_2\,I_2^{\,2}, \tag{10.65}$$

and, hence,

$$\frac{I_1}{I_2} = \sqrt{\frac{L_2}{L_1}}. \tag{10.66}$$

Equations (10.63) and (10.66) can be combined to give

$$\frac{V_1}{V_2} = \sqrt{\frac{L_1}{L_2}}. \tag{10.67}$$

Note that, although the mutual inductance of the two coils is entirely responsible for the transfer of energy between the primary and secondary circuits, it is the self inductances of the two coils which determine the ratio of the peak voltages and peak currents in these circuits.

Now, from Sect. 10.2, the self inductances of the primary and secondary coils are given by $L_1 = \mu_0\, N_1^{\,2}\, A/l$ and $L_2 = \mu_0\, N_2^{\,2}\, A/l$, respectively. It follows that

$$\frac{L_1}{L_2} = \left(\frac{N_1}{N_2}\right)^2, \tag{10.68}$$

and, hence, that

$$\frac{V_1}{V_2} = \frac{I_2}{I_1} = \frac{N_1}{N_2}. \tag{10.69}$$

In other words, the ratio of the peak voltages and peak currents in the primary and secondary circuits is determined by the ratio of the number of turns in the primary and secondary coils. This latter ratio is usually called the *turns-ratio* of the transformer. If the secondary coil contains *more* turns than the primary coil then the peak voltage in the secondary circuit *exceeds* that in the primary circuit. This type of transformer is called a *step-up transformer,* because it steps up the voltage of an AC signal. Note that in a step-up transformer the peak current in the secondary circuit is *less* than the peak current in the primary circuit (as must be the case if energy is to be conserved). Thus, a step-up transformer actually steps down the current. Likewise, if the secondary coil contains *less* turns than the primary coil then the peak voltage in the secondary circuit is *less* than that in the primary circuit. This type of transformer is called a *step-down transformer*. Note that a step-down transformer actually steps up the current (*i.e.*, the peak current in the secondary circuit exceeds that in the primary circuit).

AC electricity is generated in power stations at a fairly low peak voltage (*i.e.*, something like 440 V), and is consumed by the domestic user at a peak voltage of 110 V (in the U.S.). However, AC electricity is transmitted from the power station to the location where it is consumed at a very high peak voltage (typically 50 kV). In fact, as soon as an AC signal comes out of a generator in a power station it is fed into a step-up transformer which boosts its peak voltage from a few hundred volts to many tens of kilovolts. The output from the step-up transformer is fed into a high tension transmission line, which typically transports the electricity over many tens of kilometers, and, once the electricity has reached its point of consumption, it is fed through a series of step-down transformers until, by the time it emerges from a domestic power socket, its peak voltage is only 110 V. But, if AC electricity is both generated and consumed at comparatively low peak voltages, why go to the trouble of stepping up the peak voltage to a very high value at the power station, and then stepping down the voltage again once the electricity has reached its point of consumption? Why not generate, transmit, and distribute the electricity at a peak voltage of 110 V? Well, consider an electric power line which transmits a peak electric power P between a power station and a city. We can think of P, which depends on the number of consumers in the city, and the nature of the electrical devices which they operate, as essentially a fixed number. Suppose that V and I are the peak voltage and peak current of the AC signal transmitted along the line, respectively. We can think of these numbers as being variable, since we can change them using a transformer. However, since $P = I\,V$, the product of the peak voltage and the peak current must remain constant. Suppose that the resistance of the line is R. The peak rate at which electrical energy is lost due to ohmic heating in the line is $P_R = I^2\,R$, which can be written

$$P_R = \frac{P^2\,R}{V^2}. \tag{10.70}$$

Thus, if the power P transmitted down the line is a fixed quantity, as is the resistance R of the line, then the power lost in the line due to ohmic heating varies like the *inverse square* of the peak voltage in the line. It turns out that even at very high voltages, such as 50 kV, the ohmic power losses in transmission lines which run over tens of kilometers can amount to up to 20% of the transmitted power. It can readily be appreciated that if an attempt were made to transmit AC electric power at a peak voltage of 110 V then the ohmic losses would be so severe that virtually none of the power would reach its destination. Thus, it is only possible to generate electric power at a central location, transmit it over large distances, and then distribute it at its point of consumption, if the transmission is performed at a very high peak voltages (the higher, the better). Transformers play a vital role in this process because they allow us to step-up and step-down the voltage of an AC electric signal very efficiently (a well-designed transformer typically has a power loss which is only a few percent of the total power flowing through it).

Of course, transformers do not work for DC electricity, because the magnetic flux generated by the primary coil does not vary in time, and, therefore, does not induce an emf in the secondary coil. In fact, there is no efficient method of stepping-up or stepping-down the voltage of a DC electric signal. Thus, it is impossible to efficiently transmit DC electric power over larger distances. This is the main reason why commercially generated electricity is AC, rather than DC.

10.7 Impedance Matching

The principle use of transformers is in the transmission and distribution of commercially generated electricity. However, a second, very important use of transformers is as *impedance matching* devices. Recall, from Sect. 7.9, that for maximum power delivery from a source to a load, the load must have the same resistance as the internal resistance of the source. This can be accomplished by using a transformer to match the two resistances. Suppose that the power source is connected to the primary circuit, and the load to the secondary. If the resistance of the load is R, then $R = V_2/I_2$. However, from the *transformer equation*, (10.69), we have

$$V_1 = \frac{N_1}{N_2}\,V_2, \tag{10.71}$$

and

$$I_1 = \frac{N_2}{N_1}\,I_2. \tag{10.72}$$

Now the effective resistance R' of the load in the primary circuit is given by

$$R' = \frac{V_1}{I_1} = \left(\frac{N_1}{N_2}\right)^2 \frac{V_2}{I_2}, \tag{10.73}$$

which easily reduces to

$$R' = \left(\frac{N_1}{N_2}\right)^2 R. \tag{10.74}$$

Thus, by choosing the appropriate turns ratio, the effective load resistance R' can be made equal to the internal resistance of the source, no matter what value the actual load resistance R takes. This process is called *impedance matching*.

10.8 Worked Examples

Example 10.1: Mutual induction

Question: Suppose that two insulated wires are wound onto a common cylindrical former of length $l = 0.1\,\mathrm{m}$ and cross-sectional area $A = 0.05\,\mathrm{m}^2$. The number of turns in the first wire is $N_1 = 100$, and the number of turns in the second wire is $N_2 = 300$. What is the mutual inductance of the two wires? If the current I_1 flowing in the first wire increases uniformly from 0 to 10 A in 0.1 s, what emf is generated in the second wire? Does this emf act to drive a current in the second wire which circulates in the same sense as the current in the first wire, or the opposite sense?

Answer: From Eq. (10.10), the mutual inductance of the two wires is

$$M = \frac{\mu_0\, N_1\, N_2\, A}{l} = \frac{(1.26\times 10^{-6})\,(100)\,(300)\,(0.05)}{0.1} = 0.0188\,\mathrm{H}.$$

From Eq. (10.6), the emf generated around the second loop by the changing current in the first loop is

$$\mathcal{E}_2 = -M\,\frac{dI_1}{dt} = -(0.0188)\,\frac{(10-0)}{(0.1)} = -1.88\,\mathrm{V}.$$

The minus sign indicates that this emf acts so as to drive a current in the second wire which circulates in the *opposite* sense to the current flowing in the first wire, in accordance with Lenz's law. If the current in the first wire were decreased, instead of increased, then the emf in the second wire would act to drive a current which circulates in the same sense as the former current.

Example 10.2: Energy density of electric and magnetic fields

Question: In a certain region of space, the magnetic field has a value of 1.0×10^{-2} T, and the electric field has a value of $2.0 \times 10^{6}\,\mathrm{V\,m^{-1}}$. What is the combined energy density of the electric and magnetic fields?

Answer: For the electric field, the energy density is

$$w_E = \frac{1}{2}\,\epsilon_0\,E^2 = (0.5)\,(8.85 \times 10^{-12})\,(2.0 \times 10^{6})^2 = 18\ \mathrm{J\,m^{-3}}.$$

For the magnetic field, the energy density is

$$w_B = \frac{1}{2}\,\frac{B^2}{\mu_0} = \frac{(0.5)\,(1.0 \times 10^{-2})^2}{(4\pi \times 10^{-7})} = 40\ \mathrm{J\,m^{-3}}.$$

The net energy density is the sum of the energy density due to the electric field and the energy density due to the magnetic field:

$$w = w_E + w_B = (18 + 40) = 58\ \mathrm{J\,m^{-3}}.$$

Example 10.3: The RL circuit

Question: A coil has a resistance of $R = 5.0\,\Omega$ and an inductance of $L = 100\,\mathrm{mH}$. At a particular instant in time after a battery is connected across the coil, the current is $i = 2.0\,\mathrm{A}$, and is increasing at a rate of $di/dt = 20\,\mathrm{A\,s^{-1}}$. What is the voltage V of the battery? What is the time-constant of the circuit? What is the final value of the current?

Answer: Application of Ohm's law around the circuit gives [see Eq. (10.27)]

$$V = i\,R + L\,\frac{di}{dt} = (2.0)\,(5.0) + (0.1)\,(20) = 12\,\text{V}.$$

The time-constant of the circuit is simply

$$\tau = \frac{L}{R} = \frac{(0.1)}{(5.0)} = 0.020\,\text{s}.$$

The final steady-state current I is given by Ohm's law, with the inductor acting like a conducting wire, so

$$I = \frac{V}{R} = \frac{(12)}{(5)} = 2.4\,\text{A}.$$

Example 10.4: The RC circuit

Question: A capacitor of capacitance $C = 15\,\mu\text{F}$ is charged up to a voltage of $V = 800\,\text{V}$, and then discharged by connecting a resistor of resistance $R = 8\,\text{M}\Omega$ across its terminals. How long does it take for the charge on the positive plate of the capacitor to be reduced to 10% of its original value?

Answer: Suppose that the resistor is first connected across the capacitor at $t = 0$. The charge q on the positive plate of the capacitor is given by

$$q(t) = Q\,e^{-t/R\,C},$$

which can be rearranged to give

$$\frac{Q}{q} = e^{\,t/R\,C}.$$

Taking the natural logarithm of both sides, we obtain

$$\ln\left(\frac{Q}{q}\right) = \frac{t}{R\,C}.$$

Hence,

$$t = \tau \ln\left(\frac{Q}{q}\right),$$

where

$$\tau = R\,C = (8)\,(15) = 120 \text{ s}$$

is the RC time. Since $q/Q = 0.1$, in this case, it follows that

$$t = (120)\,(\ln 10) = 276.3 \text{ s}.$$

Example 10.5: The step-up transformer

Question: An electric power plant produces $P = 1\,\text{GW}$ of AC electric power at a peak voltage of $V_1 = 500\,\text{V}$. If it is desired to transmit this power at a peak voltage of $V_2 = 50\,\text{kV}$, what is the appropriate turns-ratio of the step-up transformer? What peak current I_1 would be sent over the transmission line if the peak voltage were $V_1 = 500\,\text{V}$? What peak current I_2 would be sent over the transmission line if the peak voltage were $V_2 = 50\,\text{kV}$? What is the ratio of the ohmic powers losses in the line in the two cases?

Answer: The appropriate turns-ratio is

$$\frac{N_2}{N_1} = \frac{(5 \times 10^4)}{(500)} = 100.$$

Since the peak power is given by $P = I_1\,V_1$, it follows that

$$I_1 = \frac{P}{V_1} = \frac{(1 \times 10^9)}{(500)} = 2\,\text{MA}.$$

Since the peak power remains unchanged after the signal passes through the transformer (assuming that there are no power losses in the transformer), we have

$$I_2 = \frac{P}{V_2} = \frac{(1 \times 10^9)}{(5 \times 10^4)} = 20\,\text{kA}.$$

The ratio of the power lost to ohmic heating in the two cases is

$$\frac{P_1}{P_2} = \frac{I_1^{\,2}\,R}{I_2^{\,2}\,R} = \left(\frac{2\times 10^6}{2\times 10^4}\right)^2 = 10000,$$

where R is the resistance of the transmission line. Note that the ohmic power loss is much greater at low peak voltage than at high peak voltage.

Example 10.6: Impedance matching

Question: An audio amplifier with an internal resistance of $2.0\,\mathrm{k}\Omega$ is used to drive a loudspeaker with a resistance of $R = 5.0\,\Omega$. A transformer is used to connect the amplifier to the loudspeaker. What is the appropriate turns-ratio of the transformer for optimal power transfer between the amplifier and the loudspeaker?

Answer: We require the transformer to convert the actual resistance R of the loudspeaker into an effective resistance R' which matches the internal resistance $2.0\,\mathrm{k}\Omega$ of the amplifier. Thus, from Eq. (10.74),

$$\frac{N_1}{N_2} = \sqrt{\frac{R'}{R}} = \sqrt{\frac{2\times 10^3}{5}} = 20.$$

11 Electromagnetic Waves

11.1 Maxwell's Equations

In the latter half of the nineteenth century, the Scottish physicist James Clerk Maxwell demonstrated that all previously established experimental facts regarding electric and magnetic fields could be summed up in just *four* equations. Nowadays, these equations are generally known as *Maxwell's equations*.

The first equation is simply Gauss' law (see Sect. 4). This equation describes how electric charges generate electric fields. Gauss' law states that:

> The electric flux through any closed surface is equal to the total charge enclosed by the surface, divided by ϵ_0.

This can be written mathematically as

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \frac{Q}{\epsilon_0}, \tag{11.1}$$

where S is a closed surface enclosing the charge Q. The above expression can also be written

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\epsilon_0}\int_V \rho\, dV, \tag{11.2}$$

where V is a volume bounded by the surface S, and ρ is the *charge density*: *i.e.*, the electric charge per unit volume.

The second equation is the magnetic equivalent of Gauss' law (see Sect. 8.10). This equation describes how the non-existence of magnetic monopoles causes magnetic field-lines to form closed loops. Gauss' law for magnetic fields states that:

> The magnetic flux through any closed surface is equal to zero.

This can be written mathematically as

$$\int_S \mathbf{B}\cdot d\mathbf{S} = 0, \tag{11.3}$$

where S is a closed surface.

The third equation is Faraday's law (see Sect. 9.3). This equation describes how changing magnetic fields generate electric fields. Faraday's law states that:

> The line integral of the electric field around any closed loop is equal to minus the time rate of change of the magnetic flux through the loop.

This can be written mathematically as

$$\oint_C \mathbf{E}\cdot d\mathbf{S} = -\frac{d}{dt}\int_{S'} \mathbf{B}\cdot d\mathbf{S}', \tag{11.4}$$

where S' is a surface attached to the loop C.

The fourth, and final, equation is Ampère's circuital law (see Sect. 8.7). This equation describes how electric currents generates magnetic fields. Ampère's circuital law states that:

> The line integral of the magnetic field around any closed loop is equal to μ_0 times the algebraic sum of the currents which pass through the loop.

This can be written mathematically as

$$\oint_C \mathbf{B}\cdot d\mathbf{r} = \mu_0\, I, \tag{11.5}$$

where I is the net current flowing through loop C. This equation can also be written

$$\oint_C \mathbf{B}\cdot d\mathbf{r} = \mu_0 \int_{S'} \mathbf{j}\cdot d\mathbf{S}', \tag{11.6}$$

where S' is a surface attached to the loop C, and $\mathbf{j}$ is the *current density*: *i.e.*, the electrical current per unit area.

When Maxwell first wrote Eqs. (11.2), (11.3), (11.4), and (11.6) he was basically trying to summarize everything which was known at the time about electric and magnetic fields in mathematical form. However, the more Maxwell looked at his equations, the more convinced he became that they were incomplete. Eventually, he proposed adding a new term, called the *displacement current*, to the right-hand side of his fourth equation. In fact, Maxwell was able to show that (11.2), (11.3), (11.4), and (11.6) are *mathematically inconsistent* unless the displacement current term is added to Eq. (11.6). Unfortunately, Maxwell's demonstration of this fact requires some advanced mathematical techniques which lie well beyond the scope of this course. In the following, we shall give a highly simplified version of his derivation of the missing term.

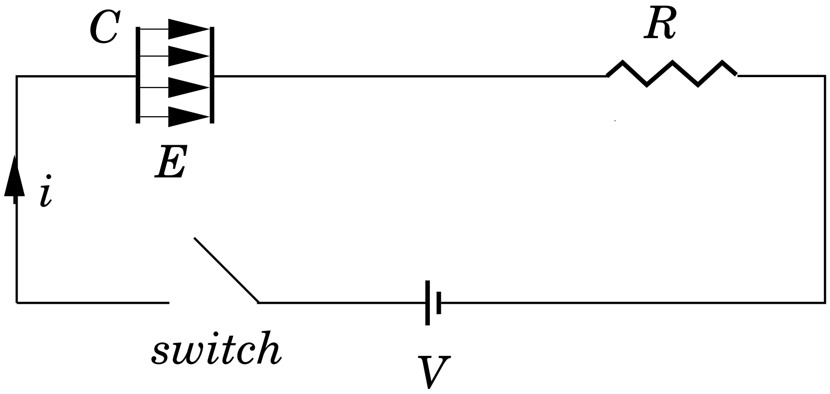


Figure 11.1: *Circuit containing a charging capacitor.*

Consider a circuit consisting of a parallel plate capacitor of capacitance C in series with a resistance R and an steady emf V, as shown in Fig. 11.1. Let A be the area of the capacitor plates, and let d be their separation. Suppose that the switch is closed at $t = 0$. The current i flowing around the circuit starts from an initial value of $I = V/R$, and gradually decays to zero on the RC time of the circuit (see Sect. 10.5). Simultaneously, the charge q on the positive plates of the capacitor starts from zero, and gradually increases to a final value of $Q = C\,V$. As the charge q varies, so does the potential difference v between the capacitor plates, since $v = q/C$.

The electric field in the region between the plates is approximately uniform, directed perpendicular to the plates (running from the positively charged plate to the negatively charged plate), and is of magnitude $E = v/d$. It follows that

$$q = C\,v = C\,d\,E. \tag{11.7}$$

In a time interval dt, the charge on the positive plate of the capacitor increases by an amount $dq = C\,d\,dE$, where dE is the corresponding increase in the electric field-strength between the plates. Note that both C and d are time-independent quantities. It follows that

$$\frac{dq}{dt} = C\,d\,\frac{dE}{dt}. \tag{11.8}$$

Now, dq/dt is numerically equal to the instantaneous current i flowing around the circuit (since all of the charge which flows around the circuit must accumulate on the plates of the capacitor). Also, $C = \epsilon_0\,A/d$ for a parallel plate capacitor. Hence, we can write

$$i = \frac{dq}{dt} = C\,d\,\frac{dE}{dt} = \epsilon_0\,A\,\frac{dE}{dt}. \tag{11.9}$$

Since the electric field E is normal to the area A, we can also write

$$i = \epsilon_0\,A\,\frac{dE_\perp}{dt}. \tag{11.10}$$

Equation (11.10) relates the instantaneous current flowing around the circuit to the time rate of change of the electric field between the capacitor plates. According to Eq. (11.6), the current flowing around the circuit generates a magnetic field. This field circulates around the current carrying wires connecting the various components of the circuit. However, since there is no actual current flowing between the plates of the capacitor, no magnetic field is generated in this region, according to Eq. (11.6). Maxwell demonstrated that for reasons of mathematical self-consistency there must, in fact, be a magnetic field generated in the region between the plates of the capacitor. Furthermore, this magnetic field must be the same as that which would be generated if the current i (*i.e.*, the same current as that which flows around the rest of the circuit) flowed between the plates. Of course, there is no actual current flowing between the plates. However, there is a changing electric field. Maxwell argued that a changing electric field constitutes an effective current (*i.e.*, it generates a magnetic field in just the same manner as an actual current). For historical reasons (which do not particularly interest us at the moment), Maxwell called this type of current a *displacement current*. Since the displacement current I_D flowing between the plates of the capacitor must equal the current i flowing around the rest of the circuit, it follows from

Eq. (11.10) that

$$I_D = \epsilon_0\, A\, \frac{dE_\perp}{dt}. \tag{11.11}$$

Equation (11.11) was derived for the special case of the changing electric field generated in the region between the plates of a charging parallel plate capacitor. Nevertheless, this equation turns out to be completely general. Note that $A\,E_\perp$ is equal to the electric flux Φ_E between the plates of the capacitor. Thus, the most general expression for the displacement current passing through some closed loop is

$$I_D = \epsilon_0\, \frac{d\Phi_E}{dt}, \tag{11.12}$$

where Φ_E is the electric flux through the loop.

According to Maxwell's argument, a displacement current is just as effective at generating a magnetic field as a real current. Thus, we need to modify Ampère's circuital law to take displacement currents into account. The modified law, which is known as the *Ampère-Maxwell law*, is written

> The line integral of the electric field around any closed loop is equal to μ_0 times the algebraic sum of the actual currents and which pass through the loop plus μ_0 times the displacement current passing through the loop.

