1) Q1 (15) 15

Solutions to problem set 1

1. There are two ways to estimate the average density of matter in the universe:

1.1. Typical mass of galaxies $\sim 10^{11} M_{\odot}$

Typical distance between galaxies ~ 1 Mpc

Hence $$\rho_m \sim \frac{10^{11}\cdot 2\cdot 10^{33}\ g}{(3\cdot 10^{16}\cdot 10^{6}\ m)^3} \sim \frac{2\cdot 10^{44}}{3\cdot 10^{67}}\frac{g}{m^3} \sim$$

$$\sim 6\cdot 10^{-24}\frac{g}{m^3} \sim 10^{-30}\frac{g}{cm^3} = 10^{-27}\frac{kg}{m^3}$$ (10) (10)

1.2. We know that $k \simeq 0$, hence

$$\rho_{TOT} \sim \frac{3H_0^2}{8\pi G} \simeq \frac{3\cdot(72)^2\ km^2}{8\cdot 3\ s^2 (3\cdot 10^{16}\cdot 10^{6} m)^2\cdot 7\cdot 10^{-8}\frac{cm^3}{s^2 g}}$$

$$\simeq \frac{5\cdot 10^3\cdot 10^6\ m^2\ g}{8\cdot 7\cdot 10^{-8}\ 10^{45}\ m^2\ cm^3} \sim \frac{5\cdot 10^{9}\ g}{5\cdot 10^{-7+45}\ cm^3} \simeq$$

$$\simeq 10^{-29}\frac{g}{cm^3} \sim \frac{10^{-23}\ g}{m^3} \simeq 10^{-26}\frac{kg}{m^3}$$

We know that only 1/4 of that in form of matter (the rest in form of dark energy). Hence

$$\rho_{mat} \sim \frac{1}{4}10^{-29}\frac{g}{cm^3} \sim 10^{-30}\frac{g}{cm^3}.$$

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$n_p \simeq \frac{\rho}{m_p} \simeq$

$\simeq 1\ m^{-3}$

Q2 (10)

$$L = \frac{v}{H} = \frac{zc}{H} = \frac{0.1\cdot 3\cdot 10^{8}\ m\ s^{-1}}{72\cdot km\ s^{-1}(Mpc)^{-1}} \simeq 416\ Mpc$$

② Q3. (30)

a) $1+z = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} = \sqrt{\frac{\left(1+\frac{v}{c}\right)\left(1+\frac{v}{c}\right)}{\left(1-\frac{v}{c}\right)\left(1+\frac{v}{c}\right)}} =$

$$= \frac{1+\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} \approx \left(1+\frac{v}{c}\right)\left(1+\frac{1}{2}\frac{v^2}{c^2}\right) \approx 1+\frac{v}{c}$$

$$1+z \approx 1+\frac{v}{c} \Rightarrow z \approx \frac{v}{c}$$ |10|

b) $R = 1.1 \cdot 10^{7}\, m^{-1}$

$$\lambda_\alpha = \frac{1}{R}\left(1-\frac{1}{4}\right)^{-1} = \frac{4}{3 \cdot 1.1} \cdot 10^{-7}\, m = 1.21 \cdot 10^{-7}\, m \approx$$

$\approx 121\, nm$

$$1+z = \frac{717}{121} = 5.93$$

$$\frac{1+\frac{v}{c}}{1-\frac{v}{c}} = (5.93)^2 \qquad 1+\frac{v}{c} = (5.93)^2\left(1-\frac{v}{c}\right)$$

$$\frac{v}{c} = \frac{(5.93)^2-1}{(5.93)^2+1} = \frac{34.16}{36.16} =$$

0.94

$v = HL$ $\quad L = \frac{0.94 \cdot 3 \cdot 10^{8} \frac{m}{s}}{72 \frac{km}{}} 1\, Mpc = \frac{3 \cdot 0.94 \cdot 10}{72\, M}$

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$L \simeq 4\,\mathrm{Gpc}$

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c) This means that the age of ~~the~~ universe is larger than $4 \cdot 10^9 \cdot 3$ years $\simeq 12 \cdot 10^9$ years.

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Q4 (15)

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$$V_p < 10\%\, V_H = \frac{1}{10} V_H = \frac{1}{10} HL$$

$$L > \frac{10 V_p}{H} \simeq \frac{10 \cdot 600\ \mathrm{km \cdot s^{-1}}}{72\ \mathrm{km \cdot s^{-1}/Mpc}} \simeq 80\,\mathrm{Mpc}$$

Q5. (30) a)

$$\dot{\rho} + \frac{3\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0$$

$$p = (\gamma - 1)\rho c^2$$

$$\dot{\rho} + \frac{3\dot{a}}{a}\rho(1 + \gamma - 1) = 0$$

$$\dot{\rho} + \frac{3\gamma\dot{a}}{a}\rho = 0$$

$$\rho \sim a^{-3\gamma} \Rightarrow \rho = \rho_0\left(\frac{a}{a_0}\right)^{-3\gamma}$$

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b) If $k = 0$

$$H^2 = \frac{8\pi G \rho}{3} \Rightarrow \frac{\dot{a}^2}{a^2} = \frac{8\pi G \rho}{3}$$

(4)

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G \rho_0}{3}\left(\frac{a}{a_0}\right)^{-3\gamma}$$

$$\dot{a} = \sqrt{\frac{8\pi G \rho_0}{3}}\left(\frac{a}{a_0}\right)^{-\frac{3\gamma}{2}+1}$$

$$\dot{a}\, a^{+\frac{3\gamma}{2}-1} = \sqrt{\frac{8\pi G \rho_0 a_0^{3\gamma}}{3}}$$

$$\frac{a^{+\frac{3\gamma}{2}}}{+\frac{3\gamma}{2}} = \left(\frac{8\pi G \rho_0 a_0^{3\gamma}}{3}\right)^{1/2} t$$

$$a = \left(\frac{2}{3\gamma}\right)^{\frac{2}{3\gamma}}\left(\frac{8\pi G \rho_0}{3}\right) a_0\, t^{\frac{2}{3\gamma}}$$

$$\boxed{\frac{a}{a_0} = \left(\frac{t}{t_0}\right)^{2/3\gamma} \qquad \rho = \rho_0\left(\frac{a}{a_0}\right)^{-3\gamma} = \rho_0\left(\frac{t}{t_0}\right)^{-2}}$$

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c) From

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \Rightarrow \ddot{a} > 0, \text{ if}$$

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$$\rho + \frac{3p}{c^2} < 0 \Rightarrow \rho + 3(\gamma - 1)\rho < 0$$

$$1 + 3\gamma - 3 < 0 \Rightarrow \gamma < \frac{2}{3}$$

For $\gamma = \frac{2}{3}$ $\quad \ddot{a} = 0$

(5) d) $\gamma = \frac{2}{3}$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2}$$

$$\rho = \rho_0\left(\frac{a}{a_0}\right)^{-3\gamma} = \rho_0\left(\frac{a}{a_0}\right)^{-2}$$

$$\frac{\dot{a}^2}{\cancel{a^2}} = \frac{8\pi G\rho_0}{3}\left(\frac{a_0}{\cancel{a}}\right)^2 - \frac{kc^2}{\cancel{a^2}}$$

$$\dot{a}^2 = \frac{8\pi G\rho_0 a_0^2}{3} - kc^2 = \text{const}$$

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$$\dot{a} = \sqrt{\frac{8\pi G\rho_0 a_0^2}{3} - kc^2}$$

$$\boxed{a = \sqrt{\frac{8\pi G\rho_0 a_0^2}{3} - kc^2} \cdot t}$$