Spacetime and Gravity: Assignment 3 Solutions

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In what follows, unless otherwise stated, we will use units such that the speed of light, $c = 1$.

1.

We are given the 2 tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{1}$$

and the vector

$$j^{\mu} = \begin{pmatrix} Q \\ j_x \\ j_y \\ j_z \end{pmatrix} \tag{2}$$

and we are asked to calculate

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{3}$$

So we make use of the summation convention and expand the equation in terms of components of the tensor

$$\partial_0 F^{0\nu} + \partial_1 F^{1\nu} + \partial_2 F^{2\nu} + \partial_3 F^{3\nu} = j^\nu \tag{4}$$

Now we pick ν manually. Pick $\nu = 0$ first,

$$\begin{aligned} \partial_0 F^{00} + \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} &= j^0 && (5) \\ \partial_x E_x + \partial_y E_y + \partial_z E_z &= Q && (6) \\ \Rightarrow \nabla \cdot E &= Q && (7) \end{aligned}$$

Now pick $\nu = 1, 2, 3$ in turn. For $\nu = 1$:

$$\begin{aligned} \partial_0 F^{01} + \partial_2 F^{21} + \partial_3 F^{31} &= j^1 && (8) \\ -\partial_t E_x + \partial_y B_z - \partial_z B_y &= j_x && (9) \\ & && (10) \end{aligned}$$

Similarly, $\nu = 2$

$$\partial_0 F^{02} + \partial_1 F^{12} + \partial_3 F^{32} = j^2 \quad (11)$$
$$-\partial_t E_y - \partial_x B_z + \partial_z B_x = j_y \quad (12)$$
$$(13)$$

and $\nu = 3$,

$$\partial_0 F^{03} + \partial_1 F^{13} + \partial_2 F^{23} = j^3 \quad (14)$$
$$-\partial_t E_z + \partial_x B_y - \partial_y B_x = j_z \quad (15)$$
$$(16)$$

Adding the last three relations together we obtain

$$-\partial_t \tilde{E} + \nabla \times \tilde{B} = \tilde{j} \quad (17)$$

We have obtained Maxwell's equations! This means that we can recast the whole of Electromagnetism in a special relativistic form, thus making lorentz covariance explicit.

2.

We are given the line element of the unit 2-sphere:

$$ds^2 = d\theta^2 + \sin^2(\theta) d\phi^2 \quad (18)$$

¿From which we can extract the metric:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{pmatrix} \quad (19)$$

The inverse metric is then

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{sin^2(\theta)} \end{pmatrix} \quad (20)$$

Using the following relation

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} (\partial_\gamma g_{\delta\beta} + \partial_\beta g_{\delta\gamma} - \partial_\delta g_{\beta\gamma}) \quad (21)$$

We calculate the Christoffel symbols

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} (\partial_\gamma g_{\delta\beta} + \partial_\beta g_{\delta\gamma} - \partial_\delta g_{\beta\gamma}) \quad (22)$$
$$= \frac{1}{2} g^{\alpha 1} (\partial_\gamma g_{1\beta} + \partial_\beta g_{1\gamma} - \partial_1 g_{\beta\gamma}) \quad (23)$$
$$+ \frac{1}{2} g^{\alpha 2} (\partial_\gamma g_{2\beta} + \partial_\beta g_{2\gamma} - \partial_2 g_{\beta\gamma}) \quad (24)$$
$$\Gamma^{1}_{\beta\gamma} = \frac{1}{2} g^{11} (\partial_\gamma g_{1\beta} + \partial_\beta g_{1\gamma} - \partial_1 g_{\beta\gamma}) \quad (25)$$

$$\Rightarrow \Gamma^1_{11} = \frac{1}{2}g^{11}(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}) \quad (26)$$
$$= \frac{1}{2}\left(\frac{\partial(1)}{\partial\theta}\right) \quad (27)$$
$$= 0 \quad (28)$$
$$\Rightarrow \Gamma^1_{22} = \frac{1}{2}g^{11}(-\partial_1 g_{22}) \quad (29)$$
$$= -\frac{1}{2}\left(\frac{\partial(\sin^2(\theta))}{\partial\theta}\right) \quad (30)$$
$$= -\sin(\theta)\cos(\theta) \quad (31)$$
$$\Rightarrow \Gamma^1_{12} = \frac{1}{2}g^{11}(\partial_2 g_{11} + \partial_1 g_{12} - \partial_2 g_{12}) \quad (32)$$
$$= 0 = \Gamma^1_{21} \quad (33)$$

And

$$\Gamma^2_{\beta\gamma} = \frac{1}{2}g^{22}(\partial_\gamma g_{2\beta} + \partial_\beta g_{2\gamma} - \partial 2 g_{\beta\gamma}) \quad (34)$$
$$\Rightarrow \Gamma^2_{22} = \frac{1}{2}g^{22}(\partial_2 g_{22} + \partial_2 g_{22} - \partial_2 g_{22}) \quad (35)$$
$$= 0 \quad (36)$$
$$\Rightarrow \Gamma^2_{11} = \frac{1}{2}g^{22}(\partial_1 g_{21} + \partial_1 g_{21} - \partial_2 g_{11}) \quad (37)$$
$$= 0 \quad (38)$$
$$\Rightarrow \Gamma^2_{21} = \frac{1}{2}g^{22}(\partial_1 g_{22} + \partial_2 g_{21} - \partial_2 g_{21}) \quad (39)$$
$$= \frac{1}{2\sin^2(\theta)}\left(\frac{\partial(\sin^2(\theta))}{\partial\theta}\right) \quad (40)$$
$$= \frac{1}{2\sin^2(\theta)} 2\sin(\theta)\cos(\theta) \quad (41)$$
$$= \frac{\cos(\theta)}{\sin(\theta)} \quad (42)$$
$$= \cot(\theta) = \Gamma^2_{12} \quad (43)$$

These are the Christoffel symbols of our space.

1 Summary of important concepts

1.We can re-express a classical theory such as Electromagnetism in a form such that its special relativistic nature is manifest. To do this one makes use of the 4-vector notation.
2.Even though in our case we found that the Christoffel symbols are symmetric under exchange of the lower indices:

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta} \quad (44)$$

this is only true in the case of vanishing torsion.