This can be written mathematically as

$$\oint_C \mathbf{B}\cdot d\mathbf{r} = \mu_0\,(I + I_D), \tag{11.13}$$

where C is a loop through which the electric current I and the displacement current I_D pass. This equation can also be written

$$\oint_C \mathbf{B}\cdot d\mathbf{r} = \mu_0 \int_{S'} \mathbf{j}\cdot d\mathbf{S}' + \mu_0\,\epsilon_0\,\frac{d}{dt}\int_{S'} \mathbf{E}\cdot d\mathbf{S}', \tag{11.14}$$

where S' is a surface attached to the loop C.

Equations (11.2), (11.3), (11.4), and (11.14) are known collectively as *Maxwell's equations*. They constitute a complete and mathematically self-consistent description of the behaviour of electric and magnetic fields.

11.2 Electromagnetic Waves

One of the first things that Maxwell did with his four equations, once he had obtained them, was to look for wave-like solutions. Maxwell knew that the wave-like solutions of the equations of gas dynamics correspond to sound waves, and the wave-like solutions of the equations of fluid dynamics correspond to gravity waves in water, so he reasoned that if his equations possessed wave-like solutions then these would correspond to a completely new type of wave, which he called an *electromagnetic wave*.

Maxwell was primarily interested in electromagnetic waves which can propagate through a vacuum (*i.e.*, a region containing no charges or currents). Now, in a vacuum, Maxwell's equations reduce to

$$\oint_S \mathbf{E}\cdot d\mathbf{S}, = 0, \tag{11.15}$$

$$\oint_S \mathbf{B}\cdot d\mathbf{S} = 0, \tag{11.16}$$

$$\oint_C \mathbf{E}\cdot d\mathbf{r} = -\frac{d}{dt}\int_{S'} \mathbf{B}\cdot d\mathbf{S}', \tag{11.17}$$

$$\oint_C \mathbf{B}\cdot d\mathbf{r} = \mu_0\,\epsilon_0\,\frac{d}{dt}\int_{S'} \mathbf{E}\cdot d\mathbf{S}', \tag{11.18}$$

where S is a closed surface, and S' a surface attached to some loop C. Note that, with the addition of the displacement current term on the right-hand side of Eq. (11.18), these equations exhibit a nice symmetry between electric and magnetic fields. Unfortunately, Maxwell's mathematical proof that the above equations possess wave-like solutions lies well beyond the scope of this course. We can, nevertheless, still write down these solutions, and comment on them.

Consider a plane electromagnetic wave propagating along the z-axis. According to Maxwell's calculations, the electric and magnetic fields associated with such a wave take the form

$$E_x = E_0\cos[2\pi\,(z/\lambda - f\,t)], \tag{11.19}$$

$$B_y = B_0\cos[2\pi\,(z/\lambda - f\,t)]. \tag{11.20}$$

Note that the fields are periodic in both time and space. The oscillation frequency (in hertz) of the fields at a given point in space is f. The equation of a wave crest is

$$\frac{z}{\lambda} - f\,t = N, \tag{11.21}$$

where N is an integer. It can be seen that the distance along the z-axis between successive wave crests is given by λ. This distance is conventionally termed the *wavelength*. Note that each wave crest *propagates* along the z-axis. In a time interval dt, the Nth wave crest moves a distance $dz = \lambda\, f\, dt$, according to Eq. (11.21). Hence, the velocity $c = dz/dt$ with which the wave propagates along the z-axis is given by

$$c = f\,\lambda. \tag{11.22}$$

Maxwell was able to establish that electromagnetic waves possess the following properties:

1. The magnetic field oscillates *in phase* with the electric field. In other words, a wave maximum of the magnetic field always coincides with a wave maximum of the electric field in both time and space.

2. The electric field is always perpendicular to the magnetic field, and both fields are directed at right-angles to the direction of propagation of the wave. In fact, the wave propagates in the direction $\mathbf{E} \times \mathbf{B}$. Electromagnetic waves are clearly a type of *transverse wave*.

3. For a z-directed wave, the electric field is free to oscillate in *any* direction which lies in the x-y plane. The direction in which the electric field oscillates is conventionally termed the direction of *polarization* of the wave. Thus, Eqs. (11.19) represent a plane electromagnetic wave which propagates along the z-axis, and is polarized in the x-direction.

4. The maximum amplitudes of the electric and the magnetic fields are related via

$$E_0 = c\,B_0. \tag{11.23}$$

5. There is no constraint on the possible frequency or wavelength of electromagnetic waves. However, the propagation velocity of electromagnetic waves is *fixed*, and takes the value

$$c = \frac{1}{\sqrt{\mu_0\,\epsilon_0}}. \tag{11.24}$$

According to Eqs. (11.17) and (11.18), a changing magnetic field generates an electric field, and a changing electric field generates a magnetic field. Thus, we can think of the propagation of an electromagnetic field through a vacuum as due to a kind of "leap-frog" effect, in which a changing electric field generates a magnetic field, which, in turn, generates an electric field, and so on. Note that the displacement current term in Eq. (11.18) plays a crucial role in the propagation of electromagnetic waves. Indeed, without this term, a changing electric field is incapable of generating a magnetic field, and so there can be no leap-frog effect. Electromagnetic waves have many properties in common with other types of wave (*e.g.*, sound waves). However, they are unique in one respect: *i.e.*, they are able to propagate through a vacuum. All other types of waves require some sort of medium through which to propagate.

Maxwell deduced that the speed of propagation of an electromagnetic wave through a vacuum is entirely determined by the constants μ_0 and ϵ_0 [see Eq. (11.24)]. The former constant is related to the strength of the magnetic field generated by a steady current, whereas the latter constant is related to the strength of the electric field generated by a stationary charge. The values of both constants were well known in Maxwell's day. In modern units, $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N\,s^2\,C^{-2}}$ and $\epsilon_0 = 8.854 \times 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}}$. Thus, when Maxwell calculated the velocity of electromagnetic waves he obtained

$$c = \frac{1}{\sqrt{(4\pi \times 10^{-7})\,(8.854 \times 10^{-12})}} = 2.998 \times 10^{8}\,\mathrm{m\,s^{-1}}. \tag{11.25}$$

Now, Maxwell knew [from the work of Fizeau (1849) and Foucault (1850)] that the velocity of light was about $3 \times 10^8\,\mathrm{m\,s^{-1}}$. The remarkable agreement between this experimentally determined velocity and his theoretical prediction for the velocity of electromagnetic waves immediately lead Maxwell to hypothesize that *light is a form of electromagnetic wave*. Of course, this hypothesis turned out to be correct. We can still appreciate that Maxwell's achievement in identifying light as a form of electromagnetic wave was quite remarkable. After all, his equations were derived from the results of bench-top laboratory experiments involving charges, batteries, coils, and currents, *etc.*, which apparently had nothing whatsoever to do with light.

Maxwell was able to make another remarkable prediction. The wavelength of light was well known in the late nineteenth century from studies of diffraction through slits, *etc*. Visible light actually occupies a surprisingly narrow range of wavelengths. The shortest wavelength blue light which is visible has a wavelength of $\lambda = 0.40$ microns (one micron is 10^{-6} meters). The longest wavelength red light which is visible has a wavelength of $\lambda = 0.76$ microns. However, there is nothing in Maxwell's analysis which suggested that this particular range of wavelengths is special. In principle, electromagnetic waves can have any wavelength. Maxwell concluded that visible light forms a small element of a vast spectrum of previously undiscovered types of electromagnetic radiation.

Since Maxwell's time, virtually all of the non-visible parts of the electromagnetic spectrum have been observed. Table 11.1 gives a brief guide to the electromagnetic spectrum. Electromagnetic waves are of particular importance because they are our only source of information regarding the Universe around us. Radio waves and microwaves (which are comparatively hard to scatter) have provided much of our knowledge about the centre of the Galaxy. This is completely unobservable in visible light, which is strongly scattered by interstellar gas and dust lying in the galactic plane. For the same reason, the spiral arms of the Galaxy can only be mapped out using radio waves. Infrared radiation is useful for detecting proto-stars which are not yet hot enough to emit visible radiation. Of course, visible radiation is still the mainstay of astronomy. Satellite based ultraviolet observations have yielded invaluable insights into the structure and distribution of distant galaxies. Finally, X-ray and γ-ray astronomy usually concentrates on exotic objects in the Galaxy such as pulsars and supernova remnants.

Radiation Type	Wavelength Range (m)
Gamma Rays	$< 10^{-11}$
X-Rays	10^{-11}–10^{-9}
Ultraviolet	10^{-9}–10^{-7}
Visible	10^{-7}–10^{-6}
Infrared	10^{-6}–10^{-4}
Microwave	10^{-4}–10^{-1}
TV-FM	10^{-1}–10^{1}
Radio	$> 10^{1}$

Table 11.1: *The electromagnetic spectrum.*

11.3 Effect of Dielectric Materials

It turns out that electromagnetic waves cannot propagate very far through a conducting medium before they are either absorbed or reflected. However, electromagnetic waves are able to propagate through transparent dielectric media without difficultly. The speed of electromagnetic waves propagating through a dielectric medium is given by

$$c' = \frac{c}{\sqrt{K}}, \tag{11.26}$$

where K is the dielectric constant of the medium in question, and c the velocity of light in a vacuum. Since $K > 1$ for dielectric materials, we conclude that:

> The velocity with which electromagnetic waves propagate through a dielectric medium is always less than the velocity with which they propagate through a vacuum.

11.4 Energy in Electromagnetic Waves

From Sect. 10.3, the energy stored per unit volume in an electromagnetic wave is given by

$$w = \frac{\epsilon_0\,E^2}{2} + \frac{B^2}{2\,\mu_0}. \tag{11.27}$$

Since, $B = E/c$, for an electromagnetic wave, and $c = 1/\sqrt{\mu_0\,\epsilon_0}$, the above expression yields

$$w = \frac{\epsilon_0\,E^2}{2} + \frac{E^2}{2\,\mu_0\,c^2} = \frac{\epsilon_0\,E^2}{2} + \frac{\epsilon_0\,E^2}{2}, \tag{11.28}$$

or

$$w = \epsilon_0\,E^2. \tag{11.29}$$

It is clear, from the above, that half the energy in an electromagnetic wave is carried by the electric field, and the other half is carried by the magnetic field.

As an electromagnetic field propagates it transports energy. Let P be the power per unit area carried by an electromagnetic wave: *i.e.*, P is the energy transported per unit time across a unit cross-sectional area perpendicular to the direction in which the wave is traveling. Consider a plane electromagnetic wave propagating along the z-axis. The wave propagates a distance $c\,dt$ along the z-axis in a time interval dt. If we consider a cross-sectional area A at right-angles to the z-axis, then in a time dt the wave sweeps through a volume dV of space, where $dV = A\,c\,dt$. The amount of energy filling this volume is

$$dW = w\,dV = \epsilon_0\,E^2\,A\,c\,dt. \tag{11.30}$$

It follows, from the definition of P, that the power per unit area carried by the wave is given by

$$P = \frac{dW}{A\,dt} = \frac{\epsilon_0\,E^2\,A\,c\,dt}{A\,dt}, \tag{11.31}$$

so that

$$P = \epsilon_0\,E^2\,c. \tag{11.32}$$

Since half the energy in an electromagnetic wave is carried by the electric field, and the other half is carried by the magnetic field, it is conventional to convert the above expression into a form involving both the electric and magnetic field strengths. Since, $E = c\,B$, we have

$$P = \epsilon_0\,c\,E\,(c\,B) = \epsilon_0\,c^2\,E\,B = \frac{E\,B}{\mu_0}. \tag{11.33}$$

Thus,

$$P = \frac{E\,B}{\mu_0}. \tag{11.34}$$

Equation (11.34) specifies the power per unit area transported by an electromagnetic wave at any given instant of time. The *peak* power is given by

$$P_0 = \frac{E_0\,B_0}{\mu_0}, \tag{11.35}$$

where E_0 and B_0 are the peak amplitudes of the oscillatory electric and magnetic fields, respectively. It is easily demonstrated that the *average* power per unit area transported by an electromagnetic wave is *half* the peak power, so that

$$S = \bar{P} = \frac{E_0\,B_0}{2\,\mu_0} = \frac{\epsilon_0\,c\,E_0^{\,2}}{2} = \frac{c\,B_0^{\,2}}{2\,\mu_0}. \tag{11.36}$$

The quantity S is conventionally termed the *intensity* of the wave.

11.5 Worked Examples

Example 11.1: Electromagnetic waves

Question: Consider electromagnetic waves of wavelength $\lambda = 30\,\text{cm}$ in air. What is the frequency of such waves? If such waves pass from air into a block of quartz, for which $K = 4.3$, what is their new speed, frequency, and wavelength?

Answer: Since, $f\,\lambda = c$, assuming that the dielectric constant of air is approximately unity, it follows that

$$f = \frac{c}{\lambda} = \frac{(3\times 10^8)}{(0.3)} = 1\times 10^9\,\text{Hz}.$$

The new speed of the waves as they pass propagate through the quartz is

$$c' = \frac{c}{\sqrt{K}} = \frac{(3\times 10^8)}{\sqrt{4.3}} = 1.4\times 10^8\,\text{m}\,\text{s}^{-1}.$$

The frequency of electromagnetic waves does not change when the medium through which the waves are propagating changes. Since $c' = f\,\lambda$ for electromagnetic waves propagating through a dielectric medium, we have

$$\lambda_{\text{quartz}} = \frac{c'}{f} = \frac{(1.4\times 10^8)}{(1\times 10^9)} = 14\,\text{cm}.$$

Example 11.2: Intensity of electromagnetic radiation

Question: Suppose that the intensity of the sunlight falling on the ground on a particular day is $140\,\mathrm{W\,m^{-2}}$. What are the peak values of the electric and magnetic fields associated with the incident radiation?

Answer: According to Eq. (11.36), the peak electric field is given by

$$E_0 = \sqrt{\frac{2\,S}{\epsilon_0\,c}} = \sqrt{\frac{(2)\,(140)}{(8.85\times 10^{-12})\,(3\times 10^8)}} = 324.7\ \mathrm{V\,m^{-1}}.$$

Likewise, the peak magnetic field is given by

$$B_0 = \sqrt{\frac{2\,\mu_0\,S}{c}} = \sqrt{\frac{(2)\,(4\pi\times 10^{-7})\,(140)}{(3\times 10^8)}} = 1.083\times 10^{-6}\,\mathrm{T}.$$

Note, of course, that $B_0 = E_0/c$.

12 Geometric Optics

12.1 Introduction

Optics deals with the propagation of light through transparent media, and its interaction with mirrors, lenses, slits, *etc*. Optical effects can be divided into two broad classes. Firstly, those which can be explained without reference to the fact that light is fundamentally a wave phenomenon, and, secondly, those which can only be explained on the basis that light is a wave phenomenon. Let us, for the moment, consider the former class of effects. It might seem somewhat surprising that any optical effects at all can be accounted for without reference to waves. After all, as we saw in Sect. 11, light really is a wave phenomenon. It turns out, however, that wave effects are only crucially important when the wavelength of the wave is either comparable to, or much larger than, the size of the objects with which it interacts (see Sect. 14). When the wavelength of the wave becomes much smaller than the size of the objects with which it interacts then the interactions can be accounted for in a very simple geometric manner, as explained in this section. Since the wavelength of visible light is only of order a micron, it is very easy to find situations in which its wavelength is very much smaller than the size of the objects with which it interacts. Thus, "wave-less" optics, which is usually called *geometric optics*, has a very wide range of applications.

In geometric optics, light is treated as a set of *rays*, emanating from a source, which propagate through transparent media according to a set of *three* simple laws. The first law is the *law of rectilinear propagation*, which states that light rays propagating through a homogeneous transparent medium do so in straight-lines. The second law is the *law of reflection*, which governs the interaction of light rays with conducting surfaces (*e.g.*, metallic mirrors). The third law is the *law of refraction*, which governs the behaviour of light rays as they traverse a sharp boundary between two different transparent media (*e.g.*, air and glass).

12.2 History of Geometric Optics

Let us first consider the law of *rectilinear propagation*. The earliest surviving optical treatise, Euclid's *Catoptrics*[1] (280 BC), recognized that light travels in straight-lines in homogeneous media. However, following the teachings of Plato, Euclid (and all other ancient Greeks) thought that light rays emanate from the eye, and intercept external objects, which are thereby "seen" by the observer. The ancient Greeks also thought that the speed with which light rays emerge from the eye is very high, if not infinite. After all, they argued, an observer with his eyes closed can open them and immediately see the distant stars.

Hero of Alexandria, in his *Catoptrics* (first century BC), also maintained that light travels with infinite speed. His argument was by analogy with the free fall of objects. If we throw an object horizontally with a relatively small velocity then it manifestly does not move in a straight-line. However, if we throw an object horizontally with a relatively large velocity then it appears to move in a straight-line to begin with, but eventually deviates from this path. The larger the velocity with which the object is thrown, the longer the initial period of apparent rectilinear motion. Hero reasoned that if an object were thrown with an infinite velocity then it would move in a straight-line forever. Thus, light, which travels in a straight-line, must move with an infinite velocity. The erroneous idea that light travels with an *infinite* velocity persisted until 1676, when the Danish astronomer Olaf Römer demonstrated that light must have a *finite* velocity, using his timings of the successive eclipses of the satellites of Jupiter, as they passed into the shadow of the planet.

The first person to realize that light actually travels from the object seen to the eye was the Arab philosopher "Alhazan" (whose real name was Abu'ali al-hasan ibn al-haytham), who published a book on optics in about 1000 AD.

The law of *reflection* was correctly formulated in Euclid's book. Hero of Alexandria demonstrated that, by adopting the rule that light rays always travel between two points by the *shortest path* (or, more rigorously, the extremal path), it is possible to derive the law of reflection using geometry.

[1] Catoptrics is the ancient Greek word for reflection.

The law of *refraction* was studied experimentally by Claudius Ptolemy (100-170 AD), and is reported in Book V of his *Catoptrics*. Ptolemy formulated a very inaccurate version of the law of refraction, which only works when the light rays are almost normally incident on the interface in question. Despite its obvious inaccuracy, Ptolemy's theory of refraction persisted for nearly 1500 years. The true law of refraction was discovered empirically by the Dutch mathematician Willebrord Snell in 1621. However, the French philosopher René Descartes was the first to publish, in his *La Dioptrique* (1637), the now familiar formulation of the law of refraction in terms of sines. Although there was much controversy at the time regarding plagiarism, Descartes was apparently unaware of Snell's work. Thus, in English speaking countries the law of refraction is called "Snell's law", but in French speaking countries it is called "Descartes' law".

In 1658, the French mathematician Pierre de Fermat demonstrated that all three of the laws of geometric optics can be accounted for on the assumption that light always travels between two points on the path which takes the *least time* (or, more rigorously, the extremal time). Fermat's ideas were an extension of those of Hero of Alexandria. Fermat's (correct) derivation of the law of refraction depended crucially on his (correct) assumption that light travels *more slowly* in dense media than it does in air. Unfortunately, many famous scientists, including Newton, maintained that light travels *faster* in dense media than it does in air. This erroneous idea held up progress in optics for over one hundred years, and was not conclusively disproved until the mid-nineteenth century. Incidentally, Fermat's principle of least time can only be justified using wave theory.

12.3 Law of Geometric Propagation

According to geometric optics, an opaque object illuminated by a point source of light casts a sharp shadow whose dimensions can be calculated using geometry. The method of calculation is very straightforward. The source emits *light-rays* uniformly in all directions. These rays can be represented as straight lines radiating from the source. The light-rays propagate away from the source until they encounter an opaque object, at which point they stop. This is illustrated in

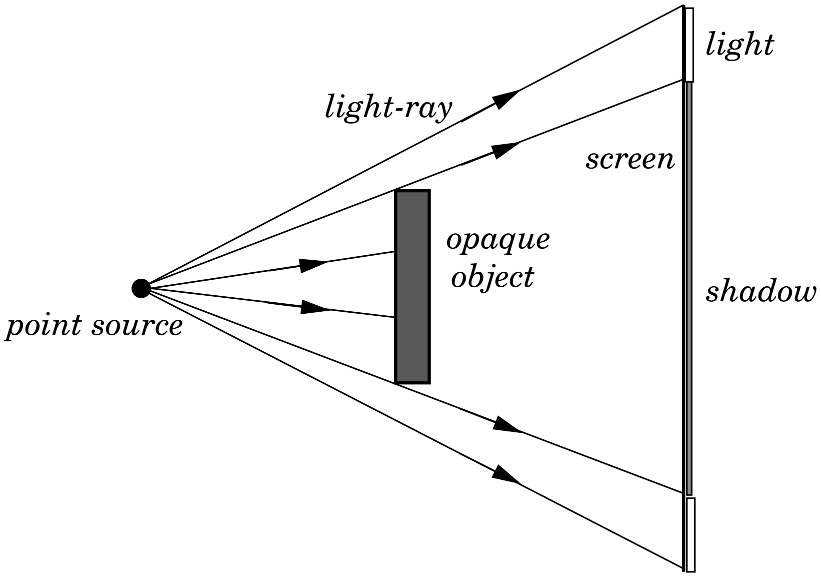


Figure 12.1: *An opaque object illuminated by a point light source.*

Fig. 12.1.

