V

EXERCISE V

$H^2 = \frac{8\pi G\rho}{3} + \frac{8\pi G\Lambda}{3}$ for $k=0$

$\Omega_m = \frac{8\pi G\rho}{3H^2}$ & $\Omega_\Lambda = \frac{8\pi G\Lambda}{3H^2}$. Since $\rho = \rho_0\frac{a_0^3}{a^3} \Rightarrow H^2 = \frac{8\pi G\rho_0 a_0^3}{3a^3} + \frac{8\pi G\Lambda}{3}$

$$\Rightarrow H^2 = \left(\frac{8\pi G\rho_0}{3H_0^2}\right)\frac{H_0^2 a_0^3}{a^3} + \left(\frac{8\pi G\Lambda}{3H_0^2}\right)H_0^2 = \Omega_{m0}\frac{H_0^2 a_0^3}{a^3} + \Omega_{\Lambda0}H_0^2$$

Age: $$t_0 = \int_0^{a_0}\frac{da}{aH} = \int_0^{a_0}\frac{da}{a}\frac{1}{\left[\frac{H_0^2 a_0^3}{a^3}\Omega_{m0} + \Omega_{\Lambda0}H_0^2\right]^{1/2}} = \frac{1}{H_0}\int_0^{a_0}\frac{da\;a^{1/2}}{\left[a_0^3\Omega_{m0} + \Omega_{\Lambda0}a^3\right]^{1/2}}$$

Let $y = \left(\frac{\Omega_{\Lambda0}}{\Omega_{m0}a_0^3}\right)^{\frac{1}{2}} a^{3/2} \Rightarrow dy = \frac{3}{2}\left(\frac{\Omega_{\Lambda0}}{\Omega_{m0}a_0^3}\right)^{\frac{1}{2}} a^{1/2}\,da$

$$\Rightarrow t_0 = \frac{1}{H_0}\int_0^{y_0}\frac{2}{3}\left(\frac{\Omega_{m0}a_0^3}{\Omega_{\Lambda0}}\right)^{\frac{1}{2}} dy\,\frac{1}{\left[a_0^3\Omega_{m0} + \left(\Omega_{\Lambda0}\cdot\frac{\Omega_{m0}a_0^3}{\Omega_{\Lambda0}}\right)y^2\right]^{1/2}}$$

$$t_0 = \frac{1}{H_0}\frac{2}{3}\left(\frac{\Omega_{m0}a_0^3}{\Omega_{\Lambda0}}\right)^{\frac{1}{2}}\int_0^{y_0}\frac{dy}{\left[a_0^3\Omega_{m0} + \Omega_{m0}a_0^3y^2\right]^{1/2}} = \frac{2}{3H_0}\frac{1}{\sqrt{\Omega_{\Lambda0}}}\int_0^{y_0}\frac{dy}{\left[1+y^2\right]^{1/2}}$$

$$\Rightarrow t_0 = \frac{2}{3H_0}\frac{1}{\sqrt{\Omega_{\Lambda0}}}\left[\ln\left[y+\sqrt{1+y^2}\right]\right]_0^{y_0} = \frac{2}{3H_0}\frac{1}{\sqrt{\Omega_{\Lambda0}}}\ln\left[y_0+\sqrt{1+y_0^2}\right]$$

Definition of $y \Rightarrow$ $y_0 = \left(\frac{\Omega_{\Lambda0}}{\Omega_{m0}}\right)^{\frac{1}{2}} = \left(\frac{\Omega_{\Lambda0}}{1-\Omega_{\Lambda0}}\right)^{\frac{1}{2}}$

$$\Rightarrow t_0 = \frac{2}{3H_0}\frac{1}{\sqrt{\Omega_{\Lambda0}}}\ln\left[\left(\frac{\Omega_{\Lambda0}}{\Omega_{m0}}\right)^{\frac{1}{2}} + \sqrt{1+\frac{\Omega_{\Lambda0}}{\Omega_{m0}}}\right]$$

$$\Rightarrow t_0 = \frac{2}{3H_0}\frac{1}{\sqrt{\Omega_{\Lambda0}}}\ln\left[\sqrt{\frac{\Omega_{\Lambda0}}{\Omega_{m0}}} + \frac{1}{\sqrt{1-\Omega_{\Lambda0}}}\right] = \frac{2}{3H_0}\frac{1}{\sqrt{\Omega_{\Lambda0}}}\ln\left\{\frac{1+\sqrt{\Omega_{\Lambda0}}}{\sqrt{1-\Omega_{\Lambda0}}}\right\}$$

$\Rightarrow$ Since $\frac{2}{3H_0} = \frac{6.5}{h}$ Gyr $\Rightarrow$ $t_0 = \frac{6.5}{h}\frac{1}{\sqrt{\Omega_{\Lambda0}}}\ln\left\{\frac{1+\sqrt{\Omega_{\Lambda0}}}{\sqrt{1-\Omega_{\Lambda0}}}\right\}$

~~If the Universe is open => older than the flat models~~