Spacetime and Gravity: Assignment 1

In what follows, unless otherwise stated, we will use units such that the speed of light, c=1.

1

The Lorentz transformations are given by:

$$x' = \frac{x - vt}{\sqrt{1-v^2}} \qquad t' = \frac{t - vx}{\sqrt{1-v^2}} \tag{1}$$

Show that

$$-t'^2 + x'^2 = -t^2 + x^2 \,. \tag{2}$$

Set $v = tanh(\theta)$ and substitute into the Lorentz transformations (1) to find the Lorentz transformations in terms of θ given by:

$$t' = tcosh(\theta) - xsinh(\theta) \qquad x' = xcosh(\theta) - tsinh(\theta) \,. \tag{3}$$

You will need the identity for hyperbolic trigonomtery that:

$$\frac{1}{\sqrt{1 - tanh^2(\theta)}} = cosh(\theta) \,. \tag{4}$$

Now again show that

$$-t'^2 + x'^2 = -t^2 + x^2 \tag{5}$$

using the Lorentz transformations (3).

2

Energy,E and momentum,p also transform under Lorentz transformations as follows:

$$E' = \frac{E - vp}{\sqrt{1-v^2}} \qquad p' = \frac{p - vE}{\sqrt{1-v^2}} \,. \tag{6}$$

Show

$$-E^2 + p^2 \tag{7}$$

is the quantity that is left invariant by these transformations? Define this invariant quantity to be $-m^2$. In the frame where p=0 and putting back in the factors of c what famous equation have you derived?

3

Extract the metric, $g_{\mu\nu}$ from the following line element:

$$ds^2 = -f(y)dt^2 + 2f(y)\gamma dxdt + f(y)dx^2 + dy^2 + dz^2 \tag{8}$$

with γ a constant. What is $g^{\mu\nu}$? Show that in the new coordinates (T,X,y,z) given by

$$T = \sqrt{1+\gamma^2}\, t \qquad X = x + \gamma t \tag{9}$$

the metric is diagonal. Hint: calculate $-dT^2 + dX^2$.

4

For two dimensions take with coordinates, $x^\mu = (t, x)$ Take the line element to be:

$$ds^2 = -f(t,x)dt^2 + 2g(t,x)dtdx + h(t,x)dx^2 \tag{10}$$

Write out the metric. Then simply write out the full expressions for

$$x^\mu x_\mu \,. \tag{11}$$

Given vector, $E^\mu = (E, p)$ What is:

$$E_\mu \qquad E_\mu x^\mu \qquad E^\nu E_\nu \qquad x^\mu E_\nu x^\nu E_\mu \qquad x^\mu E^\nu x_\mu E_\nu \tag{12}$$