For an extended light source, each element of the source emits light-rays, just like a point source. Rays emanating from different elements of the source are assumed not to interfere with one another. Figure 12.2 shows how the shadow cast by an opaque sphere illuminated by a spherical light source is calculated using a small number of critical light-rays. The shadow consists of a perfectly black disk called the *umbra*, surrounded by a ring of gradually diminishing darkness called the *penumbra*. In the umbra, *all* of the light-rays emitted by the source are blocked by the opaque sphere, whereas in the penumbra only *some* of the rays emitted by the source are blocked by the sphere. As was well-known to the ancient Greeks, if the light-source represents the Sun, and the opaque sphere the Moon, then at a point on the Earth's surface which is situated inside the umbra the Sun is totally eclipsed, whereas at a point on the Earth's surface which is situated in the penumbra the Sun is only partially eclipsed.

In the wave picture of light, a *wave-front* is defined as a surface joining all adjacent points on a wave that have the same phase (*e.g.*, all maxima, or minima, of the electric field). A light-ray is simply a line which runs perpendicular to the wave-fronts at all points along the path of the wave. This is illustrated in

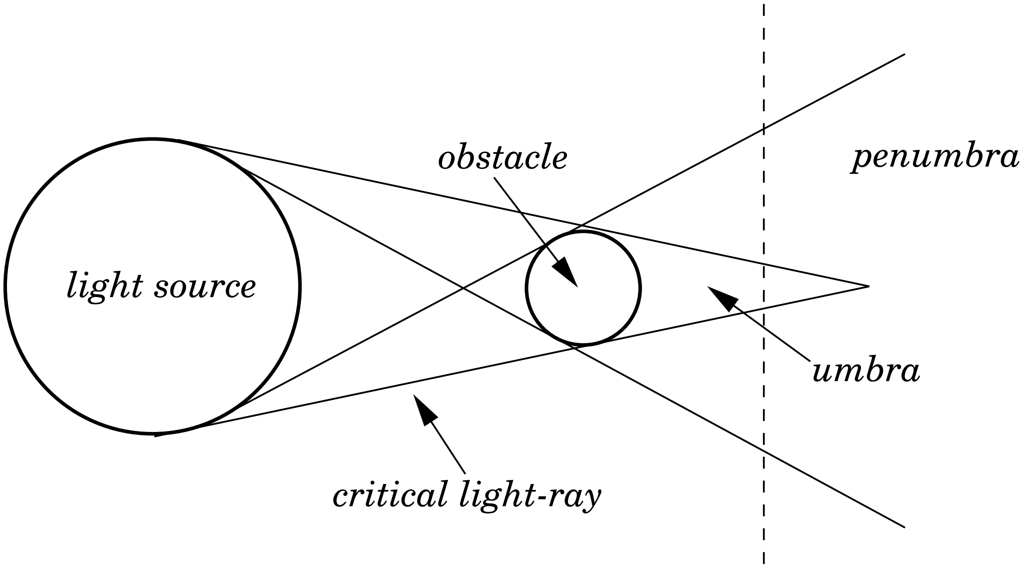


Figure 12.2: *An opaque object illuminated by an extended light source.*

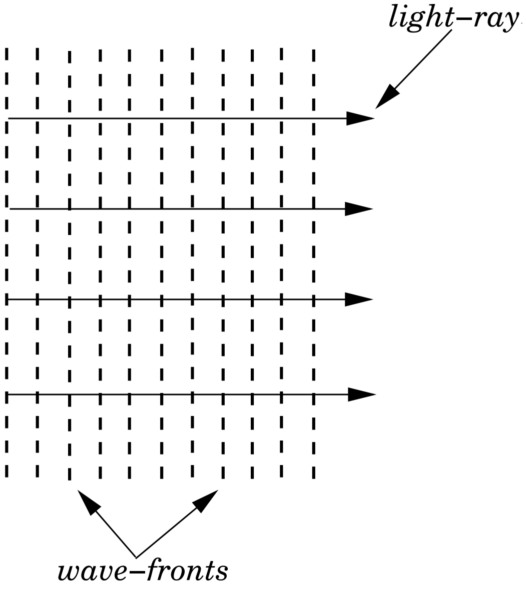


Figure 12.3: *Relationship between wave-fronts and light-rays.*

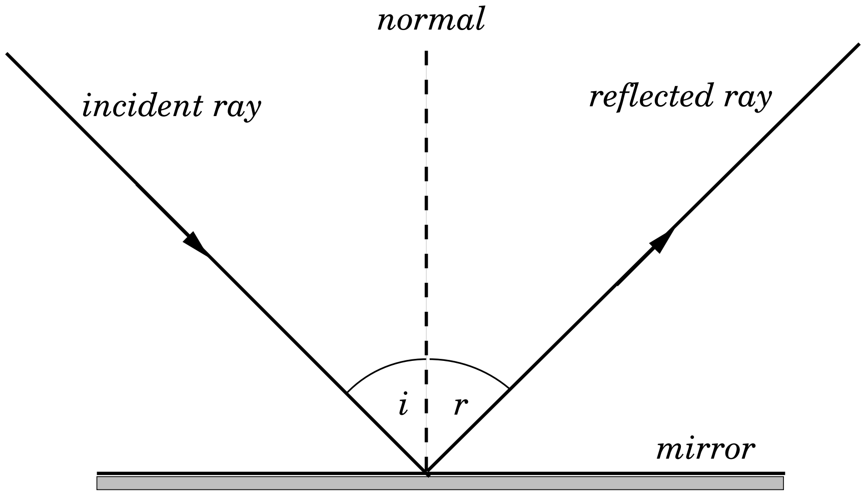


Figure 12.4: *The law of reflection*

Fig. 12.3. Thus, the law of rectilinear propagation of light-rays also specifies how wave-fronts propagate through homogeneous media. Of course, this law is only valid in the limit where the wavelength of the wave is much smaller than the dimensions of any obstacles which it encounters.

12.4 Law of Reflection

The law of reflection governs the reflection of light-rays off smooth conducting surfaces, such as polished metal or metal-coated glass mirrors.

Consider a light-ray incident on a plane mirror, as shown in Fig. 12.4. The law of reflection states that the incident ray, the reflected ray, and the normal to the surface of the mirror all lie in the *same plane*. Furthermore, the angle of reflection r is *equal* to the angle of incidence i. Both angles are measured with respect to the normal to the mirror.

The law of reflection also holds for non-plane mirrors, provided that the normal at any point on the mirror is understood to be the outward pointing normal to the local tangent plane of the mirror at that point. For rough surfaces, the law of reflection remains valid. It predicts that rays incident at slightly different points on the surface are reflected in completely different directions, because the normal to a rough surface varies in direction very strongly from point to point on the surface. This type of reflection is called *diffuse reflection*, and is what enables us to see non-shiny objects.

12.5 Law of Refraction

The law of refraction, which is generally known as *Snell's law*, governs the behaviour of light-rays as they propagate across a sharp interface between two transparent dielectric media.

Consider a light-ray incident on a plane interface between two transparent dielectric media, labelled 1 and 2, as shown in Fig. 12.5. The law of refraction states that the incident ray, the refracted ray, and the normal to the interface, all lie in the *same plane*. Furthermore,

$$n_1\,\sin\theta_1 = n_2\,\sin\theta_2, \tag{12.1}$$

where θ_1 is the angle subtended between the incident ray and the normal to the interface, and θ_2 is the angle subtended between the refracted ray and the normal to the interface. The quantities n_1 and n_2 are termed the *refractive indices* of media 1 and 2, respectively. Thus, the law of refraction predicts that a light-ray always deviates more towards the normal in the optically denser medium: *i.e.*, the medium with the higher refractive index. Note that $n_2 > n_1$ in the figure. The law of refraction also holds for non-planar interfaces, provided that the normal to the interface at any given point is understood to be the normal to the local tangent plane of the interface at that point.

By definition, the refractive index n of a dielectric medium of dielectric constant K is given by

$$n = \sqrt{K}. \tag{12.2}$$

Table 12.1 shows the refractive indices of some common materials (for yellow light of wavelength $\lambda = 589\,\text{nm}$).

The law of refraction follows directly from the fact that the speed v with which light propagates through a dielectric medium is *inversely proportional* to the re-

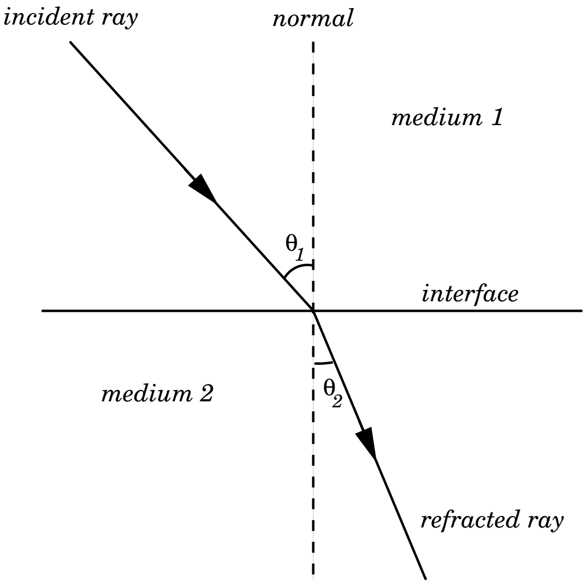


Figure 12.5: *The law of refraction.*

Material	n
Air (STP)	1.00029
Water	1.33
Ice	1.31
Glass:	
Light flint	1.58
Heavy flint	1.65
Heaviest flint	1.89
Diamond	2.42

Table 12.1: *Refractive indices of some common materials at $\lambda = 589$ nm.*

fractive index of the medium (see Sect. 11.3). In fact,

$$v = \frac{c}{n}, \tag{12.3}$$

where c is the speed of light in a vacuum. Consider two parallel light-rays, a and b, incident at an angle θ_1 with respect to the normal to the interface between two dielectric media, 1 and 2. Let the refractive indices of the two media be n_1 and n_2 respectively, with $n_2 > n_1$. It is clear from Fig. 12.6 that ray b must move from point B to point Q, in medium 1, in the same time interval, Δt, in which ray a moves between points A and P, in medium 2. Now, the speed of light in medium 1 is $v_1 = c/n_1$, whereas the speed of light in medium 2 is $v_2 = c/n_2$. It follows that the length BQ is given by $v_1\,\Delta t$, whereas the length AP is given by $v_2\,\Delta t$. By trigonometry,

$$\sin\theta_1 = \frac{BQ}{AQ} = \frac{v_1\,\Delta t}{AQ}, \tag{12.4}$$

and

$$\sin\theta_2 = \frac{AP}{AQ} = \frac{v_2\,\Delta t}{AQ}. \tag{12.5}$$

Hence,

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2} = \frac{n_2}{n_1}, \tag{12.6}$$

which can be rearranged to give Snell's law. Note that the lines AB and PQ represent wave-fronts in media 1 and 2, respectively, and, therefore, cross rays a and b at right-angles.

When light passes from one dielectric medium to another its velocity v changes, but its frequency f remains *unchanged*. Since, $v = f\,\lambda$ for all waves, where λ is the wavelength, it follows that the wavelength of light must also change as it crosses an interface between two different media. Suppose that light propagates from medium 1 to medium 2. Let n_1 and n_2 be the refractive indices of the two media, respectively. The ratio of the wave-lengths in the two media is given by

$$\frac{\lambda_2}{\lambda_1} = \frac{v_2/f}{v_1/f} = \frac{v_2}{v_1} = \frac{n_1}{n_2}. \tag{12.7}$$

Thus, as light moves from air to glass its wavelength *decreases*.

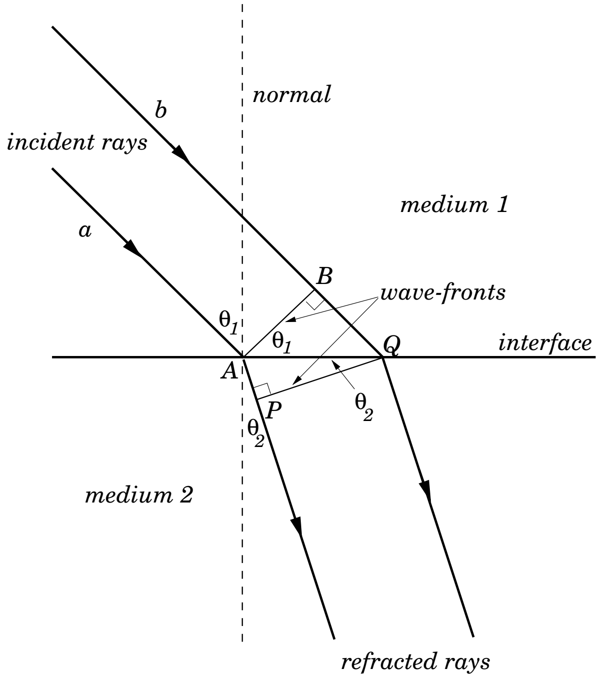


Figure 12.6: *Derivation of Snell's law.*

12.6 Total Internal Reflection

An interesting effect known as *total internal reflection* can occur when light attempts to move from a medium having a given refractive index to a medium having a *lower* refractive index. Suppose that light crosses an interface from medium 1 to medium 2, where $n_2 < n_1$. According to Snell's law,

$$\sin\theta_2 = \frac{n_1}{n_2}\,\sin\theta_1. \tag{12.8}$$

Since $n_1/n_2 > 1$, it follows that $\theta_2 > \theta_1$. For relatively small angles of incidence, part of the light is refracted into the less optically dense medium, and part is reflected (there is always some reflection at an interface). When the angle of incidence θ_1 is such that the angle of refraction $\theta_2 = 90^\circ$, the refracted ray runs along the interface between the two media. This particular angle of incidence is called the *critical angle*, θ_c. For $\theta_1 > \theta_c$, there is *no* refracted ray. Instead, all of the light incident on the interface is reflected—see Fig. 12.7. This effect is called *total internal reflection*, and occurs whenever the angle of incidence exceeds the critical angle. Now when $\theta_1 = \theta_c$, we have $\theta_2 = 90^\circ$, and so $\sin\theta_2 = 1$. It follows

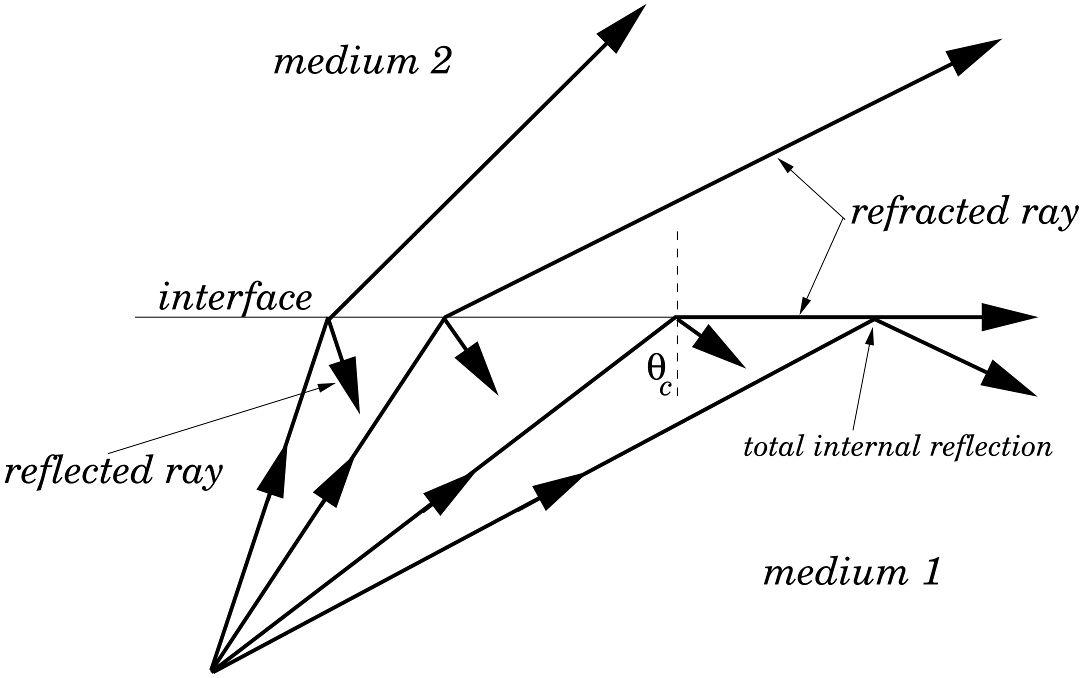


Figure 12.7: *Total internal reflection.*

from Eq. (12.8) that

$$\sin\theta_c = \frac{n_2}{n_1}. \tag{12.9}$$

Consider a fish (or a diver) swimming in a clear pond. As Fig. 12.8 makes clear, if the fish looks upwards it sees the sky, but if it looks at too large an angle to the vertical it sees the bottom of the pond reflected on the surface of the water. The critical angle to the vertical at which the fish first sees the reflection of the bottom of the pond is, of course, equal to the critical angle θ_c for total internal reflection at an air-water interface. From Eq. (12.9), this critical angle is given by

$$\theta_c = \sin^{-1}(1.00/1.33) = 48.8^\circ, \tag{12.10}$$

since the refractive index of air is approximately unity, and the refractive index of water is 1.33.

When total internal reflection occurs at an interface the interface in question acts as a *perfect reflector*. This allows 45° crown glass prisms to be used, in place of mirrors, to reflect light in binoculars. This is illustrated in Fig. 12.9. The angles

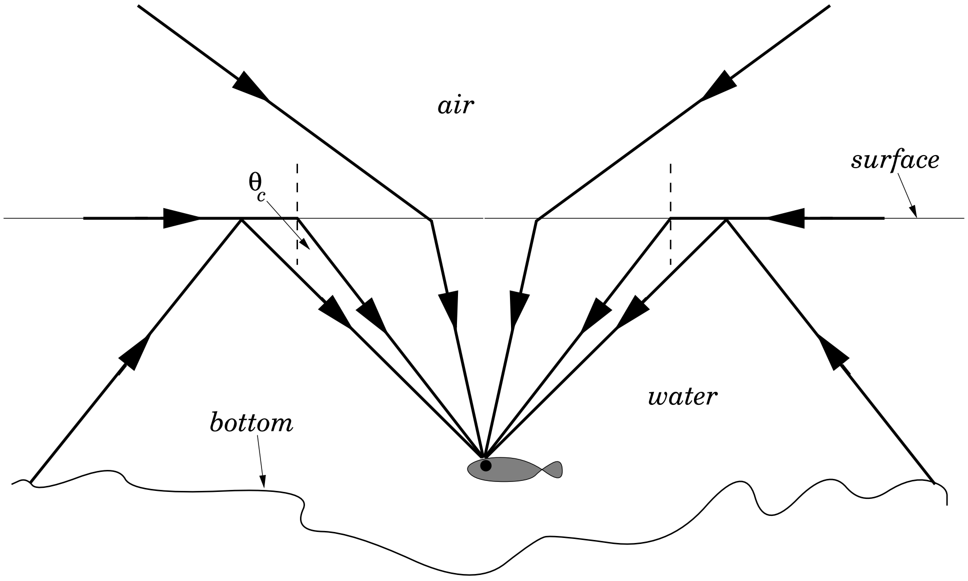


Figure 12.8: *A fish's eye view.*

of incidence on the sides of the prism are all 45°, which is greater than the critical angle 41° for crown glass (at an air-glass interface).

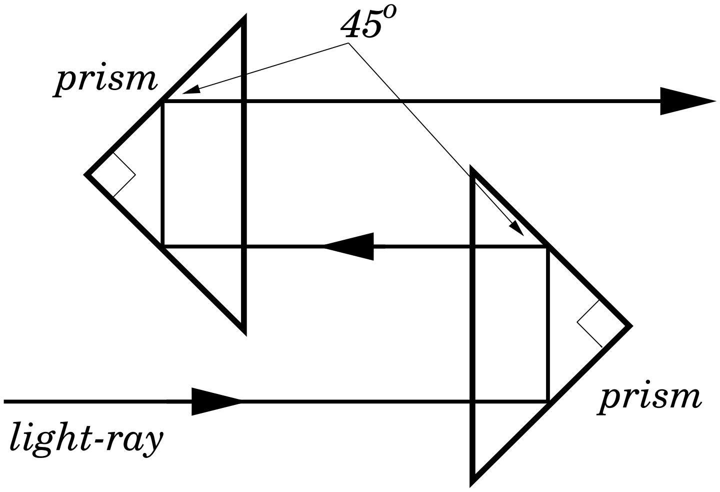


Figure 12.9: *Arrangement of prisms used in binoculars.*

Diamonds, for which $n = 2.42$, have a critical angle θ_c which is only 24°. The facets on a diamond are cut in such a manner that much of the incident light on the diamond is reflected many times by successive total internal reflections before it escapes. This effect gives rise to the characteristic sparkling of cut diamonds.

