APPENDIX 15 (NON EXAMINABLE)

EVALUATION OF THE INTEGRAL $I = \int_0^\infty dx \frac{x^3}{e^x - 1}$

First consider integral $J = \int \frac{dx}{e^x - 1}$. Let $y = -e^{-x} \Rightarrow dx = -\frac{dy}{y}$

$$\Rightarrow J = \int \frac{dy}{1+y} = \ln(1 - e^{-x})$$

Now integrate I by parts: $I = \int u\,dv = uv - \int v\,du$ where $u = x^3$ & $dv = \frac{dx}{e^x - 1}$

$$\Rightarrow I = \left[x^3 \ln(1 - e^{-x})\right]_0^\infty - 3\int_0^\infty dx.\, x^2 \ln(1 - e^{-x}) \quad \ldots (1)$$

Now $\ln(1+y) \approx y$ for $y \ll 1 \Rightarrow \lim_{x \to \infty} x^3 \ln(1 - e^{-x}) = -x^3 e^{-x} \to 0$

& $1 - e^{-x} \simeq x$ for $x \ll 1 \Rightarrow \lim_{x \to 0} x^3 \ln(1 - e^{-x}) = x^3 \ln x \to 0$

So first term in Eq. (1) vanishes $\Rightarrow I = -3\int_0^\infty dx.x^2 \ln(1 - e^{-x})$

Now expand the logarithm: $\ln(1-y) = -y - \frac{1}{2}y^2 - \frac{1}{3}y^3 - \ldots = -\sum_{n=1}^\infty \frac{1}{n} y^n$

$$\Rightarrow I = 3\int_0^\infty dx.x^2 \sum_{n=1}^\infty \frac{1}{n} e^{-nx} = 3\sum_{n=1}^\infty \frac{1}{n} \int_0^\infty dx.x^2 e^{-nx}$$

Let $z = nx \Rightarrow I = 3\sum_{n=1}^\infty \frac{1}{n} \int_0^\infty \frac{dz}{n}.\frac{z^2}{n^2} e^{-z} = 3\sum_{n=1}^\infty \frac{1}{n^4} \int_0^\infty dz.z^2 e^{-z}$

Integrate by parts: $I = 3\sum_{n=1}^\infty \frac{1}{n^4} \left\{ \underbrace{\left[-z^2 e^{-z}\right]_0^\infty}_{=0} + 2\int_0^\infty dz.z e^{-z} \right\}$

& by parts again: $I = 3\sum_{n=1}^\infty \frac{1}{n^4} \left\{ \underbrace{2\left[-z e^{-z}\right]_0^\infty}_{=0} + 2\int_0^\infty dz.e^{-z} \right\}$

$\Rightarrow I = 6\sum_{n=1}^\infty \frac{1}{n^4} \int_0^\infty dz e^{-z} = 6\sum_{n=1}^\infty \frac{1}{n^4}$ i.e. $I = 6\zeta(4)$ where

$\zeta(p) = \sum_{n=1}^\infty \frac{1}{n^p}$ is the Riemann Zeta function.

Its numerical values are tabulated & $\zeta(4) = \frac{\pi^4}{90}$. Hence $I = \frac{\pi^4}{15}$ as stated