Queen Mary
University of London

M.Sci. EXAMINATION

MAS401 Advanced Cosmology

22 May 2008
Time 18:15-21:15

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best FOUR questions answered will be counted.

You must not start reading the question paper until instructed to do so.

Calculators ARE permitted in this examination, but no programming, graph plotting or algebraic facility may be used. The unauthorised use of material stored in a pre-programmable memory constitutes an examination offence. Please state on your answer book the name and type of machine used.

The following constants may be assumed:

Speed of light, $c = 3.0 \times 10^{8}\,\mathrm{m\,s^{-1}}$

Gravitational constant, $G = 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$

Boltzmann's constant, $k_B = 1.38 \times 10^{-23}\,\mathrm{JK^{-1}}$

Radiation constant, $\alpha = 7.565 \times 10^{-16}\,\mathrm{J\,m^{-3}\,K^{-4}}$

Proton mass–energy, $m_p c^2 = 938.3\,\mathrm{MeV}$

Neutron mass–energy, $m_n c^2 = 939.6\,\mathrm{MeV}$

Mega Parsec, $1\,\mathrm{Mpc} = 3.09 \times 10^{22}\,\mathrm{m}$

Hubble time, $H_0^{-1} = 9.8 \times 10^{9} h^{-1}\,\mathrm{yr} = 3.09 \times 10^{17} h^{-1}\,\mathrm{s}$

The Conversion Factor, $1\,\mathrm{eV} = 1.602 \times 10^{-19}\,\mathrm{J}$

The Conversion Factor, $1\,\mathrm{J} = 1\,\mathrm{kg\,m^2\,s^{-2}}$

The following formulae may be assumed:

Friedmann Equation

$$H^2 = \frac{8\pi G}{3}\rho + \frac{8\pi G}{3}\Lambda - \frac{kc^2}{a^2},$$

where $H = \dot{a}/a$ is the Hubble parameter, a is the scale factor of the universe, ρ is the mass density, Λ is the cosmological constant, k is a constant and overdots denote time derivatives.

Conservation Equation

$$\dot{\rho} + 3H\left(\rho + \frac{p}{c^2}\right) = 0,$$

where p represents the pressure of the matter in the universe.

1. (a) *[5 marks]* Using Newtonian physics and the Cosmological Principle, apply energy conservation to a small mass of fluid in an homogeneous, expanding medium to derive the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}.$$

[You may assume the cosmological constant vanishes, $\Lambda = 0$.]

(b) *[6 marks]* A group of cosmologists assume that the universe is dominated by a matter source with an equation of state $p = (\gamma - 1)\rho c^2$, where γ is a constant. They decide they would like the universe to be undergoing a phase of accelerated expansion at the present time such that $a \propto t^{10/3}$. Assuming $k = \Lambda = 0$ in the Friedmann equation, what value of γ is required to achieve this?

(c) *[5 marks]* Assuming that the universe has always expanded as $a \propto t^{10/3}$, derive an expression for its age in terms of the present-day value of the Hubble constant. Is such a universe older than the oldest known globular clusters?

(d) *[5 marks]* Explain why, for a given value of the Hubble constant, a positively-curved universe is necessarily younger than a spatially flat universe of the same matter content.

(e) *[4 marks]* Explain why the real universe could not have expanded as $a \propto t^{10/3}$ for its entire history.

2. (a) *[4 marks]* Give the definitions for the critical density, $\rho_c(t)$, and the Ω-parameter. Show that when $\Lambda = 0$,

$$\Omega - 1 = \frac{kc^2}{a^2H^2}.$$

[This question continues overleaf ...]

(b) *[5 marks]* Observations indicate that the present-day value of Ω is $\Omega_0 = 1.02$. Assuming the universe contains only pressureless matter, explain how Ω will evolve in the future. Can Ω become infinite in this case? [Explain your reasoning.]

(c) *[6 marks]* Assuming $\Omega_0 = 1.02$ and $\Lambda = 0$, estimate the value of Ω at the epoch of matter-radiation equality, $t_{\mathrm{eq}} \approx 8 \times 10^{10}$ sec.

(d) *[6 marks]* Explain, qualitatively, how inflation is able to solve the flatness problem. [You should include in your answer a sketch of how Ω varies with time before, during and after inflation.]

(e) *[4 marks]* If the value of Ω at the start of inflation was $\Omega_b = 2$ and its value at the end of inflation was $\Omega_f = 1 + 10^{-52}$, calculate by what factor the volume of the universe increased during inflation.