Total internal reflection enables light to be transmitted inside thin glass fibers. The light is internally reflected off the sides of the fiber, and, therefore, follows the path of the fiber. Light can actually be transmitted around corners using a glass fiber, provided that the bends in the fiber are not too sharp, so that the light always strikes the sides of the fiber at angles greater than the critical angle. The whole field of *fiber optics*, with its many useful applications, is based on this effect.

12.7 Dispersion

When a wave is refracted into a dielectric medium whose refractive index *varies* with wavelength then the angle of refraction also varies with wavelength. If the incident wave is not monochromatic, but is, instead, composed of a mixture of waves of different wavelengths, then each component wave is refracted through a *different* angle. This phenomenon is called *dispersion*.

Figure 12.10 shows the refractive indices of some common materials as functions of wavelength in the visible range. It can be seen that the refractive index always *decreases* with increasing wavelength in the visible range. In other words, violet light is always refracted *more strongly* than red light.

Suppose that a parallel-sided glass slab is placed in a beam of white light. Dispersion takes place inside the slab, but, since the rays which emerge from the slab all run *parallel* to one another, the dispersed colours recombine to form white light again, and no dispersion is observed except at the very edges of the beam. This is illustrated in Fig. 12.11. It follows that the dispersion of white light through a parallel-sided glass slab is not generally a noticeable effect.

Suppose that a glass prism is placed in a beam of white light. Dispersion takes place inside the prism, and, since the emerging rays are *not parallel* for different colours, the dispersion is clearly noticeable, especially if the emerging rays are projected onto a screen which is placed a long way from the prism. This is illustrated in Fig. 12.12. It is clear that a glass prism is far more effective at separating white light into its component colours than a parallel-sided glass slab

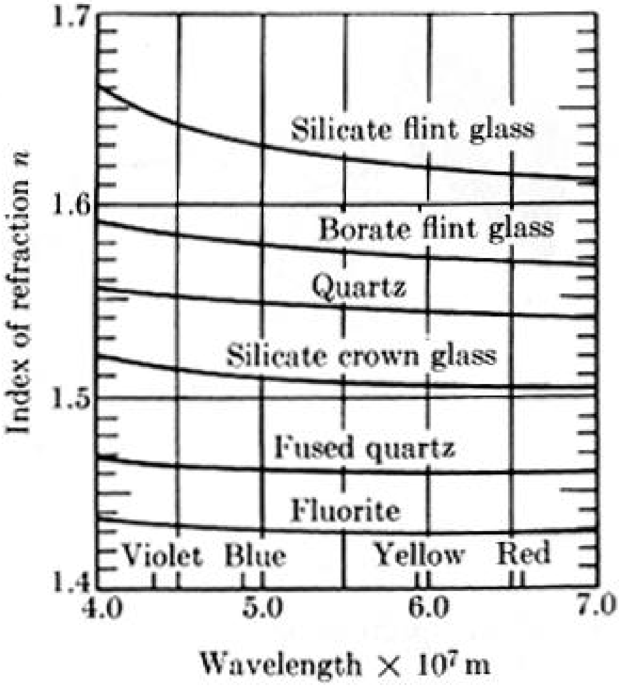


Figure 12.10: *Refractive indices of some common materials as functions of wavelength.*

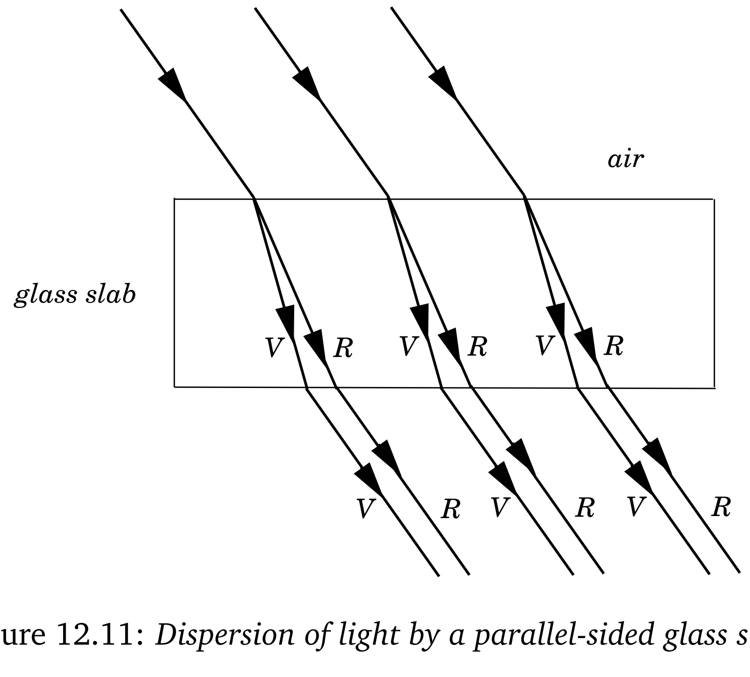


Figure 12.11: *Dispersion of light by a parallel-sided glass slab.*

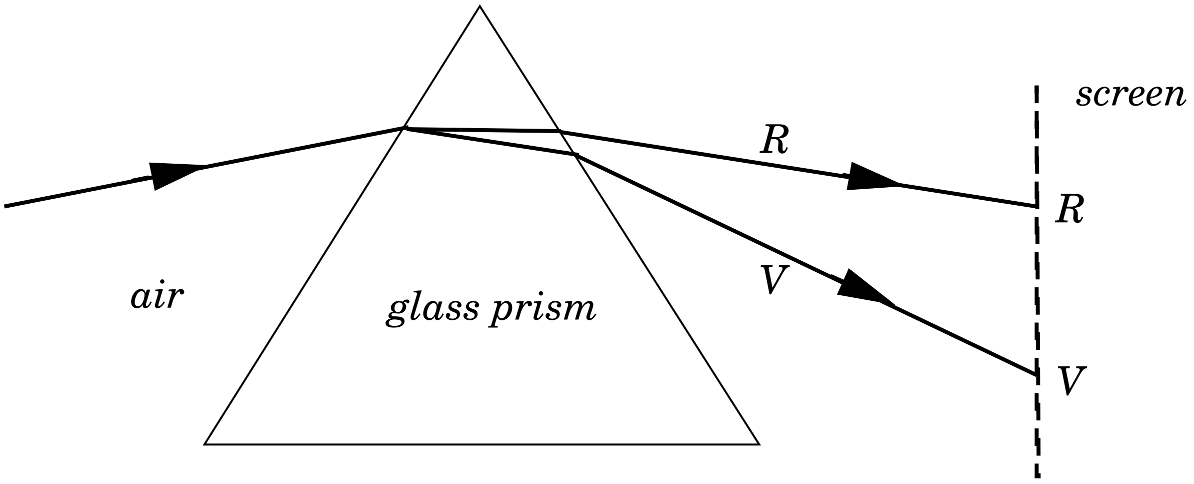


Figure 12.12: *Dispersion of light by a glass prism.*

(which explains why prisms are generally employed to perform this task).

12.8 Rainbows

The most well-known, naturally occurring phenomenon which involves the dispersion of light is a *rainbow*. A rainbow is an *arc* of light, with an angular radius of 42°, centred on a direction which is *opposite* to that of the Sun in the sky (*i.e.*, it is centred on the direction of propagation of the Sun's rays)—see Fig. 12.13. Thus, if the Sun is low in the sky (*i.e.*, close to the horizon) we see almost a full semi-circle. If the Sun is higher in the sky we see a smaller arc, and if the Sun is more than 42° above the horizon then there is no rainbow (for viewers on the Earth's surface). Observers on a hill may see parts of the rainbow below the horizontal: *i.e.*, an arc greater than a semi-circle. Passengers on an airplane can sometimes see a full circle.

The colours of a rainbow vary smoothly from red on the outside of the arc to violet on the inside. A rainbow has a diffuse inner edge, and a sharp outer edge. Sometimes a *secondary arc* is observed. This is fainter and larger (with an angular radius of 50°) than the primary arc, and the order of the colours is reversed (*i.e.*, red is on the inside, and violet on the outside). The secondary arc has a diffuse outer edge, and a sharp inner edge. The sky between the two arcs sometimes

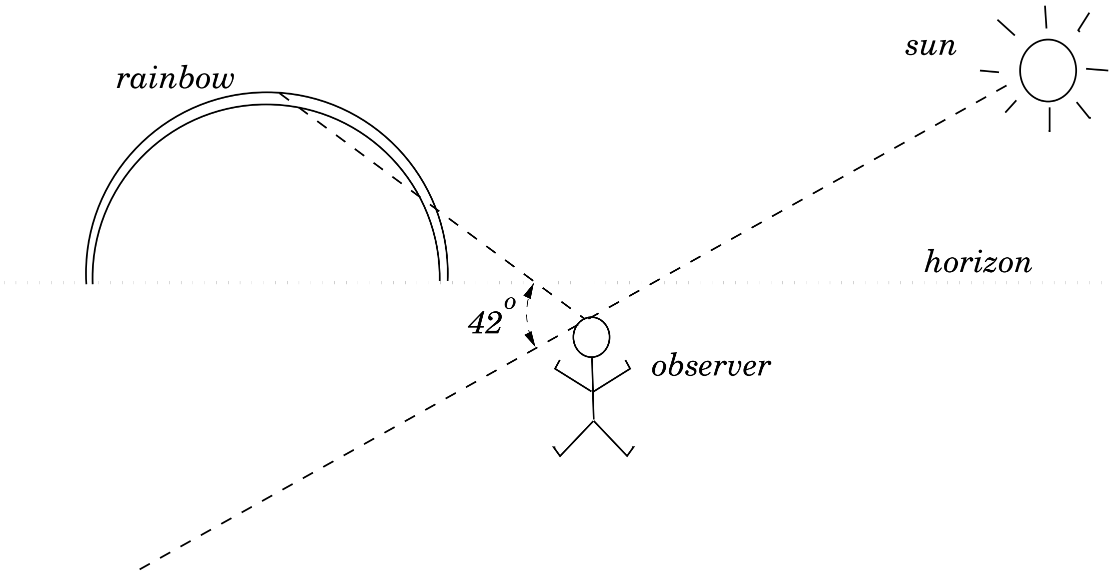


Figure 12.13: *A rainbow.*

appears to be less bright than the sky elsewhere. This region is called *Alexander's dark band*, in honour of Alexander of Aphrodisias who described it some 1800 years ago.

Rainbows have been studied since ancient times. Aristotle wrote extensively on rainbows in his *De Meteorologica*,[2] and even speculated that a rainbow is caused by the reflection of sunlight from the drops of water in a cloud.

The first scientific study of rainbows was performed by Theodoric, professor of theology at Freiburg, in the fourteenth century. He studied the path of a light-ray through a spherical globe of water in his laboratory, and suggested that the globe be thought of as a model of a single falling raindrop. A ray, from the Sun, entering the drop, is refracted at the air-water interface, undergoes internal reflection from the inside surface of the drop, and then leaves the drop in a backward direction, after being again refracted at the surface. Thus, looking away from the Sun, towards a cloud of raindrops, one sees an enhancement of light due to these rays. Theodoric did not explain why this enhancement is concentrated at a particular angle from the direction of the Sun's rays, or why the light is split into different colours.

[2] "On Weather".

The first person to give a full explanation of how a rainbow is formed was René Descartes. He showed mathematically that if one traces the path through a spherical raindrop of parallel light-rays entering the drop at different points on its surface, each emerges in a different direction, but there is a concentration of emerging rays at an angle of 42° from the reverse direction to the incident rays, in exact agreement with the observed angular size of rainbows. Furthermore, since some colours are refracted more than others in a raindrop, the "rainbow angle" is slightly different for each colour, so a raindrop disperses the Sun's light into a set of nearly overlapping coloured arcs.

Figure 12.13 illustrated Descartes' theory in more detail. It shows parallel light-rays entering a spherical raindrop. Only rays entering the upper half contribute to the rainbow effect. Let us follow the rays, one by one, from the top down to the middle of the drop. We observe the following pattern. Rays which enter near the top of the drop emerge going in almost the reverse direction, but a few degrees below the horizontal. Rays entering a little further below the top emerge at a greater angle below the horizontal. Eventually, we reach a critical ray, called the *rainbow ray*, which emerges in an angle 42° below the horizontal. Rays entering the drop lower than the rainbow ray emerge at an angle less than 42°. Thus, the rainbow ray is the one which deviates *most* from the reverse direction to the incident rays. This variation, with 42° being the maximum angle of deviation from the reverse direction, leads to a bunching of rays at that angle, and, hence, to an unusually bright arc of reflected light centred around 42° from the reverse direction. The arc has a sharp outer edge, since reflected light *cannot* deviate by more than 42° from the reverse direction, and a diffuse inner edge, since light *can* deviate by less than 42° from the reverse direction: 42° is just the *most likely* angle of deviation. Finally, since the rainbow angle varies slightly with wavelength (because the refractive index of water varies slightly with wavelength), the arcs corresponding to each colour appear at slightly different angles relative to the reverse direction to the incident rays. We expect violet light to be refracted more strongly than red light in a raindrop. It is, therefore, clear, from Fig. 12.14, that the red arc deviates slightly more from the reverse direction to the incident rays than the violet arc. In other words, violet is concentrated on the inside of the rainbow, and red is concentrated on the outside.

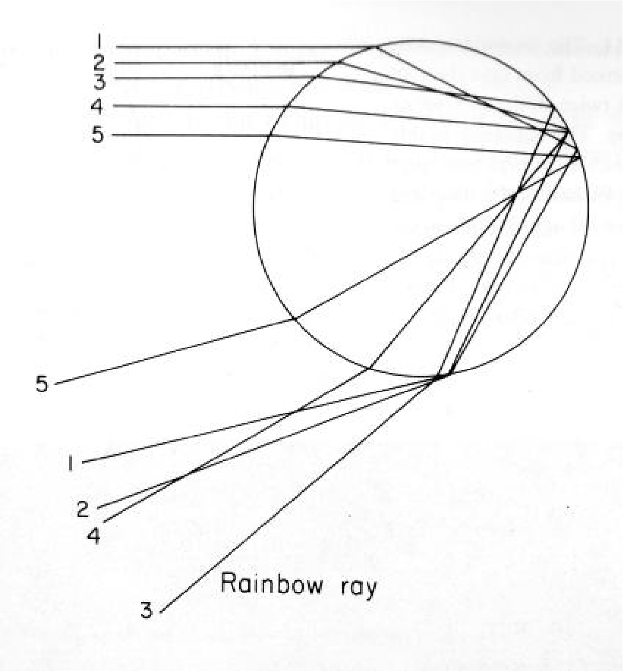


Figure 12.14: *Descarte's theory of the rainbow.*

Descartes was also able to show that light-rays which are internally reflected *twice* inside a raindrop emerge concentrated at an angle of 50° from the reverse direction to the incident rays. Of course, this angle corresponds exactly to the angular size of the secondary rainbow sometimes seen outside the first. This rainbow is naturally less intense than the primary rainbow, since a light-ray loses some of its intensity at each reflection or refraction event. Note that 50° represents the angle of *maximum* deviation of doubly reflected light from the reverse direction (*i.e.*, doubly reflected light can deviate by more than this angle, but not by less). Thus, we expect the secondary rainbow to have a diffuse outer edge, and a sharp inner edge. We also expect doubly reflected violet light to be refracted more strongly in a raindrop than doubly reflected red light. It follows, from Fig. 12.15, that the red secondary arc deviates slightly less from the reverse direction to the incident rays than the violet secondary arc. In other words, red is concentrated on the inside of the secondary rainbow, and violet on the outside. Since no reflected light emerges between the primary and secondary rainbows (*i.e.*, in the angular range 42° to 50°, relative to the reverse direction), we naturally expect this region of the sky to look slightly less bright than the other

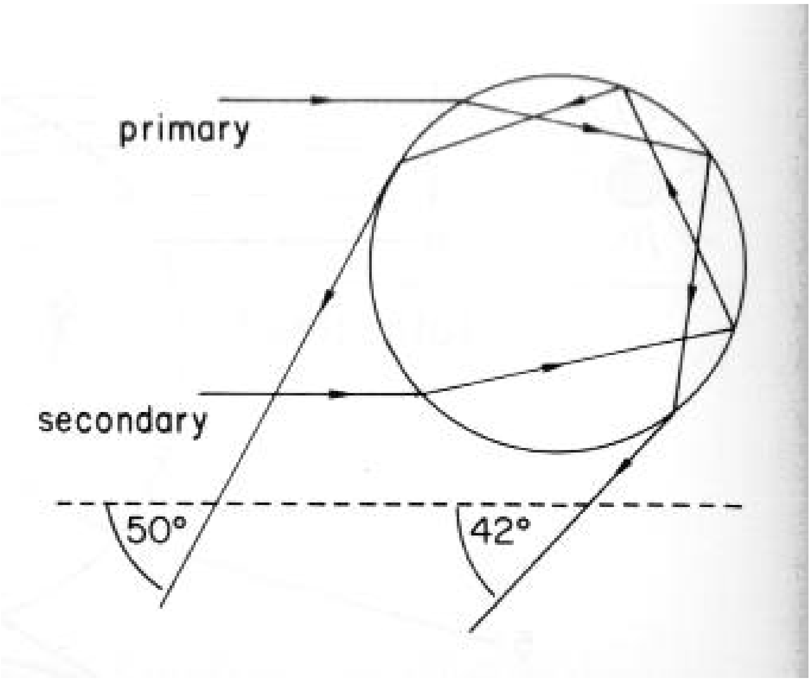


Figure 12.15: *Rainbow rays for the primary and secondary arcs of a rainbow.*

surrounding regions of the sky, which explains Alexander's dark band.

12.9 Worked Examples

Example 12.1: The corner-cube reflector

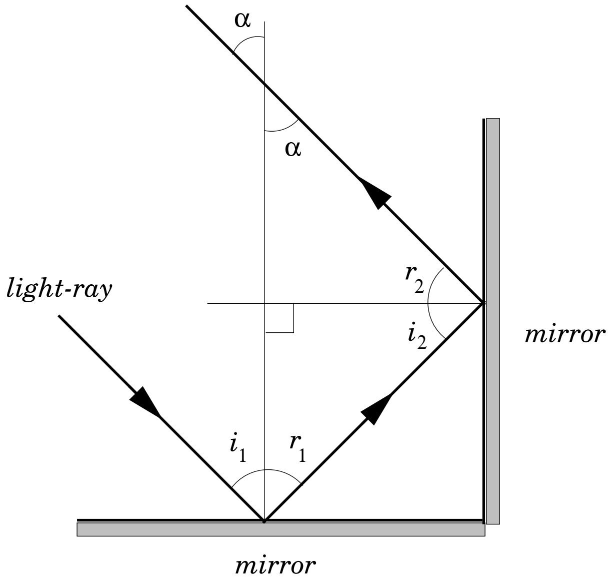
Question: Two mirrors are placed at right-angles to one another. Show that a light-ray incident from *any* direction in the plane perpendicular to both mirrors is reflected through 180°.

Answer: Consider the diagram. We are effectively being asked to prove that $\alpha = i_1$, for any value of i_1. Now, from trigonometry,

$$i_2 = 90^\circ - r_1.$$

But, from the law of reflection, $r_1 = i_1$ and $i_2 = r_2$, so

$$r_2 = 90^\circ - i_1.$$



Trigonometry also yields

$$\alpha = 90^\circ - r_2.$$

It follows from the previous two equations that

$$\alpha = 90^\circ - (90^\circ - i_1) = i_1.$$

Hence, $\alpha = i_1$, for all values of i_1.

It can easily be appreciated that a combination of *three* mutually perpendicular mirrors would reflect a light-ray incident from *any* direction through 180°. Such a combination of mirrors is called a *corner-cube reflector*. Astronauts on the Apollo 11 mission (1969) left a panel of corner-cube reflectors on the surface of the Moon. These reflectors have been used ever since to measure the Earth-Moon distance via laser range finding (basically, a laser beam is fired from the Earth, reflects off the corner-cube reflectors on the Moon, and then returns to the Earth. The time of travel of the beam can easily be converted into the Earth-Moon distance). The Earth-Moon distance can be measured to within an accuracy of 3 cm using this method.

Example 12,2: Refraction

Question: A light-ray of wavelength $\lambda_1 = 589\,\mathrm{nm}$ traveling through air is incident on a smooth, flat slab of crown glass (refractive index 1.52) at an angle of $\theta_1 = 30.0^\circ$ to the normal. What is the angle of refraction? What is the wavelength λ_2 of the light inside the glass? What is the frequency f of the light inside the glass?

Answer: Snell's law can be written

$$\sin\theta_2 = \frac{n_1}{n_2}\,\sin\theta_1.$$

In this case, $\theta_1 = 30^\circ$, $n_1 \simeq 1.00$ (here, we neglect the slight deviation of the refractive index of air from that of a vacuum), and $n_2 = 1.52$. Thus,

$$\sin\theta_2 = \frac{(1.00)}{(1.52)}\,(0.5) = 0.329,$$

giving

$$\theta_2 = 19.2^\circ$$

as the angle of refraction (measured with respect to the normal).

