Problem set 3. Solutions

25 Q1. From Friedmann eq.

$$H^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2}$$ and difintion of Ω

$$\Omega = \frac{8\pi G\rho}{3H^2} \Rightarrow H^2 = \frac{8\pi G\rho}{3\Omega}$$ and

$$\frac{8\pi G\rho}{3\Omega} = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2}$$

$$\frac{8\pi G\rho}{3}\left(1-\Omega^{-1}\right) = \frac{kc^2}{a^2} \Rightarrow \rho a^2\left(1-\Omega^{-1}\right) = \frac{3kc^2}{8\pi G} = \text{const} \qquad (10)$$

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From (10) $\Rightarrow$

$$\rho_{eq} a_{eq}^2\left(1-\Omega^{-1}\right)_{eq} = \rho_0 a_0^2\left(1-\Omega^{-1}\right)_0 \qquad (11)$$

If $\rho_{rad} \ll \rho_{matter} \Rightarrow$

$$\rho_{eq} = \rho_0\left(\frac{a_0}{a_{eq}}\right)^3 \qquad (12)$$

From (11) and (12) and $\Omega_0 = 0.3$

$$\rho_0 a_0^3 a_{eq}^{-3} a_{eq}^2\left(1-\Omega^{-1}\right)_{eq} = \rho_0 a_0^2\left(1-\frac{10}{3}\right)$$

$$\left(1-\Omega^{-1}\right)_{eq} = -\frac{a_{eq}}{a_0}\cdot\frac{7}{3} = -\frac{7}{3(1+z_{eq})}$$

$$1-\Omega_{eq}^{-1} = -\frac{7}{3(1+z_{eq})} \qquad \Omega_{eq}^{-1} = 1+\frac{7}{3(1+z_{eq})}$$

$$\Omega_{eq} = \left[1+\frac{7}{3(1+z_{eq})}\right]^{-1} \approx 1-\frac{7}{3\cdot 6000} \approx 1-4\cdot 10^{-4}$$

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This means that at the epoch of matter-radiation equility the Universe was suprisingly close to flat.

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25 Q2. From Friedmann eq. with $k=0$ and radiation domination we have

$$H^2 = \frac{8\pi G \rho_0}{3} \frac{a_0^4}{a^4} = H_0^2 \frac{a_0^4}{a^4}$$

From the general expression (5.27) [Lecture 5], we have

$$t_0 = \int_0^{a_0} \frac{da}{aH} = \int_0^{a_0} \frac{da\, a^2}{a H_0 a_0^2} = \frac{1}{H_0 a_0^2} \int_0^{a_0} a\, da =$$

$$= \frac{1}{2 H_0 a_0^2} a_0^2 = \frac{1}{2H_0} = \frac{1\ \text{Mpc}}{2 \cdot 72\ \text{km s}^{-1}} =$$

$$= \frac{3 \cdot 10^6 \cdot 3 \cdot 10^8\ \text{m} \cdot \text{s}^{-1} \cdot 1\ \text{year}}{1.44 \cdot 10^5\ \text{m} \cdot \text{s}^{-1}} \simeq 6 \cdot 10^9\ \text{years} = 6\ \text{Gyr}$$

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Repeating the same calculation for a pressureless flat universe, we have

$$H^2 = \frac{8\pi G \rho_0}{3} \frac{a_0^3}{a^3} = H_0^2 \frac{a_0^3}{a^3}$$

$$\tilde{t}_0 = \int_0^{a_0} \frac{da}{aH} = \int_0^{a_0} \frac{da\, a^{3/2}}{a H_0 a_0^{3/2}} = \frac{1}{H_0 a_0^{3/2}} \int_0^{a_0} da\, a^{1/2} =$$

$$= \frac{2}{3 H_0 a_0^{3/2}} a_0^{3/2} = \frac{2}{3H_0},$$ hence $\frac{t_0}{\tilde{t}_0} = \frac{3}{4}$.

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25 Q3. See solution in lecture 5 (eqs. 5.28 – 5.40) and demonstrate that you do understand what you are doing.

25 Q4. See solution in lecture 5 (eqs. 5.41 – 5.46). the same task as in Q3.