Exercise 4

1. a) $p=0$ & $\Omega_0=1$ $\Rightarrow$ $a \propto t^{2/3}$

$H^2 \propto \rho$ from the Friedmann eq^n $\Rightarrow$ $H^2 = H_0^2\left(\frac{\rho}{\rho_0}\right)$

But pressureless matter $\Rightarrow$ $\rho = \rho_0\left(\frac{a_0}{a}\right)^3$ $\Rightarrow$ $H = H_0\left(\frac{a_0}{a}\right)^{3/2}$

Definition of redshift, $1+z = \frac{a_0}{a}$ $\Rightarrow$ $\underline{H(z) = H_0(1+z)^{3/2}}$ (5)

b) Proper distance: $D_p = a_0\chi = c\int_0^z \frac{dz}{H(z)}$

$$\Rightarrow \quad D_p = \frac{c}{H_0}\int_0^z dz\,(1+z)^{-3/2} = -\frac{2c}{H_0}\left[(1+z)^{-1/2}\right]_0^z = -\frac{2c}{H_0}\left[\frac{1}{(1+z)^{1/2}}-1\right]$$

ie $\underline{D_p = \frac{2c}{H_0}\left[1-\frac{1}{\sqrt{1+z}}\right]}$ (5)

c) in general, proper distance is given by $D_p(t) = a(t)\chi$. Since χ is a comoving coordinate, it is time independent $\Rightarrow$

$$\frac{dD_p}{dt} = \frac{da}{dt}\chi = \dot{a}\chi = \frac{\dot{a}}{a}\cdot\frac{a}{a_0}a_0\chi = \frac{H}{1+z}a_0\chi$$

$$\Rightarrow \dot{D}_p = H_0(1+z)^{1/2}a_0\chi$$

we calculated $a_0\chi$ in part b) $\Rightarrow$ $\dot{D}_p = H_0(1+z)^{1/2}\frac{2c}{H_0}\left[1-\frac{1}{\sqrt{1+z}}\right]$

$$\Rightarrow \dot{D}_p = 2c\left[\sqrt{1+z}-1\right]$$ (5)

d) Expanding answer to part b @ small redshifts, $z \ll 1$

$$\Rightarrow D_p = \frac{2c}{H_0}\left[1-(1+z)^{-1/2}\right] \simeq \frac{2c}{H_0}\left[1-\left(1-\frac{1}{2}z+\frac{3}{8}z^2\right)\right] \simeq \frac{2c}{H_0}\left[\frac{1}{2}z \mp \frac{3}{8}z^2\right]$$

$$\Rightarrow D_p \simeq \frac{c}{H_0}\left[z \mp \frac{3}{4}z^2\right]$$

~~From the general (approximate) expression derived in lectures: $D_p \simeq \frac{c}{H_0}\left[z \mp \frac{1}{2}(1+q_0)z^2\right]$~~

~~$\Rightarrow -\frac{1}{2}(1+q_0) = \frac{3}{4}$ $\Rightarrow$ $q_0 = -\frac{1}{2}$~~

From the general (approximate) expression derived in the notes, $D_p \simeq \frac{c}{H_0}\left[z-\frac{1}{2}(1+q_0)z^2\right]$

$\Rightarrow$ $\frac{3}{4} = \frac{1}{2}(1+q_0)$ $\Rightarrow$ $q_0 = +\frac{1}{2}$ [In general $q_0 = \frac{1}{2}\Omega_0$ for pressureless matter]

2a) Definition : $q = -\frac{a\ddot{a}}{\dot{a}^2} = -\frac{\ddot{a}}{aH^2}$

Acceleration Equation : $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left[\rho + 3p/c^2\right]$

For the Equation of State $p = (\gamma-1)\rho c^2 \Rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left[\rho + 3(\gamma-1)\rho\right] = -\frac{4\pi G}{3}[3\gamma-2]\rho$

Definition of Ω-parameter : $\Omega \equiv \frac{8\pi G\rho}{3H^2}$

$\Rightarrow \frac{\ddot{a}}{aH^2} = -\left(\frac{3\gamma-2}{2}\right)\frac{8\pi G\rho}{3H^2} = -\left(\frac{3\gamma-2}{2}\right)\Omega$

$\Rightarrow \underline{q = \frac{3\gamma-2}{2}\Omega}$ (5)

b) If $k=0$ & we have pressureless matter & a cosmological constant:

$H^2 = \frac{8\pi G}{3}\rho_m + \frac{8\pi G\rho_\Lambda}{3}$

$\Rightarrow 1 = \Omega_m + \Omega_\Lambda$

Acceleration eqn : $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left[\rho_m + \rho_\Lambda + \frac{3p_\Lambda}{c^2}\right]$ since $p_m = 0$

But $p_\Lambda = -\rho_\Lambda c^2 \Rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left[\rho_m - 2\rho_\Lambda\right]$

$\frac{\ddot{a}}{aH^2} = -\frac{1}{2}\left[\frac{8\pi G\rho_m}{3H^2} - \frac{16\pi G}{3}\frac{\rho_\Lambda}{H^2}\right] = -\frac{1}{2}\Omega_m + \Omega_\Lambda$

$\Rightarrow \underline{q = \frac{\Omega_m}{2} - \Omega_\Lambda}$

Actually since $\Omega_\Lambda = 1-\Omega_m$ for this model, $q = \frac{\Omega_m}{2} - (1-\Omega_m) = \frac{3\Omega_m}{2} - 1$ (5)

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