The wavelength λ_2 of the light inside the glass is given by

$$\lambda_2 = \frac{n_1}{n_2}\,\lambda_1 = \frac{(1.00)}{(1.52)}\,(589) = 387.5\,\mathrm{nm}.$$

The frequency f of the light inside the glass is exactly the same as the frequency outside the glass, and is given by

$$f = \frac{c}{n_1\,\lambda_1} = \frac{(3\times10^{8})}{(1.00)\,(589\times10^{-9})} = 5.09\times10^{14}\,\mathrm{Hz}.$$

13 Paraxial Optics

13.1 Spherical Mirrors

A spherical mirror is a mirror which has the shape of a piece cut out of a spherical surface. There are two types of spherical mirrors: *concave*, and *convex*. These are illustrated in Fig. 13.1. The most commonly occurring examples of concave mirrors are shaving mirrors and makeup mirrors. As is well-known, these types of mirrors magnify objects placed close to them. The most commonly occurring examples of convex mirrors are the passenger-side wing mirrors of cars. These type of mirrors have wider fields of view than equivalent flat mirrors, but objects which appear in them generally look smaller (and, therefore, farther away) than they actually are.

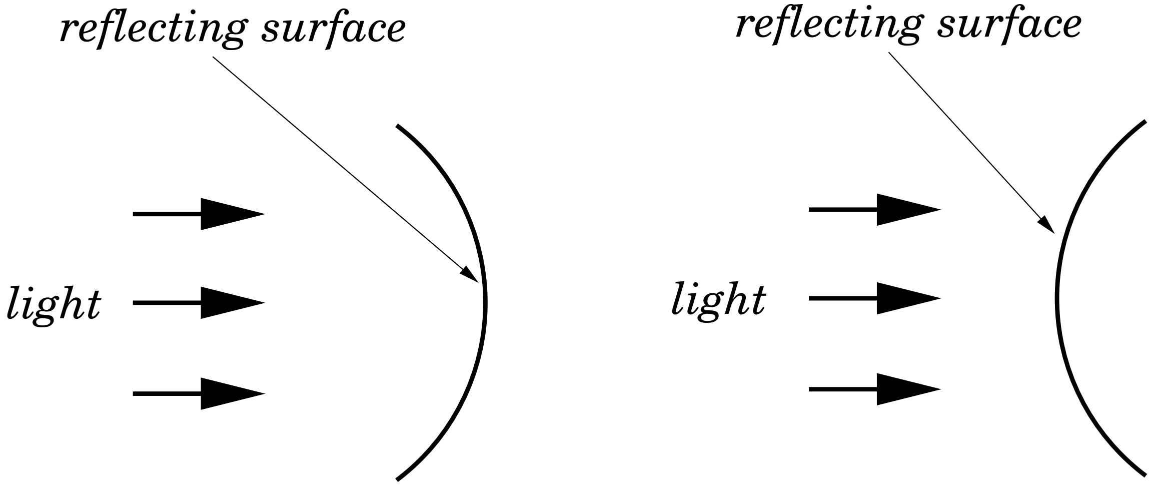


Figure 13.1: *A concave (left) and a convex (right) mirror*

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Let us now introduce a few key concepts which are needed to study image formation by a concave spherical mirror. As illustrated in Fig. 13.2, the normal to the centre of the mirror is called the *principal axis*. The mirror is assumed to be *rotationally symmetric* about this axis. Hence, we can represent a three-dimensional mirror in a two-dimensional diagram, without loss of generality. The point V at which the principal axis touches the surface of the mirror is called the *vertex*. The point C, on the principal axis, which is equidistant from all points on the reflecting surface of the mirror is called the *centre of curvature*. The distance along the principal axis from point C to point V is called the *radius of curvature* of the mirror, and is denoted R. It is found experimentally that rays striking a concave mirror parallel to its principal axis, and not too far away from this axis, are reflected by the mirror such that they all pass through the same point F on the principal axis. This point, which is lies between the centre of curvature and the vertex, is called the *focal point*, or *focus*, of the mirror. The distance along the principal axis from the focus to the vertex is called the *focal length* of the mirror, and is denoted f.

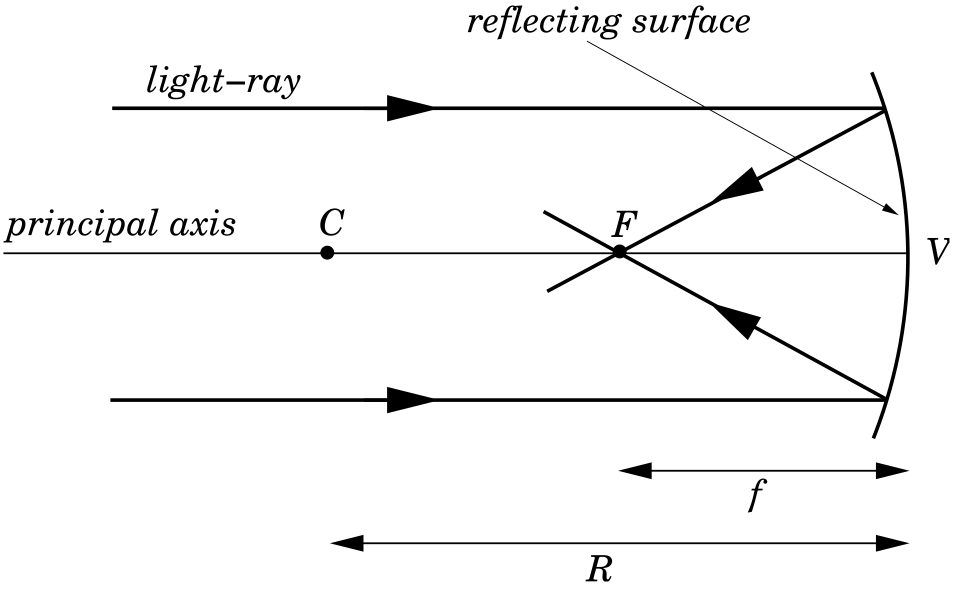

Figure 13.2: *Image formation by a concave mirror.*

In our study of concave mirrors, we are going to assume that all light-rays which strike a mirror parallel to its principal axis (*e.g.*, all rays emanating from a distant object) are brought to a focus at the same point F. Of course, as mentioned above, this is only an approximation. It turns out that as rays from a distant object depart further from the principal axis of a concave mirror they are brought to a focus ever closer to the mirror, as shown in Fig. 13.3. This lack of perfect focusing of a spherical mirror is called *spherical aberration*. The approximation in which we neglect spherical aberration is called the *paraxial approximation*.[3]

[3] “Paraxial” is derived from ancient Greek roots, and means “close to the axis”.

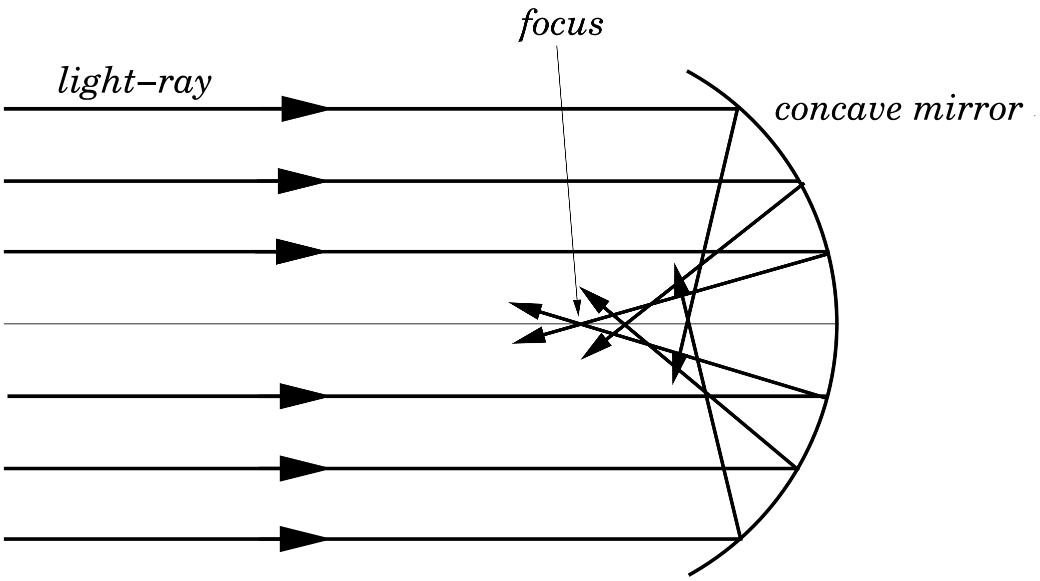

Figure 13.3: *Spherical aberration in a concave mirror.*

Likewise, the study of image formation under this approximation is known as *paraxial optics*. This field of optics was first investigated systematically by the famous German mathematician Karl Friedrich Gauss in 1841.

It can be demonstrated, by geometry, that the only type of mirror which does not suffer from spherical aberration is a *parabolic* mirror (*i.e.*, a mirror whose reflecting surface is the surface of revolution of a parabola). Thus, a ray traveling parallel to the principal axis of a parabolic mirror is brought to a focus at the same point F, no matter how far the ray is from the axis. Since the path of a light-ray is completely *reversible*, it follows that a light source placed at the focus F of a parabolic mirror yields a perfectly parallel beam of light, after the light has reflected off the surface of the mirror. Parabolic mirrors are more difficult, and, therefore, more expensive, to make than spherical mirrors. Thus, parabolic mirrors are only used in situations where the spherical aberration of a conventional spherical mirror would be a serious problem. The receiving dishes of radio telescopes are generally parabolic. They reflect the incoming radio waves from (very) distant astronomical sources, and bring them to a focus at a single point, where a detector is placed. In this case, since the sources are extremely faint, it is imperative to avoid the signal losses which would be associated with spherical aberration. A car headlight consists of a light-bulb placed at the focus of a parabolic reflector. The use of a parabolic reflector enables the headlight to cast a very straight beam of light ahead of the car. The beam would be nowhere near as well-focused were a spherical reflector used instead.

13.2 Image Formation by Concave Mirrors

There are two alternative methods of locating the image formed by a concave mirror. The first is purely graphical, and the second uses simple algebraic analysis.

The graphical method of locating the image produced by a concave mirror consists of drawing light-rays emanating from key points on the object, and finding where these rays are brought to a focus by the mirror. This task can be accomplished using just *four* simple rules:

1. An incident ray which is parallel to the principal axis is reflected through the focus F of the mirror.
2. An incident ray which passes through the focus F of the mirror is reflected parallel to the principal axis.
3. An incident ray which passes through the centre of curvature C of the mirror is reflected back along its own path (since it is normally incident on the mirror).
4. An incident ray which strikes the mirror at its vertex V is reflected such that its angle of incidence with respect to the principal axis is equal to its angle of reflection.

The validity of these rules in the paraxial approximation is fairly self-evident.

Consider an object ST which is placed a distance p from a concave spherical mirror, as shown in Fig. 13.4. For the sake of definiteness, let us suppose that the object distance p is greater than the focal length f of the mirror. Each point on the object is assumed to radiate light-rays in all directions. Consider four light-rays emanating from the tip T of the object which strike the mirror, as shown

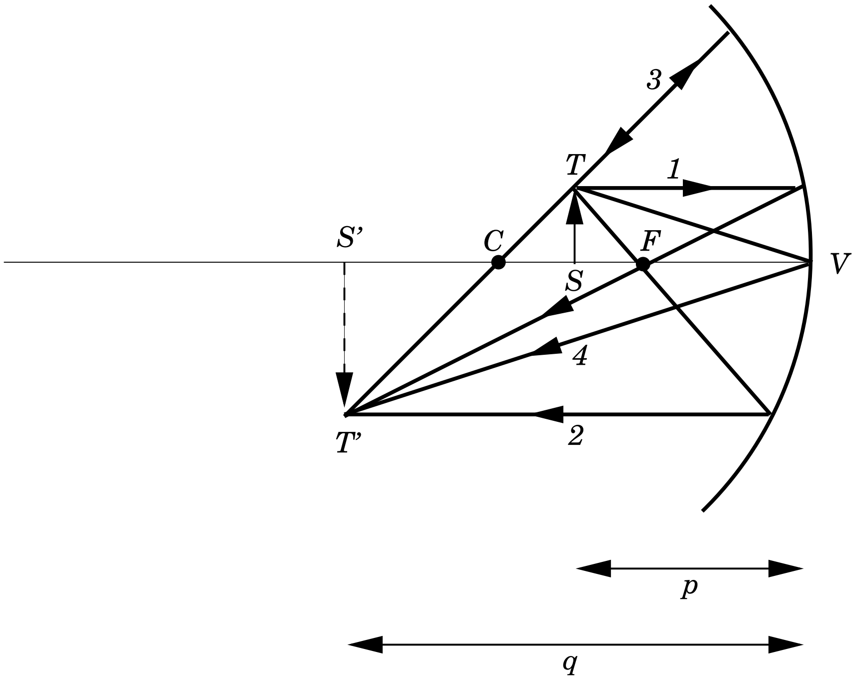

Figure 13.4: *Formation of a real image by a concave mirror.*

in the figure. The reflected rays are constructed using rules 1–4 above, and the rays are labelled accordingly. It can be seen that the reflected rays all come together at some point T'. Thus, T' is the image of T (*i.e.*, if we were to place a small projection screen at T' then we would see an image of the tip on the screen). As is easily demonstrated, rays emanating from other parts of the object are brought into focus in the vicinity of T' such that a complete image of the object is produced between S' and T' (obviously, point S' is the image of point S). This image could be viewed by projecting it onto a screen placed between points S' and T'. Such an image is termed a *real image*. Note that the image $S'T'$ would also be directly visible to an observer looking straight at the mirror from a distance greater than the image distance q (since the observer's eyes could not tell that the light-rays diverging from the image were in anyway different from those which would emanate from a real object). According to the figure, the image is *inverted* with respect to the object, and is also *magnified*.

Figure 13.5 shows what happens when the object distance p is less than the focal length f. In this case, the image appears to an observer looking straight at the mirror to be located *behind* the mirror. For instance, rays emanating from

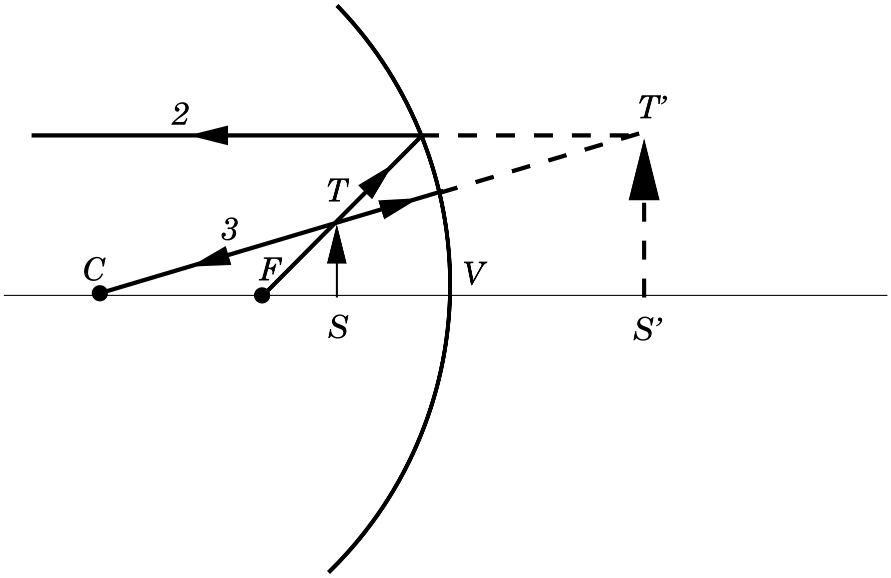

Figure 13.5: *Formation of a virtual image by a concave mirror.*

the tip T of the object appear, after reflection from the mirror, to come from a point T' which is behind the mirror. Note that only two rays are used to locate T', for the sake of clarity. In fact, *two* is the minimum number of rays needed to locate a point image. Of course, the image behind the mirror cannot be viewed by projecting it onto a screen, because there are no real light-rays behind the mirror. This type of image is termed a *virtual image*. The characteristic difference between a real image and a virtual image is that, immediately after reflection from the mirror, light-rays emitted by the object *converge* on a real image, but *diverge* from a virtual image. According to Fig. 13.5, the image is *upright* with respect to the object, and is also *magnified*.

The graphical method described above is fine for developing an intuitive understanding of image formation by concave mirrors, or for checking a calculation, but is a bit too cumbersome for everyday use. The analytic method described below is far more flexible.

Consider an object ST placed a distance p in front of a concave mirror of radius of curvature R. In order to find the image $S'T'$ produced by the mirror, we draw two rays from T to the mirror—see Fig. 13.6. The first, labelled 1, travels from T to the vertex V and is reflected such that its angle of incidence θ equals its angle

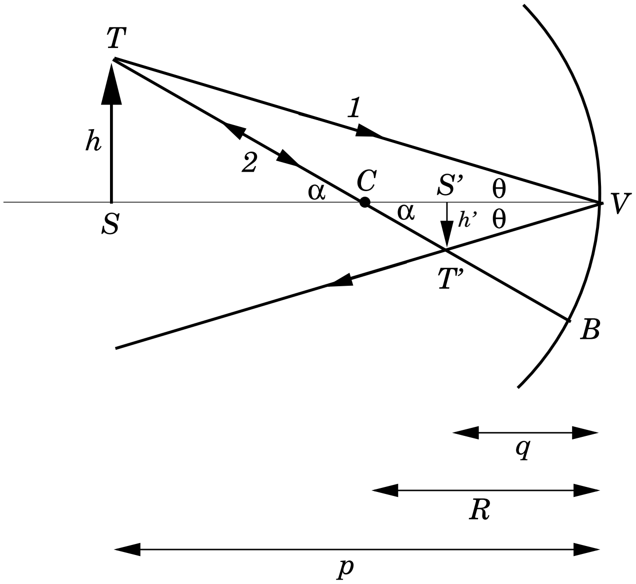

Figure 13.6: *Image formation by a concave mirror.*

of reflection. The second ray, labelled 2, passes through the centre of curvature C of the mirror, strikes the mirror at point B, and is reflected back along its own path. The two rays meet at point T'. Thus, $S'T'$ is the image of ST, since point S' must lie on the principal axis.

In the triangle STV, we have $\tan\theta = h/p$, and in the triangle $S'T'V$ we have $\tan\theta = -h'/q$, where p is the object distance, and q is the image distance. Here, h is the height of the object, and h' is the height of the image. By convention, h' is a negative number, since the image is inverted (if the image were upright then h' would be a positive number). It follows that

$$\tan\theta = \frac{h}{p} = \frac{-h'}{q}. \tag{13.1}$$

Thus, the *magnification* M of the image with respect to the object is given by

$$M = \frac{h'}{h} = -\frac{q}{p}. \tag{13.2}$$

By convention, M is negative if the image is inverted with respect to the object, and positive if the image is upright. It is clear that the magnification of the image is just determined by the ratio of the image and object distances from the vertex.

From triangles STC and S′T′C, we have $\tan\alpha = h/(p-R)$ and $\tan\alpha = -h'/(R-q)$, respectively. These expressions yield

$$\tan\alpha = \frac{h}{p-R} = -\frac{h'}{R-q}. \tag{13.3}$$

Equations (13.2) and (13.3) can be combined to give

$$\frac{-h'}{h} = \frac{R-q}{p-R} = \frac{q}{p}, \tag{13.4}$$

which easily reduces to

$$\frac{1}{p} + \frac{1}{q} = \frac{2}{R}. \tag{13.5}$$

This expression relates the object distance, the image distance, and the radius of curvature of the mirror.

For an object which is very far away from the mirror (*i.e.*, $p \rightarrow \infty$), so that light-rays from the object are parallel to the principal axis, we expect the image to form at the focal point F of the mirror. Thus, in this case, $q = f$, where f is the focal length of the mirror, and Eq. (13.5) reduces to

$$0 + \frac{1}{f} = \frac{2}{R}. \tag{13.6}$$

The above expression yields

$$f = \frac{R}{2}. \tag{13.7}$$

In other words, in the paraxial approximation, the focal length of a concave spherical mirror is *half* of its radius of curvature. Equations (13.5) and (13.7) can be combined to give

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}. \tag{13.8}$$

The above expression was derived for the case of a real image. However, as is easily demonstrated, it also applies to virtual images provided that the following sign convention is adopted. For real images, which always form *in front* of the mirror, the image distance q is *positive*. For virtual images, which always form

Position of object	*Position of image*	*Character of image*
At ∞	At F	Real, zero size
Between ∞ and C	Between F and C	Real, inverted, diminished
At C	At C	Real, inverted, same size
Between C and F	Between C and ∞	Real, inverted, magnified
At F	At ∞	
Between F and V	From $-\infty$ to V	Virtual, upright, magnified
At V	At V	Virtual, upright, same size

Table 13.1: *Rules for image formation by concave mirrors.*

behind the mirror, the image distance q is *negative*. It immediately follows, from Eq. (13.2), that real images are always inverted, and virtual images are always upright. Table 13.1 shows how the location and character of the image formed in a concave spherical mirror depend on the location of the object, according to Eqs. (13.2) and (13.8). It is clear that the *modus operandi* of a shaving mirror, or a makeup mirror, is to place the object (*i.e.*, a face) between the mirror and the focus of the mirror. The image is upright, (apparently) located behind the mirror, and magnified.

