FI 2201 Electromagnetism

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Techniques in solving Electric Potentials

LAPLACE'S EQUATIONS IN SPHERICAL COORDINATE SYSTEM

Laplace's Equation in Spherical Coord.

- In spherical coord., the Laplacian is given by

$$\boxed{\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0}$$

- For azimuthal symmetric problem, V is independent of ϕ, thus the Laplacian reduces to

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0$$

- By separation of variables method, we write

$$V(r,\theta)=R(r)\Theta(\theta)$$

- Substitution and dividing through by $V(r,\theta)$ yields

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=0$$

Laplace's Equation in Spherical Coord.

- Since the first term only depends on r, while the second term only depends on θ, and the equation must be satisfied for all (r,θ), it follows that each term in the above equation must be a constant.

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=l(l+1), \qquad \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=-l(l+1)$$

- The constant is chosen to be $l(l+1)$ so that the angular differential equation is readily recognized.
- The radial part can be solved by series solution to obtain

$$R(r)=Ar^l+\frac{B}{r^{l+1}}$$

with A and B are constants of integration.

Laplace's Equation in Spherical Coord.

- The angular part

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+l(l+1)\Theta=0$$

- With transformation of variable $z=\cos\theta$, the above differential equation transforms into

$$\frac{d}{dz}\left[\left(1-z^2\right)\frac{d\Theta}{dz}\right]+l(l+1)\Theta=0 \rightarrow \left(1-z^2\right)\frac{d^2\Theta}{dz^2}-2z\frac{d\Theta}{dz}+l(l+1)\Theta=0$$

this is a well-known ordinary differential equation called Legendre differential equation, whose solution is the Legendre polynomials

$$\Theta(\theta)=P_l(\cos\theta)$$

Laplace's Equation in Spherical Coord.

- Thus, the solution of Laplace's equation in spherical coord. with azimuthal symmetry can be written as

$$V(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

Legendre Polynomials

- The Legendre polynomials $\Theta(\theta) = P_l(\cos\theta)$ can also be found using the Rodrigues formula

$$P_l(z) = \frac{1}{2^l\, l!}\left(\frac{d}{dz}\right)^l \left(z^2 - 1\right)^l$$

- The first few Legendre polynomials are

$$P_0(z) = 1 \qquad P_1(z) = z$$

$$P_2(z) = \tfrac{1}{2}\left(3z^2 - 1\right) \qquad P_3(z) = \tfrac{1}{2}\left(5z^3 - 3z\right)$$

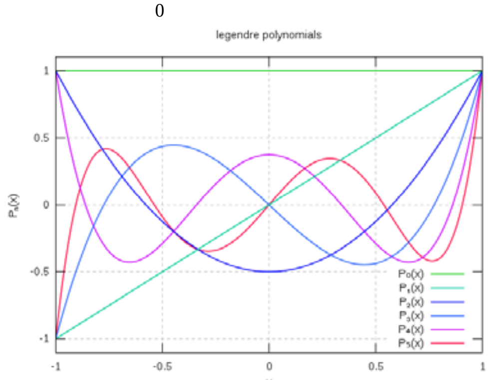
- Other ways of generating these Legendre polynomials is using a recursive relation, e.g.

$$lP_l(z) = \left(2l-1\right)zP_{l-1}(z) - \left(l-1\right)P_{l-2}(z)$$

Legendre Polynomials

- These Legendre polynomials form a complete set of function with orthogonality relation given by

$$\int_{-1}^{1} P_l(z)P_{l'}(z)dz = \int_{0}^{\pi} P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta = \frac{2}{2l+1}\delta_{ll'}$$



Legendre Polynomials

- Note that these Legendre polynomials are regular (finite) at $z = 0$.
- As Legendre equation is a second order differential equation, we would expect that we have two independent solution (for each l), one of them being $P_l(z)$.
- There is a second solution of this second order differential equation, known as the second type Legendre polynomials, $Q_l(z)$, however this solution is singular (infinite) at $z = 0$, hence it is discarded for the problem at hand.

Legendre Polynomials

- The function

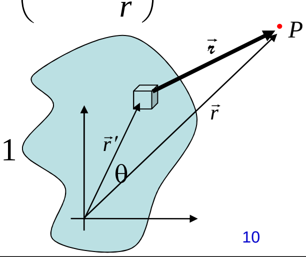
$$\Phi(x,u) = \frac{1}{\left[1+u^2-2ux\right]^{1/2}} = \sum_{n=0}^{\infty} u^n P_n(x), \quad 0<u<1$$

is called the generating function of the Legendre Polynomials.

- Recall that

$$V(\vec{r}) \propto \frac{1}{\mathscr{r}} = \frac{1}{r\left[1+\left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta\right]^{1/2}} = \frac{1}{r}\Phi\left(\cos\theta, \frac{r'}{r}\right)$$

$$= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta), \quad 0<\left(\frac{r'}{r}\right)<1$$



Laplace's Equation in Spherical Coord.

- **Example 3.6** and **Example 3.7**
- **Example 3.8**
- See also **Example 3.9**