COSMOLOGY ASTM108

EXERCISE SHEET 6

1. $$\epsilon_i(p) = \frac{4\pi c}{h^3}\left(\frac{7}{8}g_i\right)\int_0^\infty \frac{p^3\,dp}{e^{pc/k_BT}-1}$$

Let $x = \frac{pc}{k_BT}$ $\Rightarrow$ $dx = \frac{c}{k_BT}\,dp$ $\Rightarrow$ $\epsilon_i = \frac{4\pi c}{h^3}\left(\frac{7}{8}g_i\right)\frac{k_B^4T^4}{c^4}\underbrace{\int_0^\infty \frac{x^3\,dx}{e^x-1}}_{=\frac{\pi^4}{15}}$

$$\Rightarrow \epsilon_i = \frac{4\pi}{c^3h^3}k_B^4T^4\left(\frac{7}{8}g_i\right)\frac{\pi^4}{15}$$

Radiation constant, $\alpha \equiv \frac{\pi^2 k_B^4}{15\hbar^3c^3}$, $(\hbar \equiv h/2\pi)$ $\Rightarrow$ $\epsilon_i = \frac{7}{16}g_i\,\alpha T^4$

Degeneracy factor of neutrinos, $g_\nu = 1$. There are 3 types of neutrino & each has an antiparticle, $\Rightarrow$ total number of neutrino types is 6, so combined energy density is $\epsilon_{\nu\bar{\nu}} = 6\times\frac{7}{16}\times 1\times\alpha T^4 = \frac{21}{8}\alpha T^4$. But $\epsilon_{rad} = \alpha T^4$ is the energy density of the photons at temperature T (5)

2. $$\epsilon_{tot} = \epsilon_\gamma + \epsilon_{e^-} + \epsilon_{e^+} + \epsilon_\nu$$

~~$\epsilon_\gamma = \alpha T^4$, $\epsilon_{e^-} = \frac{7}{8}$~~ Photons: $\epsilon_\gamma = \alpha T^4$

Degeneracy factor of e^- is $g_{e^-} = 2$. Since they are fermions, $\epsilon_{e^-} = \frac{7}{8}\cdot\alpha T^4$

" " " e^+ " $g_{e^+} = 2$. " " " " $\epsilon_{e^+} = \frac{7}{16}\cdot 2\cdot\alpha T^4 = \frac{7}{8}\alpha T^4$

" " " ν is $g_\nu = 1$. Since a neutrino is fermionic, $\epsilon_\nu = \frac{7}{16}\cdot\alpha T^4$

But from above there are 3 types of neutrino & 3 antineutrinos $\Rightarrow$

$$\epsilon_\nu = 6\times\frac{7}{16}\alpha T^4 = \frac{21}{8}\alpha T^4.$$

$\Rightarrow$ Total Energy density is $\epsilon_{tot} = \left(1 + \frac{7}{8} + \frac{7}{8} + \frac{21}{8}\right)\alpha T^4 = \underline{\underline{\frac{43}{8}}}\,\alpha T^4$ (5) (10)

Let $g_* = \frac{43}{8}$. Since $a \propto t^{1/2}$ $\Rightarrow$ $H^2 = \frac{1}{4t^2} = \frac{8\pi G}{3c^2}\epsilon_{tot} = \frac{8\pi G\,g_*\alpha}{3c^2}\cdot T^4$

call this something else, cf. eq (7.21) & def $= \alpha$ got

$$\Rightarrow \text{age} = t = \underline{\underline{\left(\frac{32\pi G\,g_*\alpha}{3c^2}\right)^{-\frac{1}{2}} T^{-2}}}$$

The only difference is the extra factor of $g_*^{-1/2}$. In the lectures we just assumed the universe is dominated by photons for simplicity. The numerical value of the correction is $(43/8)^{-1/2} \simeq 0.4$ ie about $1/2$.

5. Mass energy of the electrons is $m_e c^2 \simeq 0.51\ \mathrm{MeV}$. The fall out of eq^{n} when energy of photons < mass energy. Typical energy of photon @ temperature T is $k_B T$
$\Rightarrow$ $e^+ e^-$ fall out of eq^{n} when

$$k_B T \simeq m_e c^2 \Rightarrow T \simeq \frac{0.51\times10^6}{8.6\times10^{-5}}\ \mathrm{K} \simeq 6\times10^9\ \mathrm{K}$$

age of universe $\left(\frac{t}{\mathrm{sec}}\right)^{-\frac{1}{2}} \simeq \frac{T}{1.5\times10^{10}\mathrm{K}} \Rightarrow t \simeq \left(\frac{6\times10^9}{1.5\times10^{10}}\right)^{-2} s \simeq 6$ seconds old $\boxed{10}$

3. From Question 1, Problem Sheet 3, we saw that $(1-\Omega^{-1})_{eq} = \frac{(1-\Omega^{-1})_0}{1+z_{eq}}$

For $\Omega_0 \simeq 0.3$, $\Rightarrow \Omega_{eq} = 0.9996$ is the density of the universe was very close to critical density (to with 99.96%). Since critical density corresponds to density of the flat universe ($k=0$) $\Rightarrow$ curvature term negligible at early times (certainly before epoch of matter-radiation equality). $\boxed{10}$

5. $\epsilon_{tot} = \frac{1}{2} g_* \alpha T^4$

In early universe $|\Omega - 1| << 1 \Rightarrow$ OK to set $k=0$ in Friedmann eq^{n}.
Before matter-rad^{n} equality, universe dominated by radiation (photons) & relativistic particles $\Rightarrow a \propto t^{1/2} \Rightarrow H = \frac{1}{2t}$

$$H^2 = \frac{8\pi G}{3}\frac{\epsilon_{tot}}{c^2} = \frac{1}{4t^2} = \frac{8\pi G}{3c^2}\frac{\alpha g_*}{2} T^4$$

$$\Rightarrow \frac{1}{t^2} = \frac{16\pi G \alpha g_*}{3c^2} T^4 = g_* T^4 \left[\frac{16\pi}{3} \cdot \frac{6.7\times10^{-11} \times 7.6\times10^{-16}}{(3\times10^8)^2}\right] \quad \frac{m^3 kg^{-1} \cancel{s^{-2}}\, J\, \cancel{m^{-3}} K^{-4}}{m^2 \cancel{s^{-2}}}$$

$$\Rightarrow \frac{1}{t^2} = \frac{2.5\times10^{-24}}{2.7\times10^{17}} g_* \left(\frac{T}{K}\right)^4 s^{-2}$$ $(1\,\mathrm{J} = 1\,\mathrm{kg\,m^2 s^{-2}})$

$$\frac{t}{\mathrm{sec}} \simeq \frac{(3\times10^{20})}{\sqrt{g_*}} \cdot \left(\frac{T}{K}\right)^{-2}$$ $\boxed{10}$