13.3 Image Formation by Convex Mirrors

The definitions of the principal axis, centre of curvature C, radius of curvature R, and the vertex V, of a convex mirror are analogous to the corresponding definitions for a concave mirror. When parallel light-rays strike a convex mirror they are reflected such that they appear to emanate from a single point F located behind the mirror, as shown in Fig. 13.7. This point is called the *virtual focus* of the mirror. The focal length f of the mirror is simply the distance between V and F. As is easily demonstrated, in the paraxial approximation, the focal length of a convex mirror is half of its radius of curvature.

There are, again, two alternative methods of locating the image formed by a convex mirror. The first is graphical, and the second analytical.

According to the graphical method, the image produced by a convex mirror can always be located by drawing a ray diagram according to *four* simple rules:

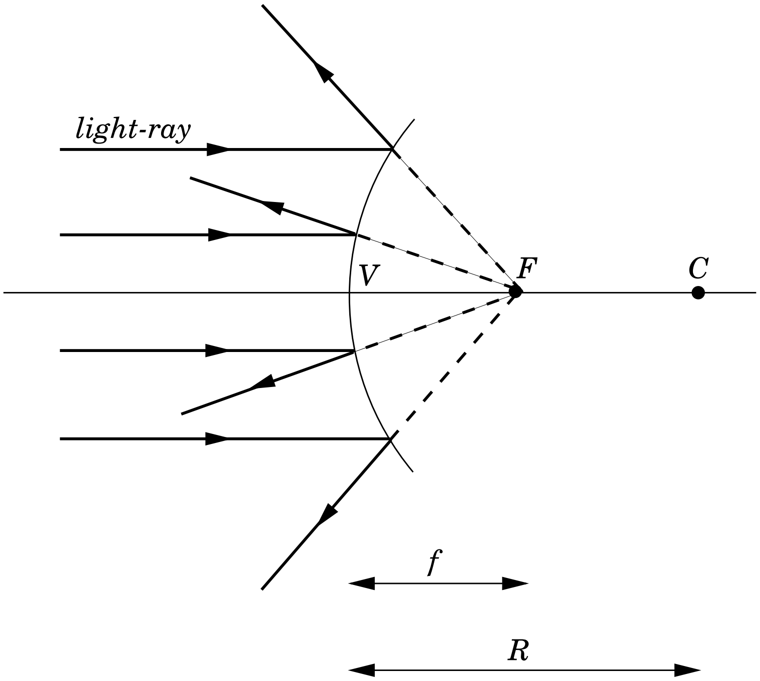

Figure 13.7: *The virtual focus of a convex mirror.*

1. An incident ray which is parallel to the principal axis is reflected as if it came from the virtual focus F of the mirror.
2. An incident ray which is directed towards the virtual focus F of the mirror is reflected parallel to the principal axis.
3. An incident ray which is directed towards the centre of curvature C of the mirror is reflected back along its own path (since it is normally incident on the mirror).
4. An incident ray which strikes the mirror at its vertex V is reflected such that its angle of incidence with respect to the principal axis is equal to its angle of reflection.

The validity of these rules in the paraxial approximation is, again, fairly self-evident.

In the example shown in Fig. 13.8, two rays are used to locate the image $S'T'$ of an object ST placed in front of the mirror. It can be seen that the image is virtual, upright, and diminished.

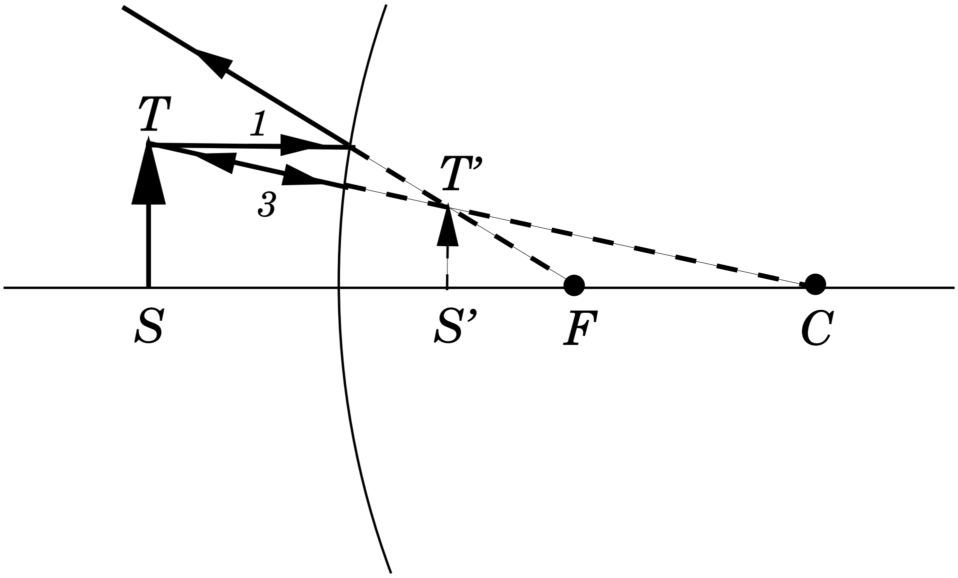

Figure 13.8: *Image formation by a convex mirror.*

Position of object	*Position of image*	*Character of image*
At ∞	At F	Virtual, zero size
Between ∞ and V	Between F and V	Virtual, upright, diminished
At V	At V	Virtual, upright, same size

Table 13.2: *Rules for image formation by convex mirrors.*

As is easily demonstrated, application of the analytical method to image formation by convex mirrors again yields Eq. (13.2) for the magnification of the image, and Eq. (13.8) for the location of the image, provided that we adopt the following sign convention. The focal length f of a convex mirror is redefined to be *minus* the distance between points V and F. In other words, the focal length of a concave mirror, with a real focus, is always positive, and the focal length of a convex mirror, with a virtual focus, is always negative. Table 13.2 shows how the location and character of the image formed in a convex spherical mirror depend on the location of the object, according to Eqs. (13.2) and (13.8) (with $f < 0$).

In summary, the formation of an image by a spherical mirror involves the *crossing* of light-rays emitted by the object and reflected off the mirror. If the light-rays actually cross in front of the mirror then the image is real. If the light-rays do not actually cross, but appear to cross when projected backwards behind the mirror, then the image is virtual. A real image can be projected onto a screen, a virtual image cannot. However, both types of images can be seen by an observer, and photographed by a camera. The magnification of the image is specified by Eq. (13.2), and the location of the image is determined by Eq. (13.8). These two formulae can be used to characterize both real and virtual images formed by either concave or convex mirrors, provided that the following sign conventions are observed:

1. The height h' of the image is positive if the image is upright, with respect to the object, and negative if the image is inverted.
2. The magnification M of the image is positive if the image is upright, with respect to the object, and negative if the image is inverted.
3. The image distance q is positive if the image is real, and, therefore, located in front of the mirror, and negative if the image is virtual, and, therefore, located behind the mirror.
4. The focal length f of the mirror is positive if the mirror is concave, so that the focal point F is located in front of the mirror, and negative if the mirror is convex, so that the focal point F is located behind the mirror.

Note that the front side of the mirror is defined to be the side from which the light is incident.

13.4 Image Formation by Plane Mirrors

Both concave and convex spherical mirrors asymptote to plane mirrors in the limit in which their radii of curvature R tend to infinity. In other words, a plane mirror can be treated as either a concave or a convex mirror for which $R \rightarrow \infty$. Now, if $R \rightarrow \infty$, then $f = \pm R/2 \rightarrow \infty$, so $1/f \rightarrow 0$, and Eq. (13.8) yields

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f} = 0, \tag{13.9}$$

or

$$q = -p. \tag{13.10}$$

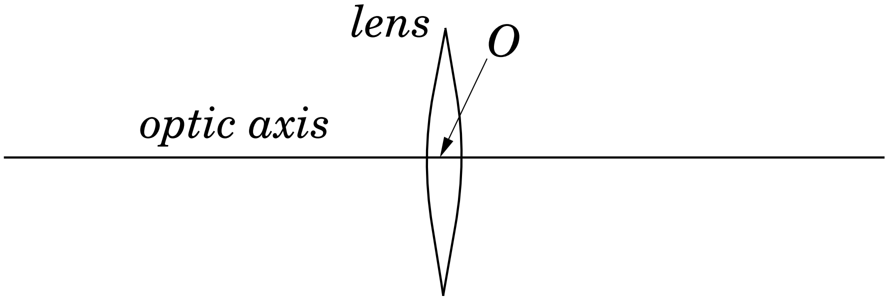

Figure 13.9: *The optic axis of a lens.*

Thus, for a plane mirror the image is *virtual*, and is located as far behind the mirror as the object is in front of the mirror. According to Eq. (13.2), the magnification of the image is given by

$$M = -\frac{q}{p} = 1. \tag{13.11}$$

Clearly, the image is *upright*, since $M > 0$, and is the *same size* as the object, since $|M| = 1$. However, an image seen in a plane mirror does differ from the original object in one important respect: *i.e.*, left and right are *swapped over*. In other words, a right-hand looks like a left-hand in a plane mirror, and *vice versa*.

13.5 Thin Lenses

A lens is a transparent medium (usually glass) bounded by two curved surfaces (generally either spherical, cylindrical, or plane surfaces). As illustrated in Fig. 13.9, the line which passes normally through both bounding surfaces of a lens is called the *optic axis*. The point O on the optic axis which lies midway between the two bounding surfaces is called the *optic centre*.

There are two basic kinds of lenses: *converging*, and *diverging*. A converging lens brings all incident light-rays parallel to its optic axis together at a point F, behind the lens, called the *focal point*, or *focus*, of the lens. A diverging lens spreads out all incident light-rays parallel to its optic axis so that they appear to diverge from a *virtual focal point* F in front of the lens. Here, the front side of the lens is conventionally defined to be the side from which the light is incident.

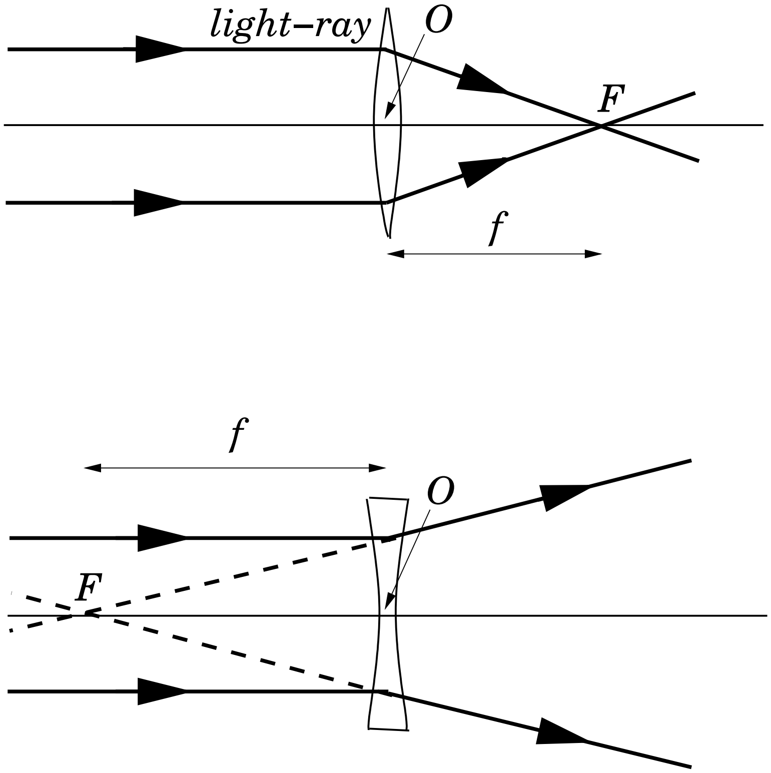

Figure 13.10: *The focii of converging (top) and diverging (bottom) lens.*

The differing effects of a converging and a diverging lens on incident light-rays parallel to the optic axis (*i.e.*, emanating from a distant object) are illustrated in Fig. 13.10.

Lenses, like mirrors, suffer from *spherical aberration*, which causes light-rays parallel to the optic axis, but a relatively long way from the axis, to be brought to a focus, or a virtual focus, *closer* to the lens than light-rays which are relatively close to the axis. It turns out that spherical aberration in lenses can be completely cured by using lenses whose bounding surfaces are *non-spherical*. However, such lenses are more difficult, and, therefore, more expensive, to manufacture than conventional lenses whose bounding surfaces are spherical. Thus, the former sort of lens is only employed in situations where the spherical aberration of a conventional lens would be a serious problem. The usual method of curing spherical aberration is to use *combinations* of conventional lenses (*i.e.*, compound lenses). In the following, we shall make use of the *paraxial approximation*, in which spherical aberration is completely ignored, and all light-rays parallel to the optic axis

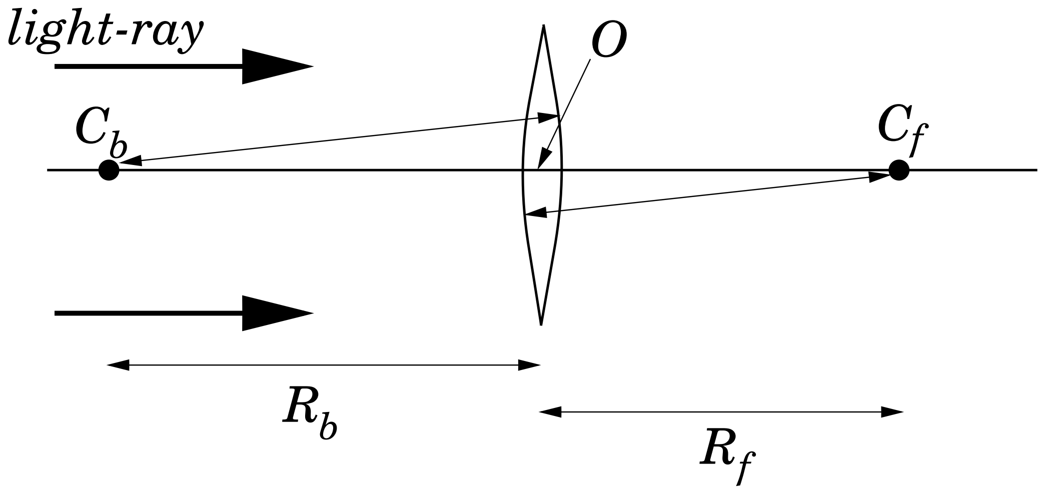

Figure 13.11: *A thin lens.*

are assumed to be brought to a focus, or a virtual focus, at the same point F. This approximation is valid as long as the radius of the lens is small compared to the object distance and the image distance.

The *focal length* of a lens, which is usually denoted f, is defined as the distance between the optic centre O and the focal point F, as shown in Fig. 13.10. However, by convention, *converging* lenses have *positive* focal lengths, and *diverging* lenses have *negative* focal lengths. In other words, if the focal point lies behind the lens then the focal length is positive, and if the focal point lies in front of the lens then the focal length is negative.

Consider a conventional lens whose bounding surfaces are *spherical*. Let C_f be the centre of curvature of the front surface, and C_b the centre of curvature of the back surface. The radius of curvature R_f of the front surface is the distance between the optic centre O and the point C_f. Likewise, the radius of curvature R_b of the back surface is the distance between points O and C_b. However, by convention, the radius of curvature of a bounding surface is *positive* if its centre of curvature lies *behind* the lens, and *negative* if its centre of curvature lies *in front* of the lens. Thus, in Fig. 13.11, R_f is positive and R_b is negative.

In the paraxial approximation, it is possible to find a simple formula relating the focal length f of a lens to the radii of curvature, R_f and R_b, of its front and back bounding surfaces. This formula is written

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_f} - \frac{1}{R_b}\right), \tag{13.12}$$

where n is the refractive index of the lens. The above formula is usually called the *lens-maker's formula*, and was discovered by Descartes. Note that the lens-maker's formula is only valid for a *thin lens* whose thickness is small compared to its focal length. What Eq. (13.12) is basically telling us is that light-rays which pass from air to glass through a *convex* surface are *focused*, whereas light-rays which pass from air to glass through a *concave* surface are *defocused*. Furthermore, since light-rays are *reversible*, it follows that rays which pass from glass to air through a *convex* surface are *defocused*, whereas rays which pass from air to glass through a *concave* surface are *focused*. Note that the net focusing or defocusing action of a lens is due to the *difference* in the radii of curvature of its two bounding surfaces.

Suppose that a certain lens has a focal length f. What happens to the focal length if we turn the lens around, so that its front bounding surface becomes its back bounding surface, and *vice versa*? It is easily seen that when the lens is turned around $R_f \rightarrow -R_b$ and $R_b \rightarrow -R_f$. However, the focal length f of the lens is invariant under this transformation, according to Eq. (13.12). Thus, the focal length of a lens is the same for light incident from either side. In particular, a converging lens remains a converging lens when it is turned around, and likewise for a diverging lens.

The most commonly occurring type of converging lens is a *bi-convex*, or *double-convex*, lens, for which $R_f > 0$ and $R_b < 0$. In this type of lens, both bounding surfaces have a focusing effect on light-rays passing through the lens. Another fairly common type of converging lens is a *plano-convex* lens, for which $R_f > 0$ and $R_b = \infty$. In this type of lens, only the curved bounding surface has a focusing effect on light-rays. The plane surface has no focusing or defocusing effect. A less common type of converging lens is a *convex-meniscus* lens, for which $R_f > 0$ and $R_b > 0$, with $R_f < R_b$. In this type of lens, the front bounding surface has a focusing effect on light-rays, whereas the back bounding surface has a defocusing effect, but the focusing effect of the front surface wins out.

The most commonly occurring type of diverging lens is a *bi-concave*, or *double-*

concave, lens, for which $R_f < 0$ and $R_b > 0$. In this type of lens, both bounding surfaces have a defocusing effect on light-rays passing through the lens. Another fairly common type of converging lens is a *plano-concave* lens, for which $R_f < 0$ and $R_b = \infty$. In this type of lens, only the curved bounding surface has a defocusing effect on light-rays. The plane surface has no focusing or defocusing effect. A less common type of converging lens is a *concave-meniscus* lens, for which $R_f < 0$ and $R_b < 0$, with $R_f < |R_b|$. In this type of lens, the front bounding surface has a defocusing effect on light-rays, whereas the back bounding surface has a focusing effect, but the defocusing effect of the front surface wins out.

Figure 13.12 shows the various types of lenses mentioned above. Note that, as a general rule, converging lenses are thicker at the centre than at the edges, whereas diverging lenses are thicker at the edges than at the centre.

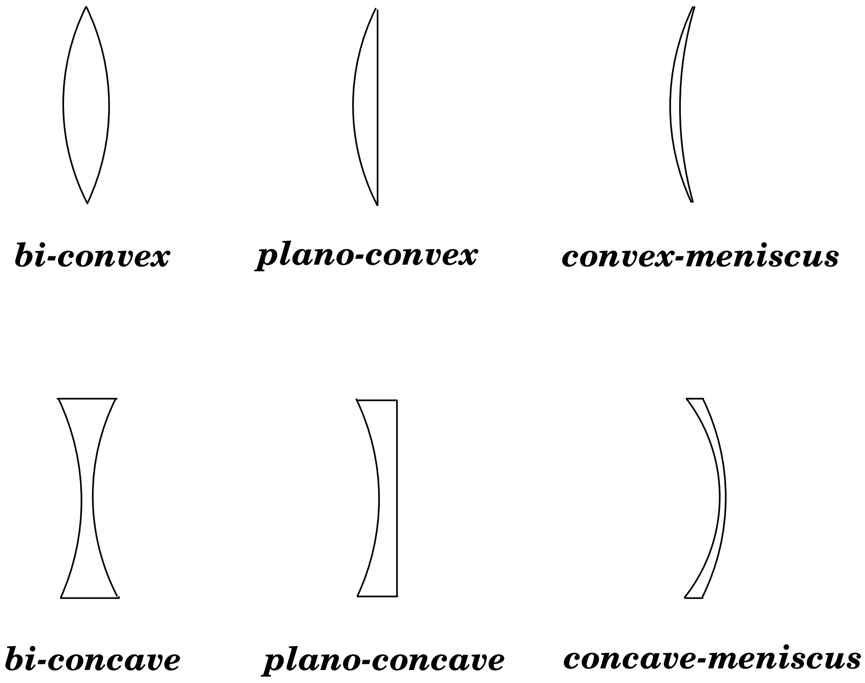

Figure 13.12: *Various different types of thin lens.*

13.6 Image Formation by Thin Lenses

There are two alternative methods of locating the image formed by a thin lens. Just as for spherical mirrors, the first method is *graphical,* and the second *analytical*.

The graphical method of locating the image formed by a thin lens involves drawing light-rays emanating from key points on the object, and finding where these rays are brought to a focus by the lens. This task can be accomplished using a small number of simple rules.