3. (a) *[5 marks]* Describe, briefly and qualitatively, how the decay of the cosmological constant at the end of inflation provides the initial conditions for the generation of temperature anisotropies in the Cosmic Microwave Background.

(b) *[3 marks]* Consider a universe containing only pressureless matter with no cosmological constant. A sphere of density ρ and radius R evolves in this universe such that

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\rho.$$

Determine how the density of the sphere varies with time. [You may assume that the sphere expands as $R \propto t^{2/3}$.]

(c) *[6 marks]* Consider now a sphere with the same mass, but with a slightly different radius and density $\rho' = \rho(1+\delta)$, where δ defines the density perturbation, which evolves as

$$\ddot{\delta} + 2\frac{\dot{R}}{R}\dot{\delta} - 4\pi G\rho\delta = 0.$$

Determine how δ depends on the radius of the sphere as R increases.

(d) *[6 marks]* Review the main observational evidence that the initial value of the density perturbation immediately after the end of inflation was approximately $\delta \approx 10^{-5}$.

(e) *[5 marks]* Given that initially $\delta \approx 10^{-5}$, explain why your answer to part (c) would be a problem in a universe that was dominated by ordinary (baryonic) matter.

4. (a) *[6 marks]* What is the observational evidence that the density of the universe today is very close to the critical density and that about 70% of the density is due to a cosmological constant?

(b) *[3 marks]* Given that the critical density today is $\rho_{c,0} = 1.9 \times 10^{-26} h^2\,\mathrm{kg\,m^{-3}}$, calculate the corresponding *energy* density of the cosmological constant.

(c) *[4 marks]* Suppose that at some time in the very far future, the cosmological constant has come to entirely dominate the universe, but then undergoes an instantaneous decay to blackbody radiation. Calculate the temperature, T, of this newly formed radiation. [You may assume that the energy density of the radiation is given by $\epsilon_{\rm rad} = \alpha T^4$, where α is the radiation constant]. How does this temperature compare to the present-day temperature of the Cosmic Microwave Background?

(d) *[4 marks]* Calculate the corresponding number density of photons that would be produced by such a decay of the cosmological constant. [You may assume that the typical energy of a photon in a blackbody distribution is $3k_BT$.] How does this number density compare to the present-day number density of photons in the Cosmic Microwave Background?

(e) *[4 marks]* If the cosmological constant never decays, describe qualitatively the ultimate fate of our universe in the cases where the density of the universe is either less than or equal to the critical density.

(f) *[4 marks]* Starting from the Friedmann equation, derive a sufficient condition on Λ in terms of the present-day value of the scale factor, a_0, for a closed universe to expand forever.

5. (a) *[4 marks]* State how the energy densities of relativistic and non-relativistic particles vary with the scale factor of the universe. How does the temperature of the universe vary with the scale factor?

(b) *[3 marks]* Why is it a good approximation to neglect the effects of the curvature term in the Friedmann equation at very early times in the universe's history?

(c) *[4 marks]* Given that the temperature of the universe was approximately 10^{10} K when it was one second old, derive an expression that relates the universe's temperature to its age which is valid for the first 10,000 years of its history.

(d) *[7 marks]* Suppose that a small fraction of matter in the universe consisted of stable, non-relativistic particles when the universe was approximately $t \approx 10^{-30}$ sec old. Calculate the limit on the density of these particles at that time relative to the critical density to ensure that they do not dominate the universe at the onset of primordial nucleosynthesis.

(e) *[7 marks]* Given that the epoch of matter-radiation equality occurred at $t_{\rm eq} \approx 8 \times 10^{10}$ sec, derive an estimate for the temperature of the universe at that time. Estimate the age of the universe when its temperature was the same as the boiling point of water.

[Next question overleaf.]

6. Write short notes on the following topics *[5 marks each]*:

(a) With the use of a diagram, explain how Hubble's law can be deduced from the Cosmological Principle.

(b) The origin of the Cosmic Microwave Background.

(c) The Horizon Problem.

(d) Explain why Primordial Nucleosynthesis represents one of the key predictions of the big bang scenario and how the instability of the deuterium nucleus affects the abundance of helium nuclei.

(e) Use the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)$$

to deduce that a universe containing only pressureless matter and radiation has a finite age.

[End of examination paper.]