Consider a converging lens. It is helpful to define *two* focal points for such a lens. The first, the so-called *image focus,* denoted F_i, is defined as the point behind the lens to which all incident light-rays parallel to the optic axis converge after passing through the lens. This is the same as the focal point F defined previously. The second, the so-called *object focus*, denoted F_o, is defined as the position in front of the lens for which rays emitted from a point source of light placed at that position would be refracted parallel to the optic axis after passing through the lens. It is easily demonstrated that the object focus F_o is as far in front of the optic centre O of the lens as the image focus F_i is behind O. The distance from the optic centre to either focus is, of course, equal to the focal length f of the lens. The image produced by a converging lens can be located using just *three* simple rules:

1. An incident ray which is parallel to the optic axis is refracted through the image focus F_i of the lens.
2. An incident ray which passes through the object focus F_o of the lens is refracted parallel to the optic axis.
3. An incident ray which passes through the optic centre O of the lens is not refracted at all.

The last rule is only an approximation. It turns out that although a light-ray which passes through the optic centre of the lens does not change direction, it is displaced slightly to one side. However, this displacement is negligible for a thin lens.

Figure 13.13 illustrates how the image $S'T'$ of an object ST placed in front of a converging lens is located using the above rules. In fact, the three rays, 1–3, emanating from the tip T of the object, are constructed using rules 1–3, respectively. Note that the image is real (since light-rays actually cross), inverted, and diminished.

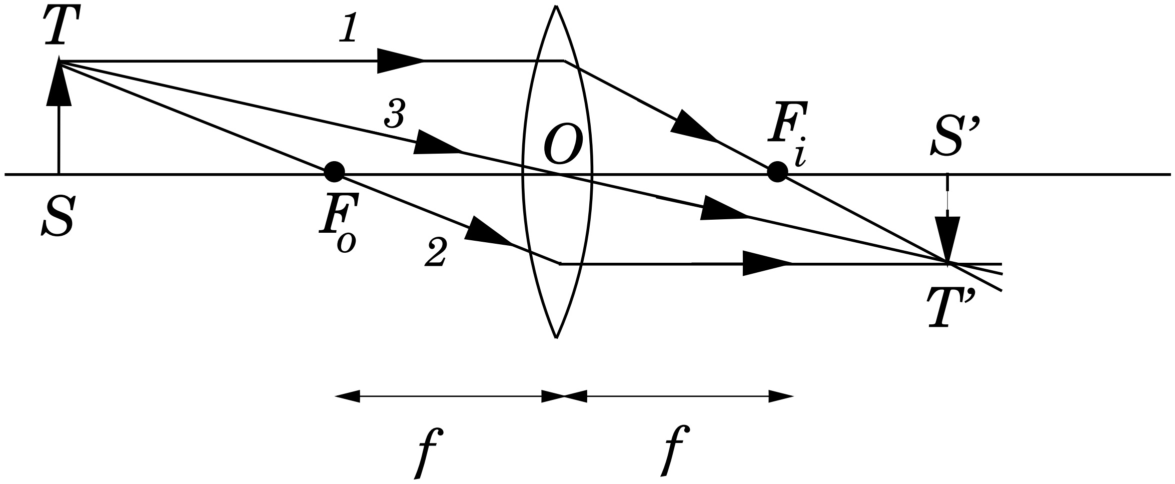

Figure 13.13: *Image formation by a converging lens.*

Consider a diverging lens. It is again helpful to define two focal points for such a lens. The image focus F_i is defined as the point in front of the lens from which all incident light-rays parallel to the optic axis appear to diverge after passing through the lens. This is the same as the focal point F defined earlier. The object focus F_o is defined as the point behind the lens to which all incident light-rays which are refracted parallel to the optic axis after passing through the lens appear to converge. Both foci are located a distance f from the optic centre, where f is the focal length of the lens. The image produced by a diverging lens can be located using the following three rules:

1. An incident ray which is parallel to the optic axis is refracted as if it came from the image focus F_i of the lens.

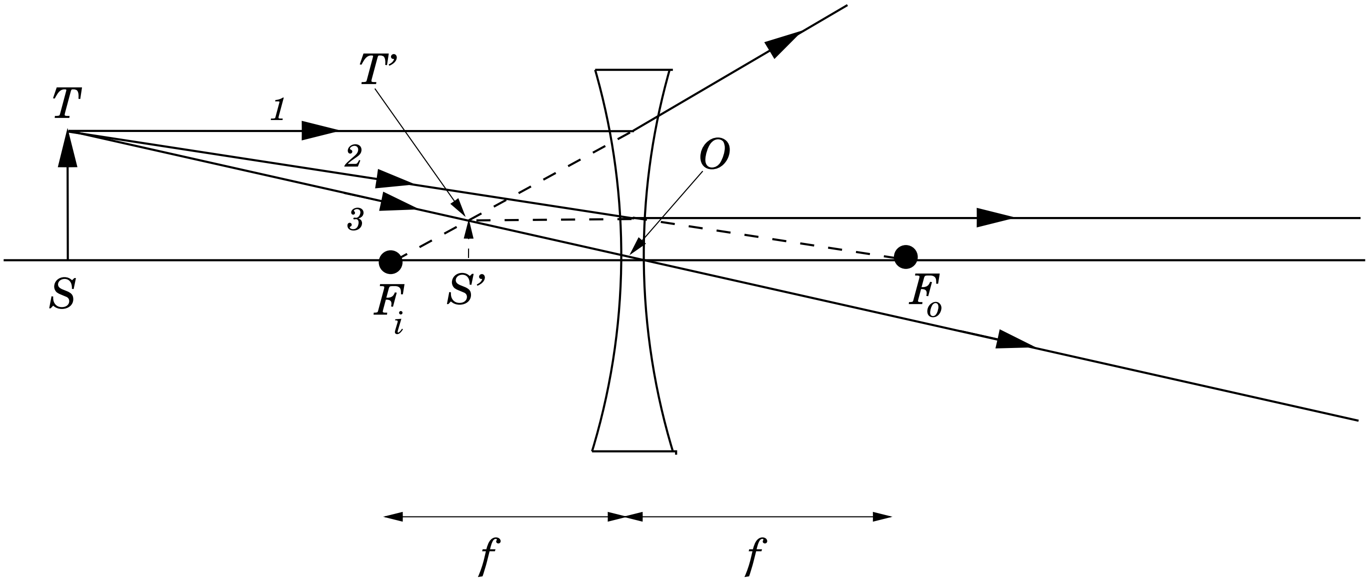

Figure 13.14: *Image formation by a diverging lens.*

2. An incident ray which is directed towards the object focus F_o of the lens is refracted parallel to the optic axis.

3. An incident ray which passes through the optic centre O of the lens is not refracted at all.

Figure 13.14 illustrates how the image $S'T'$ of an object ST placed in front of a diverging lens is located using the above rules. In fact, the three rays, 1–3, emanating from the tip T of the object, are constructed using rules 1–3, respectively. Note that the image is virtual (since light-rays do not actually cross), upright, and diminished.

Let us now investigate the analytical method. Consider an object of height h placed a distance p in front of a converging lens. Suppose that a real image of height h' is formed a distance q behind the lens. As is illustrated in Fig. 13.15, the image can be located using rules 1 and 3, discussed above.

Now, the right-angled triangles SOT and $S'OT'$ are similar, so

$$\frac{-h'}{h} = \frac{OS'}{OS} = \frac{q}{p}. \tag{13.13}$$

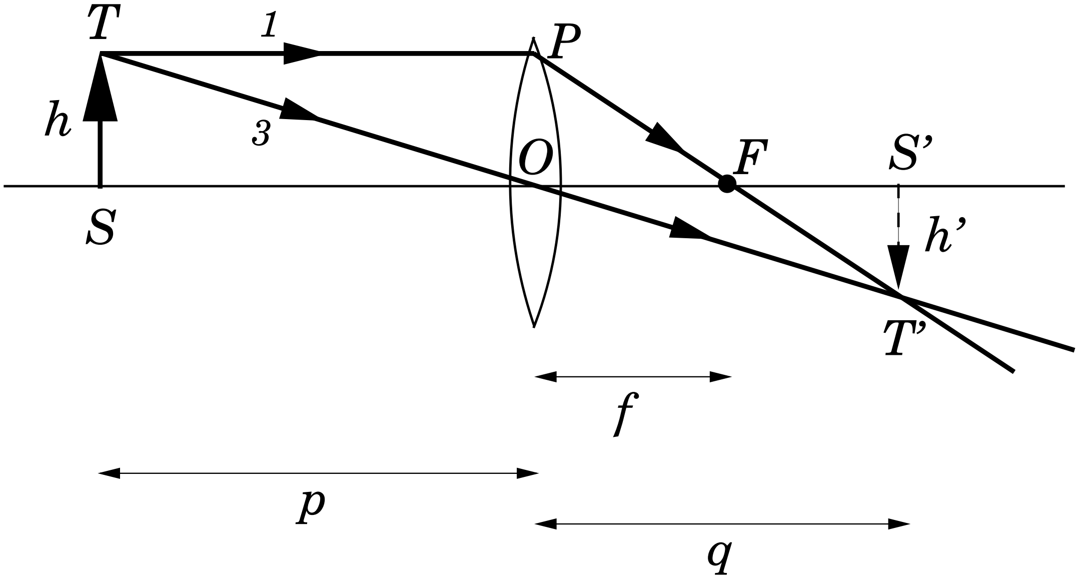

Figure 13.15: *Image formation by a converging lens.*

Here, we have adopted the convention that the image height h' is *negative* if the image is *inverted*. The magnification of a thin converging lens is given by

$$M = \frac{h'}{h} = -\frac{q}{p}. \tag{13.14}$$

This is the same as the expression (13.2) for the magnification of a spherical mirror. Note that we are again adopting the convention that the magnification is *negative* if the image is *inverted*.

The right-angled triangles OPF and $S'T'F$ are also similar, and so

$$\frac{S'T'}{OP} = \frac{FS'}{OF}, \tag{13.15}$$

or

$$\frac{-h'}{h} = \frac{q}{p} = \frac{q-f}{f}. \tag{13.16}$$

The above expression can be rearranged to give

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}. \tag{13.17}$$

Note that this is exactly the same as the formula (13.8) relating the image and object distances in a spherical mirror.

Position of object	*Position of image*	*Character of image*
At $+\infty$	At F	Real, zero size
Between $+\infty$ and V_o	Between F and V_i	Real, inverted, diminished
At V_o	At V_i	Real, inverted, same size
Between V_o and F	Between V_i and $-\infty$	Real, inverted, magnified
At F	At $-\infty$	
Between F and O	From $+\infty$ to O	Virtual, upright, magnified
At O	At O	Virtual, upright, same size

Table 13.3: *Rules for image formation by converging lenses.*

Position of object	*Position of image*	*Character of image*
At ∞	At F_i	Virtual, zero size
Between ∞ and O	Between F_i and O	Virtual, upright, diminished
At O	At O	Virtual, upright, same size

Table 13.4: *Rules for image formation by diverging lenses.*

Although formulae (13.14) and (13.17) were derived for the case of a real image formed by a converging lens, they also apply to virtual images, and to images formed by diverging lenses, provided that the following sign conventions are adopted. First of all, as we have already mentioned, the focal length f of a *converging* lens is *positive*, and the focal length of a *diverging* lens is *negative*. Secondly, the image distance q is *positive* if the image is *real*, and, therefore, located *behind* the lens, and *negative* if the image is *virtual*, and, therefore, located *in front* of the lens. It immediately follows, from Eq. (13.14), that *real* images are always *inverted*, and *virtual* images are always *upright*.

Table 13.3 shows how the location and character of the image formed by a converging lens depend on the location of the object. Here, the point V_o is located on the optic axis two focal lengths in front of the optic centre, and the point V_i is located on the optic axis two focal lengths behind the optic centre. Note the almost exact analogy between the image forming properties of a converging lens and those of a concave spherical mirror.

Table 13.4 shows how the location and character of the image formed by a diverging lens depend on the location of the object. Note the almost exact analogy between the image forming properties of a diverging lens and those of a convex spherical mirror.

Finally, let us reiterate the sign conventions used to determine the positions and characters of the images formed by thin lenses:

1. The height h' of the image is positive if the image is upright, with respect to the object, and negative if the image is inverted.
2. The magnification M of the image is positive if the image is upright, with respect to the object, and negative if the image is inverted.
3. The image distance q is positive if the image is real, and, therefore, located behind the lens, and negative if the image is virtual, and, therefore, located in front of the lens.
4. The focal length f of the lens is positive if the lens is converging, so that the image focus F_i is located behind the lens, and negative if the lens is diverging, so that the image focus F_i is located in front of the lens.

Note that the front side of the lens is defined to be the side from which the light is incident.

13.7 Chromatic aberration

We have seen that both mirrors and lenses suffer from spherical aberration, an effect which limits the clarity and sharpness of the images formed by such devices. However, lenses also suffer from another type of abberation called *chromatic abberation*. This occurs because the index of refraction of the glass in a lens is different for different wavelengths. We have seen that a prism refracts violet light more than red light. The same is true of lenses. As a result, a simple lens focuses violet light closer to the lens than it focuses red light. Hence, white light produces a slightly blurred image of an object, with coloured edges.

For many years, chromatic abberation was a sufficiently serious problem for lenses that scientists tried to find ways of reducing the number of lenses in scientific instruments, or even eliminating them all together. For instance, Isaac Newton developed a type of telescope, now called the Newtonian telescope, which uses a mirror instead of a lens to collect light. However, in 1758, John Dollond, an English optician, discovered a way to eliminate chromatic abberation. He combined two lenses, one converging, the other diverging, to make an *achromatic doublet*. The two lenses in an achromatic doublet are made of different type of glass with indices of refraction chosen such that the combination brings any two chosen colours to the same sharp focus.

Modern scientific instruments use *compound lenses* (*i.e.*, combinations of simple lenses) to simultaneously eliminate both chromatic and spherical aberration.

13.8 Worked Examples

Example 13.1: Concave mirrors

Question: An object of height $h = 4\,\mathrm{cm}$ is placed a distance $p = 15\,\mathrm{cm}$ in front of a concave mirror of focal length $f = 20\,\mathrm{cm}$. What is the height, location, and nature of the image? Suppose that the object is moved to a new position a distance $p = 25\,\mathrm{cm}$ in front of the mirror. What now is the height, location, and nature of the image?

Answer: According to Eq. (13.8), the image distance q is given by

$$q = \frac{1}{1/f - 1/p} = \frac{1}{(1/20 - 1/15)} = -60\,\mathrm{cm}.$$

Thus, the image is *virtual* (since $q < 0$), and is located $60\,\mathrm{cm}$ *behind* the mirror. According to Eq. (13.2), the magnification M of the image is given by

$$M = -\frac{q}{p} = -\frac{(-60)}{(15)} = 4.$$

Thus, the image is *upright* (since $M > 0$), and *magnified* by a factor of 4. It follows that the height h' of the image is given by

$$h' = M\,h = (4)\,(4) = 16\,\mathrm{cm}.$$

If the object is moved such that $p = 25\,\text{cm}$ then the new image distance is given by

$$q = \frac{1}{1/f - 1/p} = \frac{1}{(1/20 - 1/25)} = 100\,\text{cm}.$$

Thus, the new image is *real* (since $q > 0$), and is located 100 cm *in front of* the mirror. The new magnification is given by

$$M = -\frac{q}{p} = -\frac{(100)}{(15)} = -6.67.$$

Thus, the image is *inverted* (since $M < 0$), and *magnified* by a factor of 6.67. It follows that the new height of the image is

$$h' = M\,h = -(6.67)\,(4) = -26.67\,\text{cm}.$$

Note that the height is negative because the image is inverted.

Example 13.2: Convex mirrors

Question: How far must an object be placed in front of a convex mirror of radius of curvature $R = 50\,\text{cm}$ in order to ensure that the size of the image is ten times less than the size of the object? How far behind the mirror is the image located?

Answer: The focal length f of a convex mirror is *minus* half of its radius of curvature (taking the sign convention for the focal lengths of convex mirrors into account). Thus, $f = -25\,\text{cm}$. If the image is ten times smaller than the object then the magnification is $M = 0.1$. We can be sure that $M = +0.1$, as opposed to -0.1, because we know that images formed in convex mirrors are always virtual and upright. According to Eq. (13.2), the image distance q is given by

$$q = -M\,p,$$

where p is the object distance. This can be combined with Eq. (13.8) to give

$$p = f\left(1 - \frac{1}{M}\right) = -(25)\,(1 - 10) = 225\,\text{cm}.$$

Thus, the object must be placed 225 cm in front of the mirror. The image distance is given by

$$q = -M\,p = -(0.1)\,(225) = -22.5\,\text{cm}.$$

Thus, the image is located 22.5 cm behind the mirror.

Example 13.3: Converging lenses

Question: An object of height $h = 7$ cm is placed a distance $p = 25$ cm in front of a thin converging lens of focal length $f = 35$ cm. What is the height, location, and nature of the image? Suppose that the object is moved to a new location a distance $p = 90$ cm in front of the lens. What now is the height, location, and nature of the image?

Answer: According to Eq. (13.17), the image distance q is given by

$$q = \frac{1}{1/f - 1/p} = \frac{1}{(1/35 - 1/25)} = -87.5\,\text{cm}.$$

Thus, the image is *virtual* (since $q < 0$), and is located 87.5 cm *in front* of the lens. According to Eq. (10.24), the magnification M of the image is given by

$$M = -\frac{q}{p} = -\frac{(-87.5)}{(25)} = 3.5.$$

Thus, the image is *upright* (since $M > 0$), and *magnified* by a factor of 3.5. It follows that the height h' of the image is given by

$$h' = M\,h = (3.5)\,(7) = 24.5\,\text{cm}.$$

If the object is moved such that $p = 90$ cm then the new image distance is given by

$$q = \frac{1}{1/f - 1/p} = \frac{1}{(1/35 - 1/90)} = 57.27\,\text{cm}.$$

Thus, the new image is *real* (since $q > 0$), and is located 57.27 cm *behind* the lens. The new magnification is given by

$$M = -\frac{q}{p} = -\frac{(57.27)}{(90)} = -0.636.$$

Thus, the image is *inverted* (since $M < 0$), and *diminished* by a factor of 0.636. It follows that the new height of the image is

$$h' = M\,h = -(9.636)\,(7) = -4.45\,\text{cm}.$$

Note that the height is negative because the image is inverted.

Example 13.4: Diverging lenses

Question: How far must an object be placed in front of a diverging lens of focal length 45 cm in order to ensure that the size of the image is fifteen times less than the size of the object? How far in front of the lens is the image located?

Answer: The focal length f of a diverging lens is *negative* by convention, so $f = -45\,\text{cm}$, in this case. If the image is fifteen times smaller than the object then the magnification is $M = 0.0667$. We can be sure that $M = +0.0667$, as opposed to -0.0667, because we know that images formed in diverging lenses are always virtual and upright. According to Eq. (13.14), the image distance q is given by

$$q = -M\,p,$$

where p is the object distance. This can be combined with Eq. (13.17) to give

$$p = f\left(1 - \frac{1}{M}\right) = -(45)\,(1 - 15) = 630\,\text{cm}.$$

Thus, the object must be placed 630 cm in front of the lens. The image distance is given by

$$q = -M\,p = -(0.0667)\,(630) = -42\,\text{cm}.$$

Thus, the image is located 42 cm *in front* of the lens.

14 Wave Optics

14.1 Introduction

Geometric optics is an incredibly successful theory. Probably its most important application is in describing and explaining the operation of commonly occurring optical instruments: *e.g.*, the camera, the telescope, and the microscope. Although geometric optics does not make any explicit assumption about the nature of light, it tends to suggest that light consists of a stream of massless particles. This is certainly what scientists, including, most notably, Isaac Newton, generally assumed up until about the year 1800.

Let us examine how the particle theory of light accounts for the three basic laws of geometric optics:

1. The law of geometric propagation: This is easy. Massless particles obviously move in straight-lines in free space.
2. The law of reflection: This is also fairly easy. We merely have to assume that light particles bounce *elastically* (*i.e.*, without energy loss) off reflecting surfaces.
3. The law of refraction: This is the tricky one. Let us assume that the speed of light particles propagating through a transparent dielectric medium is proportional to the index of refraction, $n \equiv \sqrt{K}$. Let us further assume that at a general interface between two different dielectric media, light particles crossing the interface conserve momentum in the plane parallel to the interface. In general, this implies that the particle momenta normal to the interface are not conserved: *i.e.*, the interface exerts a normal reaction force on crossing particles, but no parallel force. From Fig. 14.1, parallel momentum conservation for light particles crossing the interface yields
$$v_1 \sin\theta_1 = v_2 \sin\theta_2. \tag{14.1}$$
However, by assumption, $v_1 = n_1\,c$ and $v_2 = n_2\,c$, so
$$n_1 \sin\theta_1 = n_2 \sin\theta_2. \tag{14.2}$$

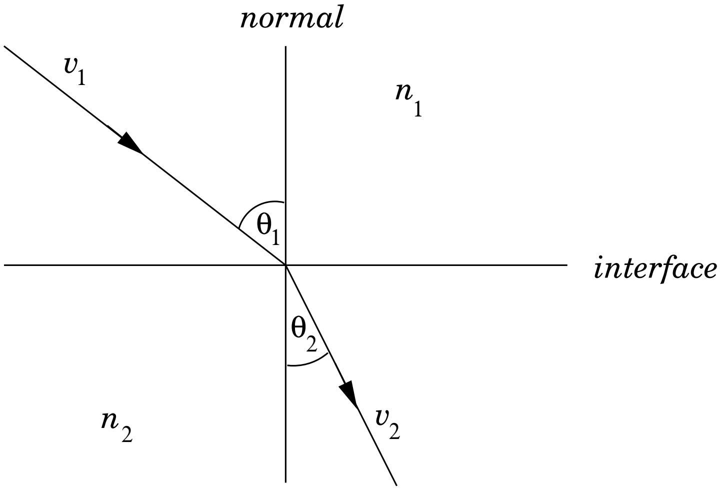

Figure 14.1: *Descartes' model of refraction*

> This highly contrived (and incorrect) derivation of the law of refraction was first proposed by Descartes in 1637. Note that it depends crucially on the (incorrect) assumption that light travels *faster* in dense media (*e.g.*, glass) than in rarefied media (*e.g.*, water). This assumption appears very strange to us nowadays, but it seemed eminently reasonable to scientists in the 17th and 18th centuries. After all, they knew that sound travels faster in dense media (*e.g.*, water) than in rarefied media (*e.g.*, air).

The wave theory of light, which became established in the first half of the 19th century, initially encountered tremendous resistance. Let us briefly examine the reasons why scientists in the early 1800s refused to think of light as a wave phenomenon? Firstly, the particle theory of light was intimately associated with Isaac Newton, so any attack on this theory was considered to be a slight to his memory. Secondly, all of the waves that scientists were familiar with at that time manifestly did not travel in straight-lines. For instance, water waves are diffracted as they pass through the narrow mouth of a harbour, as shown in Fig. 14.2. In other words, the "rays" associated with such waves are bent as they traverse the harbour mouth. Scientists thought that if light were a wave phenomenon then it would also not travel in straight-lines: *i.e.*, it would not cast straight, sharp shadows, any more than water waves cast straight, sharp "shadows." Unfortunately, they did not appreciate that if the wavelength of light

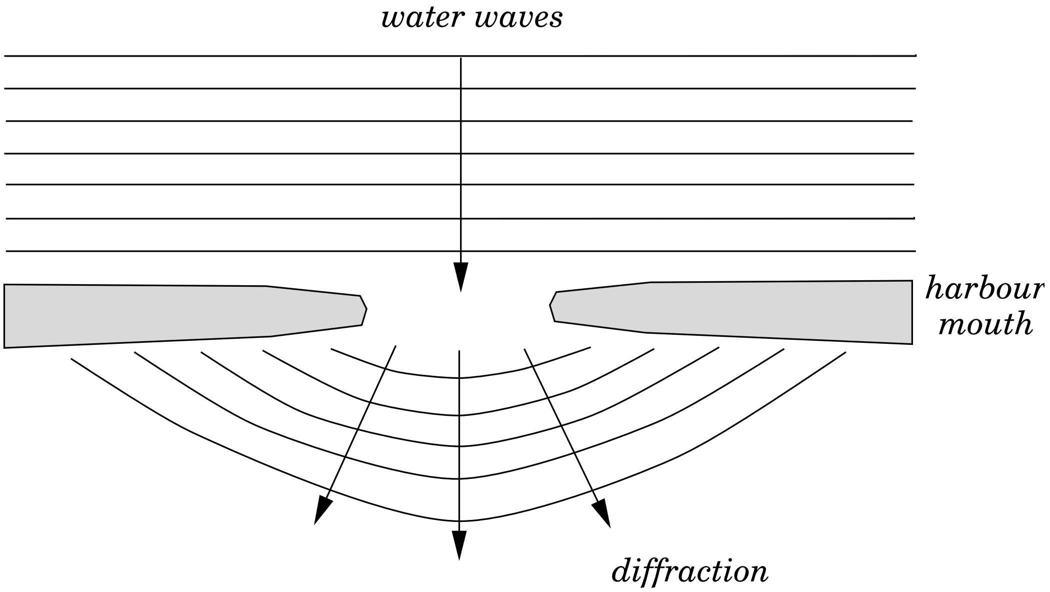

Figure 14.2: *Refraction of water waves through the entrance of a harbour.*

is much shorter than that of water waves then light can be a wave phenomenon and still propagate in a largely geometric manner.

14.2 Huygens' principle

The first person to explain how wave theory can also account for the laws of geometric optics was Christiaan Huygens in 1670. At the time, of course, nobody took the slightest notice of him. His work was later rediscovered after the eventual triumph of wave theory.

Huygens had a very important insight into the nature of wave propagation which is nowadays called *Huygens' principle*. When applied to the propagation of light waves, this principle states that:

> Every point on a wave-front may be considered a source of secondary spherical wavelets which spread out in the forward direction at the speed of light. The new wave-front is the tangential surface to all of these secondary wavelets.

According to Huygens' principle, a plane light wave propagates though free

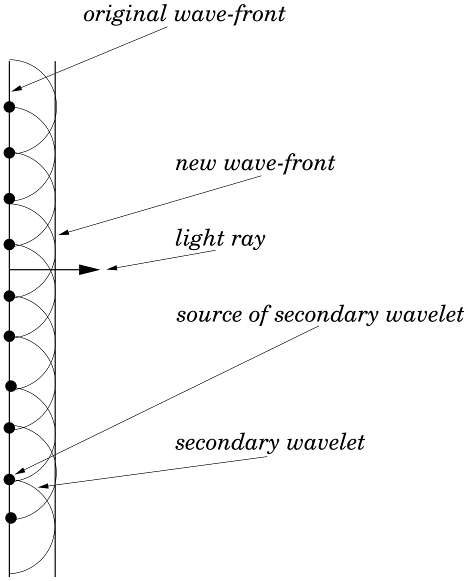

Figure 14.3: *Huygen's principle.*

space at the speed of light, c. The light rays associated with this wave-front propagate in straight-lines, as shown in Fig. 14.3. It is also fairly straightforward to account for the laws of reflection and refraction using Huygens' principle.

14.3 Young's Double-Slit Experiment

The first serious challenge to the particle theory of light was made by the English scientist Thomas Young in 1803. Young possessed one of the most brilliant minds in the history of science. A physician by training, he was the first to describe how the lens of the human eye changes shape in order to focus on objects at differing distances. He also studied Physics, and, amongst other things, definitely established the wave theory of light, as described below. Finally, he also studied Egyptology, and helped decipher the Rosetta Stone.

Young knew that sound was a wave phenomenon, and, hence, that if two sound waves of equal intensity, but 180° out of phase, reach the ear then they cancel one another out, and no sound is heard. This phenomenon is called *interference*. Young reasoned that if light were actually a wave phenomenon, as he suspected, then a similar interference effect should occur for light. This line of

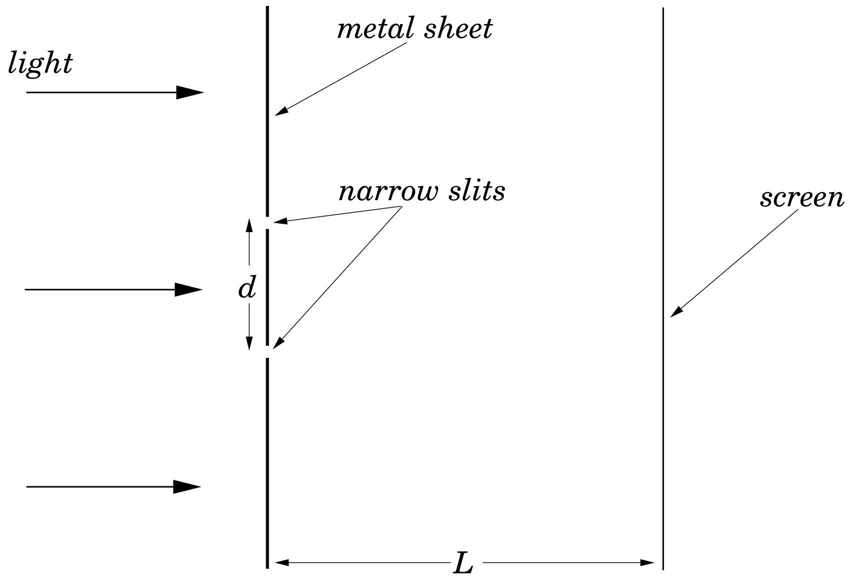

Figure 14.4: *Young's double-slit experiment.*

reasoning lead Young to perform an experiment which is nowadays referred to as *Young's double-slit experiment*.

In Young's experiment, two very narrow parallel slits, separated by a distance d, are cut into a thin sheet of metal. Monochromatic light, from a distant light-source, passes through the slits and eventually hits a screen a comparatively large distance L from the slits. The experimental setup is sketched in Fig. 14.4.

According to Huygens' principle, each slit radiates spherical light waves. The light waves emanating from each slit are superposed on the screen. If the waves are 180° out of phase then *destructive interference* occurs, resulting in a dark patch on the screen. On the other hand, if the waves are completely in phase then *constructive interference* occurs, resulting in a light patch on the screen.

The point P on the screen which lies exactly opposite to the centre point of the two slits, as shown in Fig. 14.5, is obviously associated with a bright patch. This follows because the path-lengths from each slit to this point are the same. The waves emanating from each slit are initially in phase, since all points on the incident wave-front are in phase (*i.e.*, the wave-front is straight and parallel to the metal sheet). The waves are still in phase at point P since they have traveled

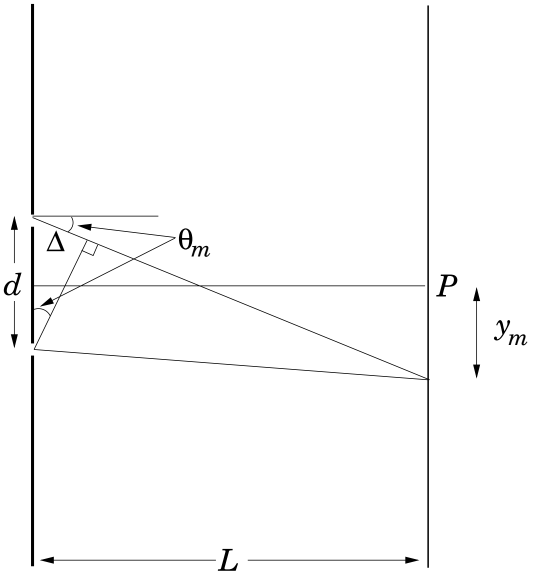

Figure 14.5: *Interference of light in Young's double-slit experiment.*

equal distances in order to reach that point.

From the above discussion, the general condition for constructive interference on the screen is simply that the difference in path-length Δ between the two waves be an *integer* number of wavelengths. In other words,

$$\Delta = m\,\lambda, \tag{14.3}$$

where $m = 0, 1, 2, \cdots$. Of course, the point P corresponds to the special case where $m = 0$. It follows, from Fig. 14.5, that the angular location of the mth bright patch on the screen is given by

$$\sin\theta_m = \frac{\Delta}{d} = \frac{m\,\lambda}{d}. \tag{14.4}$$

Likewise, the general condition for destructive interference on the screen is that the difference in path-length between the two waves be a *half-integer* number of wavelengths. In other words,

$$\Delta = (m + 1/2)\,\lambda, \tag{14.5}$$

where $m = 1, 2, 3, \cdots$. It follows that the angular coordinate of the mth dark patch on the screen is given by

$$\sin\theta_m' = \frac{\Delta}{d} = \frac{(m+1/2)\,\lambda}{d}. \tag{14.6}$$

Usually, we expect the wavelength λ of the incident light to be much less than the perpendicular distance L to the screen. Thus,

$$\sin\theta_m \simeq \frac{y_m}{L}, \tag{14.7}$$

where y_m measures position on the screen relative to the point P.

It is clear that the interference pattern on the screen consists of *alternating light and dark bands,* running parallel to the slits. The distances of the centers of the various light bands from the point P are given by

$$y_m = \frac{m\,\lambda\,L}{d}, \tag{14.8}$$

where $m = 0, 1, 2, \cdots$. Likewise, the distances of the centres of the various dark bands from the point P are given by

$$y_m' = \frac{(m+1/2)\,\lambda\,L}{d}, \tag{14.9}$$

where $m = 1, 2, 3, \cdots$. The bands are *equally spaced,* and of thickness $\lambda\,L/d$. Note that if the distance from the screen L is much larger than the spacing d between the two slits then the thickness of the bands on the screen greatly exceeds the wavelength λ of the light. Thus, given a sufficiently large ratio L/d, it should be possible to observe a banded interference pattern on the screen, despite the fact that the wavelength of visible light is only of order 1 micron. Indeed, when Young performed this experiment in 1803 he observed an interference pattern of the type described above. Of course, this pattern is a *direct consequence* of the wave nature of light, and is completely inexplicable on the basis of geometric optics.

It is interesting to note that when Young first presented his findings to the Royal Society of London he was ridiculed. His work only achieved widespread acceptance when it was confirmed, and greatly extended, by the French physicists

Augustin Fresnel and Francois Argo in the 1820s. The particle theory of light was dealt its final death-blow in 1849 when the French physicists Fizeau and Foucault independently demonstrated that light propagates *more slowly* though water than though air. Recall (from Sect. 14.1), that the particle theory of light can only account for the law of refraction on the assumption that light propagates *faster* through dense media, such as water, than through rarefied media, such as air.

14.4 Interference in Thin Films

In everyday life, the interference of light most commonly gives rise to easily observable effects when light impinges on a thin film of some transparent material. For instance, the brilliant colours seen in soap bubbles, in oil films floating on puddles of water, and in the feathers of a peacock's tail, are due to interference of this type.

Suppose that a very thin film of air is trapped between two pieces of glass, as shown in Fig. 14.6. If monochromatic light (*e.g.*, the yellow light from a sodium lamp) is incident *almost normally* to the film then some of the light is reflected from the interface between the bottom of the upper plate and the air, and some is reflected from the interface between the air and the top of the lower plate. The eye focuses these two parallel light beams at one spot on the retina. The two beams produce either destructive or constructive interference, depending on whether their path difference is equal to an odd or an even number of half-wavelengths, respectively.

Let t be the thickness of the air film. The difference in path-lengths between the two light rays shown in the figure is clearly $\Delta = 2\,t$. Naively, we might expect that constructive interference, and, hence, *brightness*, would occur if $\Delta = m\,\lambda$, where m is an integer, and destructive interference, and, hence, *darkness*, would occur if $\Delta = (m + 1/2)\,\lambda$. However, this is not the entire picture, since an additional phase difference is introduced between the two rays on reflection. The first ray is reflected at an interface between an optically dense medium (glass), through which the ray travels, and a less dense medium (air). There is no phase change on reflection from such an interface, just as there is no phase change when

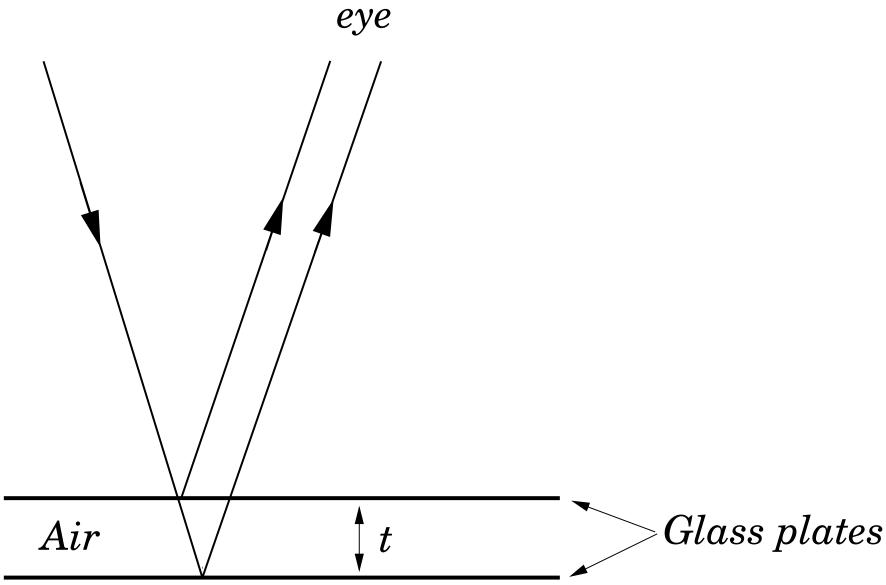

Figure 14.6: *Interference of light due to a thin film of air trapped between two pieces of glass.*

a wave on a string is reflected from a free end of the string. (Both waves on strings and electromagnetic waves are *transverse waves*, and, therefore, have analogous properties.) The second ray is reflected at an interface between an optically less dense medium (air), through which the ray travels, and a dense medium (glass). There is a 180° phase change on reflection from such an interface, just as there is a 180° phase change when a wave on a string is reflected from a fixed end. Thus, an additional 180° phase change is introduced between the two rays, which is equivalent to an additional path difference of $\lambda/2$. When this additional phase change is taken into account, the condition for constructive interference becomes

$$2\,t = (m + 1/2)\,\lambda, \tag{14.10}$$

where m is an integer. Similarly, the condition for destructive interference becomes

$$2\,t = m\,\lambda. \tag{14.11}$$

For white light, the above criteria yield constructive interference for some wavelengths, and destructive interference for others. Thus, the light reflected back from the film exhibits those colours for which the constructive interference occurs.

If the thin film consists of water, oil, or some other transparent material of refractive index n then the results are basically the same as those for an air film, except that the wavelength of the light in the film is reduced from λ (the vacuum wavelength) to λ/n. It follows that the modified criteria for constructive and destructive interference are

$$2\,n\,t = (m + 1/2)\,\lambda, \tag{14.12}$$

and

$$2\,n\,t = m\,\lambda, \tag{14.13}$$

respectively.

14.5 Worked Examples

Example 14.1: Double slit experiment

Question: Coherent light of wavelength 633 nm from a He-Ne laser falls on a double slit with a slit separation of 0.103 mm. An interference pattern is produced on a screen 2.56 m from the slits. Calculate the separation on the screen of the two fourth-order bright fringes on either side of the central image.

Solution: The easiest way to handle this problem is to calculate the distance y_4 of the fourth-order bright fringe on one side from the central image, and then double this value to obtain the distance between the two fourth-order images. From Eq. (14.8),

$$y_4 = \frac{4\,\lambda\,L}{d} = \frac{4\,(633\times 10^{-9})\,(2.65)}{(0.103\times 10^{-3})} = 6.29\,\text{cm}.$$

The distance between the two fourth-order fringes is therefore

$$2\,y_4 = 12.6\,\text{cm}.$$

Example 14.2: Interference in thin films

Question: A soap bubble 250 nm thick is illuminated by white light. The index of refraction of the soap film is 1.36. Which colours are *not* seen in the reflected light? Which colours appear strong in the reflected light? What colour does the soap film appear at normal incidence?

Solution: For destructive interference, we must have $n\,t = m\,\lambda/2$. Thus, the wavelengths that are *not* reflected satisfy

$$\lambda_m = \frac{2\,n\,t}{m},$$

where $m = 1, 2, 3, \cdots$. It follows that

$$\lambda_1 = \frac{(2)\,(1.36)\,(250\times 10^{-9})}{(1)} = 680\,\text{nm},$$

and

$$\lambda_2 = \frac{(2)\,(1.36)\,(250\times 10^{-9})}{(2)} = 340\,\text{nm}.$$

These are the only wavelengths close to the visible region of the electromagnetic spectrum for which destructive interference occurs. In fact, 680 nm lies right in the middle of the red region of the spectrum, whilst 340 nm lies in the ultraviolet region (and is, therefore, invisible to the human eye). It follows that the only non-reflected colour is *red*.

For constructive interference, we must have $n\,t = (m + 1/2)\,\lambda/2$. Thus, the wavelengths that are *strongly reflected* satisfy

$$\lambda_m' = \frac{2\,n\,t}{m + 1/2},$$

where $m = 0, 1, 2, \cdots$. It follows that

$$\lambda_1' = \frac{(2)\,(1.36)\,(250\times 10^{-9})}{(1/2)} = 1360\,\text{nm},$$

and

$$\lambda_2' = \frac{(2)\,(1.36)\,(250\times 10^{-9})}{(3/2)} = 453\,\text{nm},$$

and

$$\lambda_3' = \frac{(2)\,(1.36)\,(250 \times 10^{-9})}{(5/2)} = 272\,\text{nm}.$$

A wavelength of 272 nm lies in the ultraviolet region whereas 1360 nm lies in the infrared. Clearly, both wavelengths correspond to light which is invisible to the human eye. The only strong reflection occurs at 453 nm, which corresponds to the *blue-violet* region of the spectrum.

The reflected light is weak in the red region of the spectrum and strong in the blue-violet region. The soap film will, therefore, possess a pronounced *blue* colour.